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LARGE DEVIATIONS AND IDEMPOTENT PROBABILITY

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To my parents

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Preface

This book has grown out of an approach to establishing the large deviation principle (LDP) for probability measures that originates from viewing the LDP as an analogue of weak convergence of probability measures and develops tools for proving it along the lines of the ones used in weak convergence theory. Let us recall that, given a Hausdorff topological space E equipped with Borel σ-algebra $\mathcal{B}(E)$, a function $I : E \to [0, \infty]$ such that the sets $\{z \in E : I(z) \le a\}$ are compact for $a \in \mathbb{R}_+$, a net $\{P_\phi, \phi \in \Phi\}$ of probability measures on $(E, \mathcal{B}(E))$, and a net of non-negative numbers $\{r_\phi, \phi \in \Phi\}$ such that $r_\phi \to \infty$ as $\phi \in \Phi$, the net $\{P_\phi, \phi \in \Phi\}$ is said to obey the LDP with rate function I for scale r_ϕ if

$$\limsup_{\phi \in \Phi} \frac{1}{r_\phi} \ln P_\phi(F) \le - \inf_{z \in F} I(z) \text{ for } F \text{ being a closed subset of } E,$$

$$\liminf_{\phi \in \Phi} \frac{1}{r_\phi} \ln P_\phi(G) \ge - \inf_{z \in G} I(z) \text{ for } G \text{ being an open subset of } E.$$

The definition being modelled after the definition of weak convergence of probability measures, it is not surprising that there are similarities between methods of deriving the LDP and weak convergence, e.g., both theories make use of characteristic functionals, projective limit arguments, continuous mappings, characterisation of relative compactness in terms of certain tightness conditions, and others.

Our purpose is to explore this analogy in more depth and systematically build on it for studying properties of the LDP. The first important step is to recognise and treat the rate function as a limit case of the probability measure rather than merely as an asymptotic value. More precisely, we consider the set function $\Pi(A)$ on E defined by $\Pi(A) = \sup_{z \in A} \exp(-I(z))$ as an analogue and a limit of probabilities, so we call it "a deviability". We next look for proper-

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ties of Π that are inherited from probabilities in the hope that this will help us to identify it, e.g., we are interested in an analogue of the martingale property.

A distinctive feature of deviability is that it is "maxitive" in that $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$. Maxitive set functions have been known as possibility measures in possibility theory and idempotent measures in idempotent measure theory (also referred to by the names "max-plus calculus" and "min-plus calculus"). We adopt the name "idempotent measure"; on the other hand, the notation Π, which is used not only for deviabilities but also for general idempotent measures "of mass 1", is borrowed from possibility theory. Developing "a stochastic calculus" for idempotent measures is the subject of part I of the book. We start with basic axioms, consider extension theorems, measurability issues, idempotent expectations and conditional idempotent expectations, topologies on spaces of idempotent measures, and other analogues of the constructions of probability theory. The axioms for an idempotent measure are mostly the same as the ones used in possibility theory and idempotent measure theory so we recover some of the results of these theories. Besides, we extensively analyse the τ-smoothness property of idempotent measures that requires a certain type of "continuity from above" and has been prompted by the fact that deviabilities are τ-smooth with respect to decreasing nets of closed sets. We also undertake a study in the spirit of the general theory of stochastic processes of idempotent analogues of stopping times, filtrations, stochastic processes, Ito differential equations, martingales and semimartingales (which we call maxingales and semimaxingales, respectively), and martingale problems (referred to as maxingale problems). Being motivated by applications to large deviation theory, by no means do we consider analogues of all standard probability topics, the most notable omissions being analogues of the theory of limit theorems and theory of Markov processes. Our focus is on developing weak convergence theory for idempotent measures and those parts of "maxingale theory" that are instrumental in deriving large deviation limit theorems.

Part II studies the large deviation setting. In order to emphasise the view of a deviability as a limit of probabilities, we refer to "the LDP for the P_ϕ with rate function I" as "large deviation (LD) convergence of the P_ϕ to Π". Thus, the P_ϕ are said to LD converge to

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Π at rate r_ϕ if

$$\limsup_{\phi\in\Phi} P_\phi(F)^{1/r_\phi} \leq \Pi(F) \text{ for } F \text{ closed,} \tag{0.0.1}$$

$$\liminf_{\phi\in\Phi} P_\phi(G)^{1/r_\phi} \geq \Pi(G) \text{ for } G \text{ open.} \tag{0.0.2}$$

In our study we actually use a different form of the definition of LD convergence that states that the P_ϕ LD converge to Π (at rate r_ϕ) if

$$\lim_{\phi\in\Phi}\Bigl(\int_E h(z)^{r_\phi}\, dP_\phi(z)\Bigr)^{1/r_\phi} = \sup_{z\in E} h(z)\Pi(z) \tag{0.0.3}$$

for all $\mathbb{R}_+$-valued bounded and continuous functions h on E. For general Hausdorff spaces properties (0.0.1) and (0.0.2) are stronger than (0.0.3). One could draw the line by referring to (0.0.1) and (0.0.2) as narrow large deviation convergence and (0.0.3) as weak large deviation convergence. However, for Tihonov spaces (i.e., completely regular T_1-spaces), which we mostly consider and which seem to suffice for concrete large deviation settings, the two forms are equivalent; so we refer to the property specified by (0.0.3) as large deviation convergence. The advantage of using definition (0.0.3) is that many proofs can be significantly shortened (which fact is explained to some extent by both the limit and pre-limit objects being certain norms).

We explore general properties of LD convergence in the form (0.0.3) in the first section of part II, where our methods are similar to those of studying weak convergence of measures and idempotent measures. The rest of part II considers LD convergence of the distributions of semimartingales for the Skorohod topology on the space of right-continuous with left-hand limits $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$. Here we are able to implement the approaches used for deriving convergence in distribution for semimartingales such as characterisations of limits in terms of their finite-dimensional distributions and as solutions to martingale problems. We interpret the limit deviabilities as distributions of idempotent processes and state the results in the form of LD convergence in distribution of semimartingales to semimaxingales. For example, we formulate the LDP for diffusion processes with small diffusion terms as LD convergence in distribution to an idempotent diffusion. We give applications to LD convergence of Markov processes and processes arising in queueing systems. Our

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results for queues are in the same theme as corresponding weak convergence results. As a byproduct, the results of part II show that possibility theory can be viewed as a large deviation limit of probability theory.

The book concludes with two appendices. Appendix A proves certain auxiliary results invoked in the main body of the book. Appendix B contains additional comments on the results and bibliographical notes; the latter reflect the author's view of the related work and are by their very nature subjective, nor do we make any claim to completeness of the list of references.

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Basic notation

$\mathbb{R}_+$	$=$	$[0, \infty)$
$\overline{\mathbb{R}}_+$	$=$	$[0, \infty]$
$a \vee b$		the maximum of a and b
a^+	$=$	$a \vee 0$
$a \wedge b$		the minimum of a and b
$\lfloor a \rfloor$		the integer part of a
$f \circ g$		the composition of functions f and g
$\mathbb{N}$		the set of natural numbers
$\mathbb{Z}_+$		the set of non-negative integers
$x \cdot y$		the inner product of vectors x and y
$\|x\|$		the Euclidean norm of a vector x
σ^T		the transpose of a matrix σ
$\\|\sigma\\|$		the operator norm of a matrix σ
$\sigma^\oplus$		the pseudo inverse of a matrix σ
$\sigma^{1/2}$		the square root of a positive semi-definite symmetric matrix σ
A^c		the complement of a set A
A^B		the collection of functions from a set B to a set A
$\mathcal{P}(A)$		the power set of a set A
$\mathcal{Q}(A)$		the collection of finite subsets of a set A
$\mathbf{1}(A),\ \mathbf{1}_A$		the indicator function of a set A
$\operatorname{int} A$		the interior of a subset A of a topological space
$\operatorname{cl} A$		the closure of a subset A of a topological space
$\mathcal{B}(E)$		the Borel σ-algebra on a topological space E
$\overline{\mathcal{B}}(\mathbb{R}_+)$		the Lebesgue σ-algebra on $\mathbb{R}_+$
$\overline{\mathcal{B}}\big([0,t]\big)$		the Lebesgue σ-algebra on $[0,t]$

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Part I

Idempotent Probability Theory

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Chapter 1

Idempotent probability measures

In this chapter we introduce idempotent analogues of basic objects of probability theory such as probability measures, random variables, expectations, conditional probabilities and expectations, and others, and study their properties.

1.1 Idempotent measures

In this section we define the notion of an idempotent measure and obtain an extension theorem. We also introduce idempotent analogues of a measure space and probability space.

Let Ω be a set and $\mathcal{E}$ be a collection of subsets of Ω, which contains $\emptyset$. Let $\mathcal{P}(\Omega)$ denote the power set of Ω. We reserve symbols Φ and Ψ to denote directed sets, and J to denote arbitrary index sets.

Definition 1.1.1. *A set function $\mu : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}_+$ is an idempotent measure on Ω if the following conditions hold:*

(μ0) $\mu(\emptyset) = 0$,

(μ1) $\mu(A \cup B) = \mu(A) \vee \mu(B)$,

(μ2) $\mu(\cup_\phi A_\phi) = \sup_\phi \mu(A_\phi)$
for every increasing net $\{A_\phi, \phi \in \Phi\}$ of subsets of Ω.

If, in addition,

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(Π) $\mu(\Omega) = 1$,

the idempotent measure is called an idempotent probability measure or idempotent probability, for short, and denoted by Π.

If, in addition to $(\mu 0)$, $(\mu 1)$ *and* $(\mu 2)$,

$(\mu 3)$ $\mu(\cap_\phi F_\phi) = \inf_\phi \mu(F_\phi)$
for every decreasing net $\{F_\phi, \phi \in \Phi\}$ *of elements of* $\mathcal{E}$,

then we say that the idempotent measure is τ*-smooth relative to* $\mathcal{E}$, *or, for short, is an* $\mathcal{E}$*-idempotent measure.*

Remark 1.1.2. *Throughout, we use the terms "increasing" and "decreasing" as synonyms of "non-decreasing" and "non-increasing", respectively.*

Remark 1.1.3. *Property* $(\mu 1)$ *shows that* μ *is an increasing and subadditive set function in that* $\mu(A) \le \mu(B)$ *if* $A \subset B$ *and* $\mu(A \cup B) \le \mu(A) + \mu(B)$.

The following characterisation of idempotent measures is a straightforward consequence of the definition.

Lemma 1.1.4. *Conditions* $(\mu 1)$ *and* $(\mu 2)$ *are equivalent to the condition*

$$\mu\Big(\bigcup_{j \in J} A_j\Big) = \sup_j \mu(A_j) \tag{1.1.1}$$

for every collection $\{A_j,\, j \in J\}$ *of subsets of* Ω, *which in turn is equivalent to the representation*

$$\mu(A) = \sup_{\omega \in A} \mu(\{\omega\}), \quad A \subset \Omega. \tag{1.1.2}$$

The function $\mu(\{\omega\})$ is called the density of μ. We also refer to property $(\mu 2)$ as τ-smoothness along increasing nets (it should not be confused with τ-smoothness, which concerns decreasing nets of elements of $\mathcal{E}$) and to property (1.1.1) as τ-maxitivity. For set functions that are only defined on subsets of Ω we use a similar terminology introduced by the following definition.

Definition 1.1.5. *A set function* $\mu : \mathcal{E} \to \overline{\mathbb{R}}_+$ *is maxitive (respectively,* τ*-maxitive) on* $\mathcal{E}$ *if* $\mu(A \cup B) = \mu(A) \vee \mu(B)$ *for every* $A \in \mathcal{E}$ *and* $B \in \mathcal{E}$ *such that* $A \cup B \in \mathcal{E}$ *(respectively,* $\mu\big(\cup_{j \in J} A_j\big) = \sup_{j \in J} \mu(A_j)$ *for every collection of sets* $A_j \in \mathcal{E}$, $j \in J$, *such that* $\cup_{j \in J} A_j \in \mathcal{E}$*).*

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Given a collection $\mathcal{E}$, we denote by $\mathcal{E}_u$ (respectively, $\mathcal{E}_i$) the collection of arbitrary unions (respectively, intersections) of elements of $\mathcal{E}$. If finite unions (respectively, intersections) of sets from $\mathcal{E}$ belong to $\mathcal{E}$, then we may and often do assume that the sets in an infinite union (respectively, intersection) of elements of $\mathcal{E}$ form an increasing (respectively, decreasing) net relative to a directed set. We also denote $\mathcal{E}_{iu} = (\mathcal{E}_i)_u$ and observe that it coincides with $\mathcal{E}_{ui} = (\mathcal{E}_u)_i$. The collection $\mathcal{E}_{iu}$ is clearly closed under the formation of arbitrary unions and intersections.

We recall the following definitions.

Definition 1.1.6. *A collection $\mathcal{E}$ of subsets of Ω is called a paving on Ω if it contains $\emptyset$ and is closed under the formation of finite unions and intersections. A collection $\mathcal{E}$ of subsets of Ω is called a π-system if it is closed under the formation of finite intersections.*

The next lemma shows that if μ is an idempotent measure that is τ-smooth relative to a paving $\mathcal{E}$, then the values of μ on $\mathcal{E}_{iu}$ are uniquely specified by the values on $\mathcal{E}$ so that there is at most one extension of μ from $\mathcal{E}$ to $\mathcal{E}_{iu}$.

Theorem 1.1.7. *Let $\mathcal{E}$ be a π-system containing $\emptyset$ and μ be an $\mathcal{E}$-idempotent measure. Then*

$$\mu(A) = \sup_{\substack{B \in \mathcal{E}_i: \\ B \subset A}} \mu(B), \quad A \in \mathcal{E}_{iu},$$

$$\mu(B) = \inf_{\substack{F \in \mathcal{E}: \\ F \supset B}} \mu(F), \quad B \in \mathcal{E}_i.$$

The proof follows by τ-maxitivity and τ-smoothness of μ, and the fact that each set in $\mathcal{E}_i$ is an intersection of a decreasing net of elements of $\mathcal{E}$.

The following simple fact is useful for extension theorems (see Theorem 1.1.9 below). We give the proof to show a typical argument.

Lemma 1.1.8. *If an idempotent measure μ is τ-smooth relative to a π-system $\mathcal{E}$ containing $\emptyset$, then it is τ-smooth relative to $\mathcal{E}_i$.*

Proof. Let $\{A_\psi, \psi \in \Psi\}$ be a decreasing net of elements of $\mathcal{E}_i$, i.e., $A_\psi = \bigcap_{\phi \in \Phi_\psi} F_{\psi\phi}$, where $F_{\psi\phi} \in \mathcal{E}$. Let Δ be the collection of finite sequences $\delta = \{(\psi_{i_1}\phi_{i_1}), (\psi_{i_2}\phi_{i_2}), \ldots, (\psi_{i_k}\phi_{i_k})\}$, where $\psi_{i_j} \in \Psi$, $\psi_{i_1} \le \psi_{i_2} \le \ldots \le \psi_{i_k}$ and $\phi_l \in \Phi_{\psi_l}$ for $l = i_1, i_2, \ldots, i_k$. We say

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that $\delta \le \delta'$ if all the pairs $(\psi\,\phi)$ that appear in δ are also contained in δ'. For $\delta \in \Delta$, let $B_\delta = \bigcap_{(\psi\phi)\in\delta} F_{\psi\phi}$. Then Δ is a directed set and $\{B_\delta, \delta \in \Delta\}$ is a decreasing net. Also $B_\delta \in \mathcal{E}$ and $\bigcap_{\psi\in\Psi} A_\psi = \bigcap_{\delta\in\Delta} B_\delta$, so, since μ is an $\mathcal{E}$-idempotent measure,

$$\mu\Big(\bigcap_{\psi\in\Psi} A_\psi\Big) = \inf_{\delta\in\Delta} \mu(B_\delta). \tag{1.1.3}$$

For arbitrary $\delta = \{(\psi_{i_1}\phi_{i_1}), (\psi_{i_2}\phi_{i_2}), \ldots, (\psi_{i_k}\phi_{i_k})\}$, let $\psi_\delta \ge \psi_{i_j}$, $j = 1, \ldots, k$. Then, since $\{A_\psi, \psi \in \Psi\}$ is a decreasing net, it follows that $B_\delta \supset A_{\psi_\delta}$; hence,

$$\mu(B_\delta) \ge \mu\big(A_{\psi_\delta}\big) \ge \inf_{\psi\ge\psi_\delta} \mu(A_\psi) = \inf_{\psi\in\Psi} \mu(A_\psi).$$

Thus, in view of (1.1.3),

$$\mu\Big(\bigcap_{\psi\in\Psi} A_\psi\Big) \ge \inf_{\psi\in\Psi} \mu(A_\psi).$$

□

We consider now the issue of extending set functions to idempotent measures.

Theorem 1.1.9. *Let $\mathcal{E}$ be a paving on Ω. Let μ be an $\overline{\mathbb{R}}_+$-valued maxitive function on $\mathcal{E}$ such that $\mu(\emptyset) = 0$.*

1. *The set function μ can be extended to an idempotent measure μ^* on Ω if and only if it is τ-smooth along increasing nets, i.e., for every increasing net $\{F_\phi\}$ of elements of $\mathcal{E}$ whose union belongs to $\mathcal{E}$ we have*

   $$\mu\big(\bigcup_\phi F_\phi\big) = \sup_\phi \mu(F_\phi).$$

   The extension is uniquely specified on $\mathcal{E}_u$.

2. *The set function μ can be extended to an $\mathcal{E}$-idempotent measure μ^* if and only if the following condition holds.*

   (S) If $\{F_{1,\phi}\}$ and $\{F_{2,\psi}\}$ are respective increasing and decreasing nets of elements of $\mathcal{E}$ such that

   $$\bigcup_\phi F_{1,\phi} \supset \bigcap_\psi F_{2,\psi},$$

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then

$$\sup_\phi \mu(F_{1,\phi}) \geq \inf_\psi \mu(F_{2,\psi}).$$

The idempotent measure μ^* is then also τ-smooth relative to $\mathcal{E}_i$ and is uniquely specified on $\mathcal{E}_{iu}$.

Proof. We first consider part 1. Necessity of the condition is obvious. We prove sufficiency. We first note that in view of maxitivity of μ and the fact that $\mathcal{E}$ is closed under the formation of finite unions the condition of τ-smoothness along increasing nets implies τ-maxitivity of μ on $\mathcal{E}$. For $\omega \in \Omega$, let

$$\mu^*(\{\omega\}) = \inf_{\substack{F\in\mathcal{E}:\\ \omega\in F}} \mu(F), \tag{1.1.4}$$

and for $A \subset \Omega$ let

$$\mu^*(A) = \sup_{\omega\in A} \mu^*(\{\omega\}). \tag{1.1.5}$$

Clearly, $\mu^*(A)$ is an idempotent measure. We prove that μ^* agrees with μ on $\mathcal{E}$. Let $F \in \mathcal{E}$. By (1.1.4) $\mu^*(\{\omega\}) \leq \mu(F)$ if $\omega \in F$, so by (1.1.5) $\mu^*(F) \leq \mu(F)$. Conversely, given $\varepsilon > 0$, let $F^\omega \in \mathcal{E}$ for $\omega \in F$ be such that $\omega \in F^\omega$ and $\mu^*(\{\omega\}) \geq \mu(F^\omega) - \varepsilon$. Then by (1.1.5) $\mu^*(F) \geq \sup_{\omega\in F} \mu(F^\omega) - \epsilon$. Since $F = \bigcup_{\omega\in F}(F \cap F^\omega)$, where $F \cap F^\omega \in \mathcal{E}$ by the fact that $\mathcal{E}$ is closed under the formation of finite intersections, and μ is τ-maxitive and increasing on $\mathcal{E}$,

$$\mu(F) = \sup_{\omega\in F} \mu(F\cap F^\omega) \leq \sup_{\omega\in F} \mu(F^\omega) \leq \mu^*(F)+\varepsilon.$$

Part 1 is proved.

We prove part 2. It is obvious that if there exists an idempotent measure μ^*, which is τ-smooth relative to $\mathcal{E}$ and coincides with μ on $\mathcal{E}$, then condition (S) holds. For the converse, we note that condition (S) implies the condition of τ-smoothness of μ relative to increasing nets of elements of $\mathcal{E}$ in part 1. Therefore, by part 1 the set function μ^* defined by (1.1.4) and (1.1.5) is an idempotent measure, which extends μ. We prove that μ^* is an $\mathcal{E}_i$-idempotent measure. Note that since μ is maxitive on $\mathcal{E}$ and $\mathcal{E}$ is closed under the formation of finite unions, condition (S) extends to the case where $\{F_{1,\phi}\}$ is an

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arbitrary collection of elements of $\mathcal{E}$. Next, by Lemma 1.1.8 it suffices to check (μ3) for decreasing nets of elements of $\mathcal{E}$. Let $F_\psi \downarrow F$, where $F_\psi \in \mathcal{E}$. Given $\varepsilon > 0$, we choose for every $\omega \in F$ sets $F^\omega \in \mathcal{E}$ as in the proof of sufficiency in part 1. Since $\cup_{\omega\in F} F^\omega \supset \cap_\psi F_\psi$ and condition (S) extends to arbitrary collections $\{F_{1,j}\}$ of elements of $\mathcal{E}$, we conclude that $\sup_{\omega\in F^\omega} \mu(F^\omega) \geq \inf_\psi \mu(F_\psi)$ so that $\mu^*(F) \geq \sup_{\omega\in F} \mu^*(F^\omega) - \varepsilon \geq \inf_\psi \mu(F_\psi) - \varepsilon$, which completes the proof.

The fact that μ^* is unique on $\mathcal{E}_{iu}$ follows by Theorem 1.1.7. ☐

Remark 1.1.10. *If $\mathcal{E}$ is a ring, i.e., closed under the formation of differences, then condition (S) is equivalent to continuity of μ at 0: if $F_\psi \downarrow \emptyset$, then $\mu(F_\psi) \downarrow 0$. Theorem 1.1.9 is then an analogue of Caratheodory's theorem, see, e.g., Halmos [58].*

Remark 1.1.11. *Wang and Klir [133, Theorem 4.9] prove an extension theorem in the theme of part 1 for the case where the collection $\mathcal{E}$ is not necessarily closed under the formation of finite unions and intersections. Then the requirements on μ of maxitivity and τ-smoothness along increasing nets are replaced by the following P-consistency condition:*

if a collection $\{F_j\}$ of elements of $\mathcal{E}$ and $F \in \mathcal{E}$ are such that $F \subset \cup_j F_j$, then $\mu(F) \leq \sup_j \mu(F_j)$.

Similarly, part 2 admits a version for collections $\mathcal{E}$ that have the only property of including the empty set. Condition (S) then has to be replaced by the following condition:

(S') *If $\{F_{1,j}\}$ is a collection of elements of $\mathcal{E}$ and $\{F_{2,\psi}\}$ is a decreasing net of elements of $\mathcal{E}$ such that*

$$\bigcup_j F_{1,j} \supset \bigcap_\psi F_{2,\psi},$$

then

$$\sup_j \mu(F_{1,j}) \geq \inf_\psi \mu(F_{2,\psi}).$$

Condition (S') is necessary and sufficient for μ to be extended to an $\mathcal{E}$-idempotent measure. If $\mathcal{E}$ is a π-system, then by Lemma 1.1.8 the extension is τ-smooth relative to $\mathcal{E}_i$. We note also that condition (S') implies the P-consistency condition.

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Finally, if in part 2 we only omit the requirement that $\mathcal{E}$ be a π-system, then the extension μ^* also exists and is an $\mathcal{E}$-idempotent measure.

We will be interested in more special collections of subsets of Ω than pavings and π-systems. The following notion plays a central part in our analysis below.

Definition 1.1.12. *A collection $\mathcal{A}$ of subsets of Ω is a τ-algebra if it contains $\emptyset$ and is closed under the formation of complements and arbitrary unions. The elements of $\mathcal{A}$ are referred to as $\mathcal{A}$-measurable subsets of Ω.*

The power set $\mathcal{P}(\Omega)$ is obviously a τ-algebra, we refer to it as the discrete τ-algebra.

Definition 1.1.13. *A collection $\mathcal{E}$ of subsets of Ω is said to be atomic if it has a subcollection $\mathcal{E}' = \{A_\alpha\}$, consisting of non-empty subsets of Ω, such that either $A_\alpha \cap A_{\alpha'} = \emptyset$ or $A_\alpha = A_{\alpha'}$ for every α and α', and $F \in \mathcal{E}$ if and only if $F = \cup A_\alpha$, where the union is taken over $A_\alpha \subset F$, $A_\alpha \in \mathcal{E}'$. The elements of $\mathcal{E}'$ are called the atoms of $\mathcal{E}$.*

The structure of τ-algebras is revealed by the next theorem, which follows from the definition.

Theorem 1.1.14. *A collection $\mathcal{A}$, which contains $\emptyset$ and Ω, is a τ-algebra if and only if it is atomic.*

We denote as $[\omega]_{\mathcal{A}}$ the atom of a τ-algebra $\mathcal{A}$ that contains $\omega \in \Omega$. We note that $A \in \mathcal{A}$ if and only if $A = \cup_{\omega \in A}[\omega]_{\mathcal{A}}$, where an empty union is assumed to be empty.

Remark 1.1.15. *The relation $\mathcal{R}$ on Ω defined by $(\omega, \omega') \in \mathcal{R}$ if ω' and ω belong to the same atom of a τ-algebra $\mathcal{A}$ is obviously an equivalence relation. We denote this by $\omega' \sim \omega$ (or $\omega' \overset{\mathcal{A}}{\sim} \omega$ if we want to emphasise the τ-algebra to which the equivalence relation refers). Note that $\omega' \sim \omega$ if and only if $\omega' \in [\omega]_{\mathcal{A}}$ if and only if $\omega \in [\omega']_{\mathcal{A}}$ if and only if $[\omega]_{\mathcal{A}} = [\omega']_{\mathcal{A}}$.*

The following simple observation is frequently used below.

Corollary 1.1.16. *A set $A \subset \Omega$ is an element of a τ-algebra $\mathcal{A}$ on Ω if and only if $[\omega]_{\mathcal{A}} \subset A$ for every $\omega \in A$.*

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Definition 1.1.17. *We say that a τ-algebra $\mathcal{A}'$ is a sub-τ-algebra of a τ-algebra $\mathcal{A}$ if $\mathcal{A}' \subset \mathcal{A}$.*

Lemma 1.1.18. *A τ-algebra $\mathcal{A}'$ is a sub-τ-algebra of a τ-algebra $\mathcal{A}$ if and only if the atoms of $\mathcal{A}'$ are unions of the atoms of $\mathcal{A}$.*

We refer to the smallest τ-algebra containing a collection $\mathcal{E}$ as the τ-algebra generated by $\mathcal{E}$ and denote it as $\tau(\mathcal{E})$. It is obviously unambiguously defined.

Definition 1.1.19. *We say that a collection $\mathcal{E}$ of subsets of Ω is a semi-τ-algebra if it includes $\emptyset$, is a π-system, and $F^c \in \mathcal{E}_{iu}$ for every $F \in \mathcal{E}$.*

The structure of semi-τ-algebras is similar to that of τ-algebras.

Lemma 1.1.20. *A π-system $\mathcal{E}$, which includes $\emptyset$, is a semi-τ-algebra if and only if $\mathcal{E}_i$ is atomic and the union of the atoms of $\mathcal{E}_i$ equals Ω.*

Proof. It is obvious that if $\mathcal{E}_i$ is atomic and its atoms make up Ω, then $\mathcal{E}$ is a semi-τ-algebra. For the converse, given $\omega \in \Omega$, we take as the atom about ω the intersection of all elements of $\mathcal{E}$ that contain ω. □

The preceding proof also proves the following lemma.

Lemma 1.1.21. *If $\mathcal{E}$ is a semi-τ-algebra, then $\tau(\mathcal{E}) = \mathcal{E}_{iu}$.*

Theorem 1.1.9 and Remark 1.1.11 yield the following fact.

Corollary 1.1.22. *Let μ be a set function on a semi-τ-algebra $\mathcal{E}$ such that $\mu(\emptyset) = 0$. If condition (S') holds, then μ has a unique extension to an $\mathcal{E}_i$-idempotent measure on the τ-algebra generated by $\mathcal{E}$.*

Proof. We only need to extend μ to a maxitive set function on the collection of finite unions of elements of $\mathcal{E}$ by setting

$$\mu\Big(\bigcup_{i=1}^{k} F_i\Big) = \max_{i=1,\ldots,k} \mu(F_i) \tag{1.1.6}$$

and apply part 2 of Theorem 1.1.9. The fact that the extension (1.1.6) is unambiguously defined follows from condition (S'). □

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The notion of a τ-algebra is obviously an analogue of the notion of a σ-algebra. The next definition paraphrases the definition of complete σ-algebras.

Definition 1.1.23. *We say that a τ-algebra $\mathcal{A}$ is complete with respect to an idempotent measure μ on Ω (or μ-complete, for short) if every $\omega \in \Omega$ such that $\mu(\{\omega\}) = 0$ is an atom of $\mathcal{A}$.*

Definition 1.1.24. *We call the completion of a τ-algebra $\mathcal{A}$ with respect to idempotent measure μ the τ-algebra that has as its atoms all the elements of idempotent measure 0 and the atoms of $\mathcal{A}$ without the elements of idempotent measure zero. We denote the completion by $\mathcal{A}^\mu$.*

Remark 1.1.25. *Clearly, $\mathcal{A}^\mu$ is the smallest complete τ-algebra containing $\mathcal{A}$.*

Definition 1.1.26. *A set Ω with a τ-algebra of subsets of Ω is called a τ-measurable space and is denoted as $(\Omega, \mathcal{A})$.*

We now define an analogue of a measure space.

Definition 1.1.27. *A triplet $(\Omega, \mathcal{A}, \mu)$, where Ω is a set, $\mathcal{A}$ is a τ-algebra of subsets of Ω and μ is an idempotent measure on Ω, is called an idempotent measure space. We denote the idempotent measure space $(\Omega, \mathcal{P}(\Omega), \mu)$ as (Ω, μ). If Π is an idempotent probability, we refer to $(\Omega, \mathcal{A}, \Pi)$ as an idempotent probability space.*

Definition 1.1.28. *Given an idempotent measure μ and a τ-algebra $\mathcal{A}$ on Ω the set function $\mu_{\mathcal{A}}$ defined by $\mu_{\mathcal{A}}(A) = \mu(A)$, $A \in \mathcal{A}$, is called the restriction of μ to $\mathcal{A}$.*

Remark 1.1.29. *As we will see, it is often the case that an idempotent measure is originally specified on a τ-algebra. Though by Theorem 1.1.9 it can always be extended to an idempotent measure on $\mathcal{P}(\Omega)$, this extension might not be unique, which justifies restricting our consideration to the elements of $\mathcal{A}$. To emphasise this we refer to μ as an idempotent measure on $(\Omega, \mathcal{A})$. We note, however, that ambiguity in extending μ does not necessarily lead to ambiguity in the end results. In the sequel, we use the same symbol μ to denote some extension of μ from $\mathcal{A}$ to $\mathcal{P}(\Omega)$.*

On the other hand, given an idempotent measure space $(\Omega, \mathcal{A}, \mu)$, where μ is uniquely specified on $\mathcal{A}$, we could reduce it to a space

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with the discrete τ-algebra by introducing the factor-space of Ω with respect to the equivalence relation specified by the atoms of $\mathcal{A}$. In this sense considering arbitrary τ-algebras does not give anything new. However, it comes in useful if we need to deal with a collection of τ-algebras on the same set as in Chapter 2, where τ-algebras are used to keep track of "the history of a process".

1.2 Measurable functions

In this section we introduce measurable maps of spaces with idempotent measures. Let Ω and Ω' be sets, and $\mathcal{E}$ and $\mathcal{E}'$ be respective collections of subsets of Ω and Ω', both containing $\emptyset$.

Definition 1.2.1. *For a function $f : \Omega \to \Omega'$, we define the collection of subsets of Ω generated by f as the collection $f^{-1}(\mathcal{E}') = \{f^{-1}(B), B \subset \mathcal{E}'\}$.*

We also refer to functions defined on idempotent probability spaces as idempotent variables. The following lemma is a consequence of the definition.

Lemma 1.2.2. *If $\mathcal{E}'$ is a τ-algebra (respectively, a π-system, a paving, a semi-τ-algebra) on Ω', then $f^{-1}(\mathcal{E}')$ is a τ-algebra (respectively, a π-system, a paving, a semi-τ-algebra). The collection of the atoms of the τ-algebra $f^{-1}(\mathcal{E}')$ is the collection $\{f^{-1}(A')\}$, where A' are the atoms of the τ-algebra $\mathcal{E}'$.*

Definition 1.2.3. *A function $f : \Omega \to \Omega'$ is said to be $\mathcal{E}/\mathcal{E}'$-measurable if $f^{-1}(\mathcal{E}') \subset \mathcal{E}$.*

The following result is obvious.

Lemma 1.2.4. *A function $f : \Omega \to \Omega'$ is $\mathcal{E}_{iu}/\mathcal{E}'_{iu}$-measurable if and only if it is $\mathcal{E}_{iu}/\mathcal{E}'$-measurable.*

As a consequence, a function $f : \Omega \to \Omega'$ is $\mathcal{A}/\mathcal{A}'$-measurable, where $\mathcal{A}$ and $\mathcal{A}'$ are τ-algebras on respective sets Ω and Ω', if and only if the inverse images of the atoms of $\mathcal{A}'$ belong to $\mathcal{A}$. Thus, we have the following.

Corollary 1.2.5. *A function $f : \Omega \to \Omega'$ is $\mathcal{A}/\mathcal{A}'$-measurable if and only if every atom of $\mathcal{A}$ is mapped into a subset of some atom of $\mathcal{A}'$. In particular, f is $\mathcal{A}/\mathcal{P}(\Omega')$-measurable if and only if it is constant on the atoms of $\mathcal{A}$.*

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In the sequel, we refer to $\mathcal{A}/\mathcal{P}(\Omega')$-measurable functions as $\mathcal{A}$-measurable functions or idempotent variables on $(\Omega, \mathcal{A})$. Note that $A \subset \Omega$ is $\mathcal{A}$-measurable if and only if $\mathbf{1}(A) : \Omega \to \mathbb{R}_+$ is $\mathcal{A}$-measurable.

Lemma 1.2.6. *Let $(\Omega, \mathcal{A}, \mu)$ be an idempotent measure space and $\mathcal{A}^\mu$ be the completion of $\mathcal{A}$ with respect to μ. If $f : \Omega \to \Omega'$ is $\mathcal{A}^\mu$-measurable, then there exists an $\mathcal{A}$-measurable idempotent variable f' such that $f' = f$ μ-a.e.*

Proof. Let $[\omega]_\mathcal{A}$ be an atom of $\mathcal{A}$. We define $f'(\omega') = f(\tilde{\omega})$ for all $\omega' \in [\omega]_\mathcal{A}$, where $\tilde{\omega} \in [\omega]_\mathcal{A}$ is such that $\mu(\tilde{\omega}) > 0$ if $\mu([\omega]_\mathcal{A}) > 0$ and $\tilde{\omega}$ is an arbitrary element of $[\omega]_\mathcal{A}$ otherwise. Then f' is $\mathcal{A}$-measurable and $f' = f$ μ-a.e. by the construction of $\mathcal{A}^\mu$ (see Definition 1.1.24). $\square$

The next lemma is a version of Doob's result.

Lemma 1.2.7. *Let a τ-algebra $\mathcal{A}$ on Ω be generated by a function $f : \Omega \to \Omega'$, where Ω' is equipped with a τ-algebra $\mathcal{A}'$. A function $g : \Omega \to \Omega''$ is $\mathcal{A}$-measurable if and only if there exists an $\mathcal{A}'$-measurable function $h : \Omega' \to \Omega''$ such that $g = h \circ f$.*

Proof. Sufficiency of the condition is obvious. We prove the necessity. Since $\mathcal{A}$ is generated by f and g is $\mathcal{A}$-measurable, for arbitrary $\omega'' \in \Omega''$ there exists $A'_{\omega''} \in \mathcal{A}'$ such that $g^{-1}(\omega'') = f^{-1}(A'_{\omega''})$. Since the sets $g^{-1}(\omega'')$ are disjoint, the sets $A'_{\omega''}$, $\omega'' \in \Omega''$, are also disjoint. Therefore, letting $h(\omega') = \omega''$ for $\omega'' \in \Omega''$ and $\omega' \in A'_{\omega''}$, and $h(\omega') = \hat{\omega}$ for $\omega' \in \left(\cup_{\omega'' \in \Omega''} A'_{\omega''}\right)^c$, where $\hat{\omega}$ is a fixed element of Ω'', defines h unambiguously. Clearly, $h \circ f(\omega) = g(\omega)$, $\omega \in \Omega$. $\square$

We have the following corollary for functions assuming values on the real line.

Corollary 1.2.8. *Let $\mathcal{A}$ be a τ-algebra on Ω. If functions $f_j : \Omega \to \mathbb{R}$, $j \in J$, are $\mathcal{A}$-measurable and $F : \mathbb{R}^J \to \mathbb{R}$, then $F((f_j)_{j \in J})$ is $\mathcal{A}$-measurable. In particular, $\sup_j f_j$ and $\inf_j f_j$ are $\mathcal{A}$-measurable, and if Φ is a directed set, then $\limsup_{\phi \in \Phi} f_\phi$ and $\liminf_{\phi \in \Phi} f_\phi$ are $\mathcal{A}$-measurable.*

We now consider images of idempotent measures under mappings. Let μ be an idempotent measure on Ω. The next lemma is straightforward.

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Lemma 1.2.9. *Let $f : \Omega \to \Omega'$. Then the set function μ' on Ω' defined by $\mu'(A') = \mu\big(f^{-1}(A')\big)$ for $A' \subset \Omega'$ is an idempotent measure on Ω'.*

Definition 1.2.10. *The set function μ' as defined in the lemma is called the image of μ under f and denoted by $\mu \circ f^{-1}$.*

For the image of a τ-smooth idempotent measure to be a τ-smooth idempotent measure, we need to impose conditions on the mapping. By Luzin's theorem in measure theory a real-valued function of a real argument is Borel-measurable if and only if it is continuous on "large" sets (closed or compact). We turn the theorem into the definition of a measurability concept. The first step is to introduce an abstract analogue of the concept of a tight measure.

Definition 1.2.11. *Let μ be an $\mathcal{E}$-idempotent measure. We say that a collection $\mathcal{T}$ of subsets of Ω is tightening for μ if $T \cap F \in \mathcal{E}$ for $T \in \mathcal{T}$ and $F \in \mathcal{E}$, and for arbitrary $\varepsilon > 0$ there exists $T \in \mathcal{T}$ such that $\mu(T^c) \le \varepsilon$. We then also say that μ is tight relative to $\mathcal{T}$, or $\mathcal{T}$-tight, for short.*

We next define "Luzin measurability".

Definition 1.2.12. *Let $\mathcal{T}$ be a tightening collection for an $\mathcal{E}$-idempotent measure μ. A function $f : \Omega \to \Omega'$ is called Luzin $(\mathcal{E},\mathcal{T})/\mathcal{E}'$-measurable if the restriction of f to an arbitrary $T \in \mathcal{T}$ is $\mathcal{E}_T/\mathcal{E}'$–measurable, where $\mathcal{E}_T = \{T \bigcap F,\ F \in \mathcal{E}\}$.*

Remark 1.2.13. *Equivalently, $f : \Omega \to \Omega'$ is Luzin $(\mathcal{E},\mathcal{T})/\mathcal{E}'$–measurable if $f^{-1}(F') \bigcap T \in \mathcal{E}$ for every $F' \in \mathcal{E}'$ and $T \in \mathcal{T}$. Note also that $\mathcal{E}/\mathcal{E}'$-measurability implies Luzin $(\mathcal{E},\mathcal{T})/\mathcal{E}'$-measurability. On the other hand, the collection $\mathcal{T} = \{\Omega\}$ is trivially a tightening collection for μ so that $\mathcal{E}/\mathcal{E}'$-measurability is a specific case of Luzin $(\mathcal{E},\mathcal{T})/\mathcal{E}'$-measurability.*

We refer to Luzin $(\mathcal{E},\mathcal{T})/\mathcal{E}'$-measurable functions as Luzin measurable functions if the collections $\mathcal{E}$, $\mathcal{T}$ and $\mathcal{E}'$ are understood. The purpose of introducing the concept of Luzin measurability is seen from the following theorem.

Theorem 1.2.14. *Let μ be an $\mathcal{E}$-idempotent measure and $\mathcal{T}$ be a tightening collection for μ. If a function $f : \Omega \to \Omega'$ is Luzin*

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$(\mathcal{E},\mathcal{T})/\mathcal{E}'$-measurable, then the image $\mu'=\mu\circ f^{-1}$ of μ under f is a τ-smooth relative to $\mathcal{E}'$ idempotent measure on Ω'. If $f(T)\cap F'\in\mathcal{E}'$ whenever $T\in\mathcal{T}$ and $F'\in\mathcal{E}'$, then μ' is $f(\mathcal{T})$-tight.

Proof. By Lemma 1.2.9 μ' is an idempotent measure. We check property (μ3). Let $F'_\phi\in\mathcal{E}'$ be a decreasing net. Given $\varepsilon>0$, let $T\in\mathcal{T}$ be such that $\mu(T^c)<\varepsilon$. Then

$$\mu'(F'_\phi)=\mu(f^{-1}(F'_\phi))\le\mu\bigl(f^{-1}(F'_\phi)\bigcap T\bigr)+\varepsilon.$$

Since the F'_ϕ decrease, the $f^{-1}(F'_\phi)\bigcap T$ decrease as well. Therefore, since $f^{-1}(F'_\phi)\bigcap T\in\mathcal{E}$, by τ-smoothness of μ

$$\mu\bigl(f^{-1}(F'_\phi)\bigcap T\bigr)\to\mu\Bigl(\bigcap_\phi\bigl(f^{-1}(F'_\phi)\bigcap T\bigr)\Bigr)\le\mu'\bigl(\bigcap_\phi F'_\phi\bigr)$$

proving (μ3).

The fact that the idempotent measure μ' is $f(\mathcal{T})$-tight provided $f(T)\cap F'\in\mathcal{E}'$ whenever $T\in\mathcal{T}$ and $F'\in\mathcal{E}'$ follows since μ is $\mathcal{T}$-tight and $\mu'\bigl(f(T)^c\bigr)=\mu\bigl((f^{-1}(f(T)))^c\bigr)\le\mu(T^c)$. $\square$

We fix some more terminology.

Definition 1.2.15. *Let (Ω,Π) be an idempotent probability space and $f:\Omega\to\Omega'$ be an Ω'-valued idempotent variable. The idempotent probability $\Pi\circ f^{-1}$ is called the idempotent distribution (or idempotent law) of f under Π. If Ω' is a metric space and $\lim_{r\to\infty}\Pi(f\notin B_r(z))=0$, where z is a fixed element of E and $B_r(z)$ denotes the closed r-ball about z, then f is called a proper idempotent variable.*

1.3 Modes of convergence

We consider idempotent analogues of convergence in measure and convergence almost everywhere. Let (Ω,μ) be an idempotent measure space. Let f_ϕ and f denote idempotent variables on Ω with values in a metric space E with metric ρ.

Definition 1.3.1. *We say that a net $\{f_\phi,\phi\in\Phi\}$ converges μ-a.e. to f if*

$$\mu\bigl(\omega\in\Omega:\ f_\phi(\omega)\not\to f(\omega)\bigr)=0.$$

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More generally, we say that a property concerning elements of Ω *holds* μ*-a.e. (or a.e. if* μ *is understood) if the idempotent measure* μ *of the set where the property does not hold equals* 0.

Definition 1.3.2. *We say that a net* $\{f_\phi, \phi \in \Phi\}$ *converges to* f *in idempotent measure* μ *(or in idempotent measure if* μ *is understood) if for every* $\varepsilon > 0$

$$\lim_{\phi \in \Phi} \mu(\omega \in \Omega : \rho(f_\phi(\omega), f(\omega)) > \varepsilon) = 0.$$

Note that since $\mu(\rho(f,g) > \varepsilon) \le \mu(\rho(f, f_\phi) > \varepsilon/2) + \mu(\rho(g, f_\phi) > \varepsilon/2)$, the limit in idempotent measure is unique μ-a.e. The same fact is of course true for convergence μ-a.e. We denote convergence in idempotent measure by $f_\phi \xrightarrow{\mu} f$.

Lemma 1.3.3. *("Borel-Cantelli") Let* $\{A_\phi, \phi \in \Phi\}$ *be a net of subsets of* Ω*. If* $\mu(A_\phi) \to 0$*, then* $\mu\big(\limsup_{\phi \in \Phi} A_\phi\big) = 0$.

Proof. The claim follows since

$$\mu\big(\limsup_{\phi \in \Phi} A_\phi\big) = \mu\Big(\bigcap_{\phi' \in \Phi} \bigcup_{\phi \ge \phi'} A_\phi\Big) \le \inf_{\phi' \in \Phi} \sup_{\phi \ge \phi'} \mu(A_\phi).$$

□

Theorem 1.3.4. *("Egorov")* $f_\phi \xrightarrow{\mu} f$ *if and only if for every* $\varepsilon > 0$ *there exists a set* A_ε *such that* $\mu(A_\varepsilon^c) \le \varepsilon$ *and* $\sup_{\omega \in A_\varepsilon} \rho(f_\phi(\omega), f(\omega)) \to 0$.

Proof. If $\{f_\phi,\, \phi \in \Phi\}$ is a net such that $f_\phi \xrightarrow{\mu} f$, then for every $\delta > 0$ there exists $\phi_\delta \in \Phi$ such that $\mu(\rho(f_\phi, f) > \delta) < \varepsilon$ for all $\phi \ge \phi_\delta$; hence, $\rho(f_\phi, f) \le \delta$ on the set $A_\varepsilon = \{\omega \in \Omega : \mu(\{\omega\}) \ge \varepsilon\}$. The converse is obvious. □

Theorem 1.3.5. *If* $f_\phi \xrightarrow{\mu} f$*, then* $f_\phi \to f$ μ*-a.e.*

Proof. By Lemma 1.3.3 $\mu\big(\limsup_\phi \rho(f_\phi, f) > \varepsilon\big) = 0$. □

The next result gives a partial converse.

Theorem 1.3.6. *If* $f_\phi \to f, \phi \in \Phi$, μ*-a.e., then there exists a net* $\{h_\psi, \psi \in \Psi\}$*, which converges to* f *in idempotent measure and is*

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such that $\{h_\psi(\omega), \psi \in \Psi\}$ *is a subnet of* $\{f_\phi(\omega), \phi \in \Phi\}$ *for every* $\omega \in \Omega$.

If $\{f_n, n \in \mathbb{N}\}$ *is a sequence such that* $f_n \to f$ μ*-a.e., then for every* $\omega \in \Omega$ *there exists a subsequence* $k_n(\omega)$ *such that* $k_n(\omega) \geq n$ *and* $f_{k_n} \xrightarrow{\mu} f$.

Proof. Let $\Psi = \{(\phi, \varepsilon) : \phi \in \Phi, \epsilon \in \mathbb{R}_+\}$. We turn Ψ into a directed set by defining that $(\phi, \varepsilon) \leq (\phi', \varepsilon')$ if $\phi \leq \phi', \varepsilon \geq \varepsilon'$. For $\psi = (\phi, \epsilon)$, we define $\chi_\psi(\omega)$ as $\phi' \geq \phi$ such that $\rho(f_{\tilde{\phi}}(\omega), f(\omega)) \leq \epsilon$ for all $\tilde{\phi} \geq \phi'$ if such a ϕ' exists and $\chi_\psi(\omega) = \phi$ otherwise. Let $h_\psi(\omega) = f_{\chi_\psi(\omega)}(\omega)$. Clearly, $\{h_\psi(\omega), \psi \in \Psi\}$ is a subnet of $\{f_\phi(\omega), \phi \in \Phi\}$ for every $\omega \in \Omega$. Since $f_\phi \to f$ μ-a.e., ϕ' exists for almost every $\omega \in \Omega$; hence, $\rho(h_{\tilde{\psi}}(\omega), f(\omega)) \leq \epsilon$ for almost every $\omega \in \Omega$ if $\tilde{\psi} \geq \psi$.

In the case of sequences, we define $k_n(\omega) = \min\{l \geq n : \rho(f_l(\omega), f(\omega)) \leq 1/n\}$ and $k_n(\omega) = n$ if no such l exists. □

Remark 1.3.7. *Generally speaking, convergence* μ*-a.e. does not imply convergence in idempotent measure. Consider the following example. Let* $\Omega = [0, 1]$ *and* $\mu(\{\omega\}) = 1$ *for* $\omega \in [0, 1]$. *Let* $f_n(\omega) = n\omega, \omega \in [0, 1/n]$, $f_n(\omega) = 2 - n\omega, \omega \in [1/n, 2/n]$ *and* $f_n(\omega) = 0$ *elsewhere. Then* $f_n(\omega) \to 0$ *as* $n \to \infty$ *for every* $\omega \in [0, 1]$. *However,* $\mu(\omega : f_n(\omega) = 1) = 1$.

We now consider Cauchy nets.

Lemma 1.3.8. *If* $\mu(\rho(f_\phi, f_{\phi'}) > \varepsilon) \to 0$ *as* $\phi, \phi' \in \Phi$ *for every* $\varepsilon > 0$ *and* (E, ρ) *is complete, then* $\{f_\phi\}$ *converges* μ*-a.e.*

Proof. In analogy with the proof of Lemma 1.3.3, for $\varepsilon > 0$,

$$\mu\Big(\bigcap_{\phi} \bigcup_{\phi' \geq \phi} \{\rho(f_\phi, f_{\phi'}) > \varepsilon\}\Big) \leq \inf_{\phi} \sup_{\phi' \geq \phi} \mu\big(\rho(f_\phi, f_{\phi'}) > \varepsilon\big) = 0.$$

Thus, the net $\{f_\phi\}$ is Cauchy μ-a.e., so it converges μ-a.e. by completeness of (E, ρ). □

Theorem 1.3.9. *If* $\mu(\rho(f_\phi, f_{\phi'}) > \varepsilon) \to 0$ *as* $\phi, \phi' \in \Phi$ *and* (E, ρ) *is complete, then* $\{f_\phi\}$ *converges in idempotent measure.*

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Proof. By Lemma 1.3.8 $f_\phi \to f$ μ-a.e. Let us choose χ_ψ as in the proof of Theorem 1.3.6. Then, for $\epsilon > 0$, taking $\psi = (\phi, \epsilon/2)$ and using the inequalities $\rho(f_{\chi_\psi}, f) \le \varepsilon/2$ μ-a.e. and $\chi_\psi(\omega) \ge \phi$,

$$\mu(\rho(f_\phi, f) > \varepsilon) \le \mu(\rho(f_\phi, f_{\chi_\psi}) > \varepsilon/2) \vee \mu(\rho(f_{\chi_\psi}, f) > \varepsilon/2)$$
$$\le \mu(\sup_{\phi' \ge \phi} \rho(f_\phi, f_{\phi'}) > \varepsilon/2) = \sup_{\phi' \ge \phi} \mu(\rho(f_\phi, f_{\phi'}) > \varepsilon/2).$$

The latter term goes to 0 as $\phi \in \Phi$ by hypotheses. □

The last result of the section shows that for τ-smooth idempotent measures and decreasing nets of $\mathbb{R}_+$-valued Luzin measurable functions convergence μ-almost everywhere implies convergence in idempotent measure. Let $\mathcal{U}$ denote the paving on $\mathbb{R}_+$ consisting of intervals $[a, \infty)$, where $a \in \mathbb{R}_+$, and $\emptyset$.

Theorem 1.3.10. *Let μ be τ-smooth relative to a collection $\mathcal{E}$ and let $\mathcal{T}$ be a tightening collection for μ. If $\{\xi_\phi, \phi \in \Phi\}$ is a decreasing net of Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable $\mathbb{R}_+$-valued functions, which converges μ-a.e. to 0, then $\{\xi_\phi, \phi \in \Phi\}$ converges to 0 in idempotent measure.*

Proof. Let $\delta > 0$, $\epsilon > 0$ and $T \in \mathcal{T}$ such that $\mu(T^c) \le \varepsilon$. Then $\mu\big(\xi_\phi \ge \delta\big) \le \mu\big(\{\xi_\phi \ge \delta\} \cap T\big) + \varepsilon$. Since $\{\xi_\phi \ge \delta\} \cap T \in \mathcal{E}$, by τ-smoothness of μ relative to $\mathcal{E}$ we have that $\mu\big(\{\xi_\phi \ge \delta\} \cap T\big) \to \mu\big(\cap_\phi \{\xi_\phi \ge \delta\} \cap T\big) = 0$. □

Remark 1.3.11. *The result is a counterpart of the fact in probability theory stating that a monotonic sequence of non-negative random variables that converges to zero in probability also converges to zero almost surely.*

1.4 Idempotent integration

In this section we develop an idempotent analogue of integration theory. Let (Ω, μ) be an idempotent measure space such that $\mu(\Omega) < \infty$. We adopt the convention that $\infty \cdot 0 = 0$.

Definition 1.4.1. *For a function f on Ω with values in $\overline{\mathbb{R}}_+$ we define the idempotent integral of f with respect to μ by*

$$\bigvee_\Omega f \, d\mu = \sup_{a \in \overline{\mathbb{R}}_+} a \, \mu(f \ge a).$$

For $A \subset \Omega$, we let $\bigvee_A f \, d\mu = \bigvee_\Omega f \, \mathbf{1}(A) \, d\mu$.

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Idempotent integral is called idempotent expectation if μ is an idempotent probability. In the sequel we also denote idempotent integrals as $\bigvee_\Omega f(\omega)\, d\mu(\omega)$ and, if μ is an idempotent probability, as Sf, $Sf(\omega)$, or $S_\Pi f$, the latter notation is used to emphasise the idempotent probability Π for which the idempotent integral is evaluated. The next lemma follows by definition.

Lemma 1.4.2. *Let $f : \Omega \to \overline{\mathbb{R}}_+$. The following equivalent representations hold.*

$$\bigvee_\Omega f\, d\mu = \sup_{a \in \overline{\mathbb{R}}_+} a\mu(f = a) = \sup_{\omega \in \Omega} f(\omega)\mu(\{\omega\})$$
$$= \sup_{\omega \in \Omega} f(\omega)\mu([\omega]_{f^{-1}(\mathcal{P}(\overline{\mathbb{R}}_+))}).$$

Remark 1.4.3. *If μ is originally defined on a τ-algebra $\mathcal{A}$, then the value of the idempotent integral of a function f depends generally speaking on what extension of μ to $\mathcal{P}(\Omega)$ we consider. However, if f is $\mathcal{A}$-measurable, then $\bigvee_\Omega f\, d\mu$ is defined unambiguously, which follows by the last equality in Lemma 1.4.2. To emphasise this we sometimes denote the integral as $\bigvee_\Omega f\, d\mu_\mathcal{A}$, $\bigvee_\Omega f(\omega)\, d\mu_\mathcal{A}(\omega)$ and, if Π is an idempotent probability, as $S_{\Pi_\mathcal{A}} f$ and $S_{\Pi_\mathcal{A}} f(\omega)$, where $\mu_\mathcal{A}$ and $\Pi_\mathcal{A}$ denote the respective restrictions of μ and Π to $\mathcal{A}$. A careful examination of the proofs below shows that if we require that the functions and sets considered in the statements be $\mathcal{A}$-measurable, then the results are insensitive to the particular extension of μ to $\mathcal{P}(\Omega)$.*

The results below whose proofs are omitted directly follow from Lemma 1.4.2. We consider only $\mathbb{R}_+$-valued integrands, the corresponding properties for $\overline{\mathbb{R}}_+$-valued integrands are derived similarly.

Theorem 1.4.4. *Let f, g be $\mathbb{R}_+$-valued functions on Ω. The following properties hold.*

(JS0) $\displaystyle\bigvee_\Omega 0\, d\mu = 0$

(JS1) $\displaystyle\bigvee_\Omega f\, d\mu \le \bigvee_\Omega g\, d\mu$ *if* $f \le g$

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(JS2) $\bigvee_{\Omega}(cf)\,d\mu = c\bigvee_{\Omega} f\,d\mu,\ c \in \mathbb{R}_+$

(JS3) $\bigvee_{\Omega}(f \vee g)\,d\mu = \bigvee_{\Omega} f\,d\mu \vee \bigvee_{\Omega} g\,d\mu$

(JS4) $\bigvee_{\Omega}(f + g)\,d\mu \leq \bigvee_{\Omega} f\,d\mu + \bigvee_{\Omega} g\,d\mu$

(JS5) $|\bigvee_{\Omega} f\,d\mu - \bigvee_{\Omega} g\,d\mu| \leq \bigvee_{\Omega}|f-g|\,d\mu$ *provided the left-hand side is well defined*

(JS6) $\bigvee_{\Omega} \sup_{j \in J} f_j\,d\mu = \sup_{j\in J}\bigvee_{\Omega} f_j\,d\mu$, *where* $f_j : \Omega \to \mathbb{R}_+,\ j \in J$

The following Chebyshev-type inequality plays as important a role below as its counterpart does in probability theory.

Lemma 1.4.5. *If* $f : \Omega \to \mathbb{R}_+$, *then*

$$\mu(f \geq a) \leq \frac{1}{a}\bigvee_{\Omega} f\,\mathbf{1}(f \geq a)\,d\mu, \quad a > 0.$$

We also have an analogue of the change-of-variables formula in the Lebesgue integral.

Theorem 1.4.6. *Let* μ' *be an idempotent measure on a set* Ω' *such that* $\mu' = \mu \circ f^{-1}$ *for some* $f : \Omega \to \Omega'$. *Then, for a function* $g : \Omega' \to \mathbb{R}_+$,

$$\bigvee_{\Omega'} g\,d\mu' = \bigvee_{\Omega} g \circ f\ d\mu.$$

The following "Hölder" inequalities are also useful. For $f : \Omega \to \mathbb{R}_+$ and $p > 0$ we define $\|f\|_p = \left(\bigvee_{\Omega} f^p\ d\mu\right)^{1/p}$ and $\|f\|_\infty = \sup_{\omega:\,\mu(\{\omega\})>0} f(\omega)$.

Lemma 1.4.7. *Let* $f, g : \Omega \to \mathbb{R}_+$.

1. *Let* $p \in [1,\infty]$ *and* $q \in [1,\infty]$ *be such that* $1/p + 1/q = 1$. *Then* $\bigvee_{\Omega} fg\ d\mu \leq \|f\|_p \|g\|_q$.
2. *If* $\mu(\Omega) = 1$, *then, for* $0 < p < q$, $\|f\|_p \leq \|f\|_q$.

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We are interested in convergence properties of idempotent integrals, so we study an analogue of the concept of uniform integrability.

Definition 1.4.8. *We say that a function* $f : \Omega \to \mathbb{R}_+$ *is maximable (or* μ*-maximable if the idempotent measure needs to be emphasised) if* $\bigvee_\Omega f \, d\mu < \infty$ *and, moreover,* $\bigvee_\Omega f \, \mathbf{1}(f > a) \, d\mu \to 0$ *as* $a \to \infty$*.*

The following version of La Vallee-Poussin's theorem holds.

Theorem 1.4.9. *A function* $f : \Omega \to \mathbb{R}_+$ *is maximable if and only if there exists a monotonically increasing function* $F : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $F(x)/x \to \infty$ *as* $x \to \infty$ *and* $\bigvee_\Omega F \circ f \, d\mu < \infty$*.*

Proof. We prove that the condition is sufficient for f to be maximable. Given $\varepsilon > 0$, let $a > 0$ be such that $x/F(x) \leq \varepsilon$ for $x \geq a$. Then

$$\bigvee_\Omega f \, \mathbf{1}(f > a) \, d\mu \leq \varepsilon \bigvee_\Omega F \circ f \, \mathbf{1}(f > a) \, d\mu \leq \varepsilon \bigvee_\Omega F \circ f \, d\mu.$$

Conversely, let f be maximable. Since there is no loss of generality in assuming that $\|f\|_\infty = \infty$, we define

$$F(x) = \frac{x}{\bigvee_\Omega f \, \mathbf{1}(f \geq x) \, d\mu}.$$

Then F is monotonic and $F(x)/x \to \infty$ as $x \to \infty$ by maximability of f. Also, $\bigvee_\Omega F \circ f \, d\mu \leq 1$. ☐

Definition 1.4.10. *A collection* $\{f_j, j \in J\}$ *of* $\mathbb{R}_+$*-valued functions on* Ω *is said to be uniformly maximable (or* μ*-uniformly maximable) if*

$$\sup_{j \in J} \bigvee_\Omega f_j \, \mathbf{1}(f_j > a) \, d\mu \to 0 \text{ as } a \to \infty.$$

Theorem 1.4.11. *A collection* $\{f_j, j \in J\}$ *is uniformly maximable if and only if the following conditions hold:*

(i) $\sup_{j \in J} \bigvee_\Omega f_j \, d\mu < \infty$,

(ii) for every $\varepsilon > 0$ *there exists* $\eta > 0$ *such that* $\sup_{j \in J} \bigvee_A f_j \, d\mu < \varepsilon$ *for every set* A *such that* $\mu(A) < \eta$*.*

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Proof. Let $\{f_j, j \in J\}$ be uniformly maximable. Then (i) and (ii) follow by the inequality

$$\bigvee_A f_j \, d\mu \le \bigvee_\Omega f_j \, \mathbf{1}(f_j > a) \, d\mu + a\mu(A), \; a > 0.$$

The converse follows by the fact that since for all $j \in J$ and a large enough

$$\mu(f_j > a) \le \frac{\bigvee_\Omega f_j \, d\mu}{a} < \eta,$$

where η is chosen as in condition (ii), we have $\bigvee_\Omega f_j \, \mathbf{1}(f_j > a) \, d\mu < \varepsilon, j \in J$. □

Corollary 1.4.12. *Let $f : \Omega \to \mathbb{R}_+$ be maximable. Then the set function $A \to \bigvee_\Omega f \, \mathbf{1}(A) \, d\mu$, $A \subset \Omega$, is absolutely continuous with respect to μ in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\bigvee_\Omega f \, \mathbf{1}(A) \, d\mu < \varepsilon$ for all A such that $\mu(A) < \delta$.*

Theorem 1.4.13. *A collection $\{f_j, j \in J\}$ of $\mathbb{R}_+$-valued functions on Ω is uniformly maximable if and only if $\sup_{j \in J} f_j$ is maximable.*

Proof. Sufficiency of the condition is obvious. Conversely, let $\{f_j, j \in J\}$ be uniformly maximable. For $\epsilon > 0$, let $\eta > 0$ be chosen as in part (ii) of Theorem 1.4.11. By "the Chebyshev inequality" $\mu(\sup_j f_j \ge a) \le \sup_j \bigvee_\Omega f_j \, d\mu / a$, the latter supremum being less than η if a is large enough in view of condition (i) of Theorem 1.4.11. Then by Theorem 1.4.4 and the choice of η we have $\bigvee_\Omega \sup_j f_j \, \mathbf{1}(\sup_j f_j \ge a) d\mu = \sup_j \bigvee_\Omega f_j \, \mathbf{1}(\sup_{j'} f_{j'} \ge a) d\mu \le \epsilon$. □

The following analogue of La Vallee-Poussin's theorem is a simple consequence of Theorem 1.4.9 and Theorem 1.4.13.

Corollary 1.4.14. *A collection $\{f_j, j \in J\}$ of $\mathbb{R}_+$-valued functions on Ω is uniformly maximable if and only if there exists a monotonically increasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that $F(x)/x \to \infty$ as $x \to \infty$ and $\sup_{j \in J} \bigvee_\Omega F \circ f_j \, d\mu < \infty$.*

The next easy corollary gives simple conditions for uniform maximability.

Corollary 1.4.15. *A collection $\{f_j, j \in J\}$ is uniformly maximable if either one of the following conditions holds:*

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1. *$f_j \le f, j \in J$, where f is maximable,*

2. $\sup_{j\in J} \bigvee_\Omega f_j^{1+\varepsilon}\, d\mu < \infty$ *for some* $\varepsilon > 0$,

3. $\sup_{j\in J} \bigvee_\Omega \exp(\lambda f_j)\, d\mu < \infty$ *for some* $\lambda > 0$.

We now consider uniformly maximable nets.

Definition 1.4.16. *A net $\{f_\phi, \phi \in \Phi\}$ of $\mathbb{R}_+$-valued functions on Ω is said to be uniformly maximable if*

$$\limsup_{\phi\in\Phi} \bigvee_\Omega f_\phi\, \mathbf{1}(f_\phi > a)\, d\mu \to 0 \ \text{as } a \to \infty.$$

We have the following analogues of the properties of uniformly maximable collections of functions. Similar proofs apply.

Theorem 1.4.17. *A net $\{f_\phi, \phi \in \Phi\}$ is uniformly maximable if and only if the following conditions hold:*

(i) $\limsup_{\phi\in\Phi} \bigvee_\Omega f_\phi\, d\mu < \infty$,

(ii) for every $\varepsilon > 0$ there exists $\eta > 0$ such that $\limsup_{\phi\in\Phi} \bigvee_{A_\phi} f_\phi\, d\mu < \varepsilon$ for every net of sets $\{A_\phi,\, \phi \in \Phi\}$ such that $\limsup_{\phi\in\Phi} \mu(A_\phi) < \eta$.

Corollary 1.4.18. *A net $\{f_\phi, \phi \in \Phi\}$ is uniformly maximable if either one of the following conditions holds:*

1. $\limsup_{\phi\in\Phi} \bigvee_\Omega f_\phi^{1+\varepsilon}\, d\mu < \infty$ *for some* $\varepsilon > 0$,

2. $\limsup_{\phi\in\Phi} \bigvee_\Omega \exp(\lambda f_\phi)\, d\mu < \infty$ *for some* $\lambda > 0$.

We now study convergence of idempotent integrals.

Theorem 1.4.19. *Let $\{f_\phi, \phi \in \Phi\}$ be a net of $\mathbb{R}_+$-valued functions on Ω and f be an $\mathbb{R}_+$-valued function on Ω.*

1. *("the Fatou lemma"). If $\liminf_{\phi\in\Phi} f_\phi \ge f$ μ-a.e., then*

$$\liminf_\phi \bigvee_\Omega f_\phi\, d\mu \ge \bigvee_\Omega f\, d\mu.$$

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2. *("the Lebesgue dominated convergence theorem"). If* $f_\phi \xrightarrow{\mu} f$ *and the net* $\{f_\phi\}$ *is uniformly maximable, then*

$$\lim_\phi \bigvee_\Omega f_\phi \, d\mu = \bigvee_\Omega f \, d\mu.$$

3. *("the Lebesgue monotone convergence theorem"). If* $f_\phi \uparrow f$ μ*-a.e., then*

$$\lim_\phi \bigvee_\Omega f_\phi \, d\mu = \bigvee_\Omega f \, d\mu.$$

4. *Let* μ *be* τ*-smooth relative to a collection* $\mathcal{E}$ *of subsets of* Ω *and* $\mathcal{T}$ *be a tightening collection for* μ*. Let* f_ϕ *be Luzin* $(\mathcal{E}, \mathcal{T})/\mathcal{U}$*-measurable and maximable functions. If* $f_\phi \downarrow f$ μ*-a.e., then*

$$\lim_\phi \bigvee_\Omega f_\phi \, d\mu = \bigvee_\Omega f \, d\mu.$$

Proof. We only prove part 4. Since $\{f_\phi, \phi \in \Phi\}$ is uniformly maximable and μ is $\mathcal{T}$-tight, we can and do assume that the f_ϕ are bounded by some N and $\mathcal{E}/\mathcal{U}$-measurable. Let for $m \in \mathbb{N}$

$$g_{m,\phi} = \max_{i=1,\dots,mN} \frac{i}{m} \, \mathbf{1}\big(f_\phi \geq \frac{i}{m}\big), \quad g_m = \max_{i=1,\dots,mN} \frac{i}{m} \, \mathbf{1}\big(f \geq \frac{i}{m}\big).$$

We have that $|g_{m,\phi} - f_\phi| \leq 1/m$ and $|g_m - f| \leq 1/m$ so that

$$|\bigvee_\Omega g_{m,\phi} \, d\mu - \bigvee_\Omega f_\phi \, d\mu| \leq \frac{\mu(\Omega)}{m}, \quad |\bigvee_\Omega g_m \, d\mu - \bigvee_\Omega f \, d\mu| \leq \frac{\mu(\Omega)}{m}.$$

On the other hand, using properties of μ and the facts that the f_ϕ decrease and $\{f_\phi \geq i/m\} \in \mathcal{E}$, as $\phi \in \Phi$,

$$\bigvee_\Omega g_{m,\phi} \, d\mu = \max_{i=1,\dots,mN} \frac{i}{m} \, \mu\big(f_\phi \geq \frac{i}{m}\big) \downarrow \max_{i=1,\dots,mN} \frac{i}{m} \, \mu\big(f \geq \frac{i}{m}\big) = \bigvee_\Omega g_m \, d\mu.$$

□

We have the following useful consequence.

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Corollary 1.4.20. *Let μ be an $\mathcal{E}$-idempotent measure with a tightening collection $\mathcal{T}$. Let $f : \Omega \to \mathbb{R}_+$ be Luzin $(\mathcal{E},\mathcal{T})/\mathcal{U}$-measurable and μ-maximable. Then the set function μ' defined by $\mu'(A) = \bigvee_\Omega f\,\mathbf{1}(A)\,d\mu$, $A \subset \Omega$, is an $\mathcal{E}$-idempotent measure, which has $\mathcal{T}$ as a tightening collection.*

Proof. It is obvious that μ' is a finite idempotent measure. It is τ-smooth relative to $\mathcal{E}$ by Theorem 1.4.19. It is $\mathcal{T}$-tight since for $a \in \mathbb{R}_+$ and $T \in \mathcal{T}$

$$\mu'(T^c) \le a\mu(T^c) + \bigvee_\Omega f\,\mathbf{1}(f > a)\,d\mu.$$

□

The following lemma establishes connection between convergence in idempotent measure and in "$L_1(\mu)$".

Lemma 1.4.21. *Let $\{f_\phi, \phi \in \Phi\}$ be a net of $\mathbb{R}_+$-valued functions on Ω and $f : \Omega \to \mathbb{R}_+$.*

1. *If $\bigvee_\Omega |f - f_\phi|\,d\mu \to 0$, then $f_\phi \xrightarrow{\mu} f$.*
2. *If $f_\phi \xrightarrow{\mu} f$ and the net $\{f_\phi\}$ is uniformly maximable, then $\bigvee_\Omega |f - f_\phi|\,d\mu \to 0$.*

Proof. Part 1 follows by "the Chebyshev inequality". For part 2, note that if $f_\phi \xrightarrow{\mu} f$ and the net $\{f_\phi\}$ is uniformly maximable, then f is maximable by Theorem 1.4.19. Hence, by the inequality

$$\bigvee_\Omega |f - f_\phi|\,\mathbf{1}(|f - f_\phi| > a)\,d\mu \le \bigvee_\Omega [f\,\mathbf{1}(f > a) + f_\phi\,\mathbf{1}(f_\phi > a)]\,d\mu$$

the net $\{|f - f_\phi|\}$ is uniformly maximable, so $\bigvee_\Omega |f - f_\phi|\,d\mu \to 0$ by Theorem 1.4.19. □

We now prove an analogue of Daniell's representation theorem, see, e.g., Meyer [88], stating that idempotent integral is specified by properties (JS2) and (JS3).

Theorem 1.4.22. *Let $\mathcal{H}$ be a set of $\mathbb{R}_+$-valued functions on Ω, which contains the zero function and is closed under multiplication by non-negative scalars and the formation of maximums and minimums. Let $\mathcal{E}$ denote the paving on Ω consisting of the sets $\{f \ge a\}, f \in \mathcal{H}, a \in \mathbb{R}_+$, and $\emptyset$. Let $V : \mathcal{H} \to \mathbb{R}_+$ be a non-negative homogeneous maxitive functional, i.e., V has the properties*

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(V1) $V(cf) = cV(f),\ c \in \mathbb{R}_+,\ f \in \mathcal{H},$

(V2) $V(f \vee g) = V(f) \vee V(g),\ f, g \in \mathcal{H}.$

Then the following holds.

1. *There exists an idempotent measure μ on Ω such that*

$$V(f) = \bigvee_{\Omega} f\, d\mu,\ f \in \mathcal{H},$$

if and only if V is τ-smooth along increasing nets in the sense that for every increasing net $\{f_\phi\}$ of bounded functions from $\mathcal{H}$ such that $\sup_\phi f_\phi \in \mathcal{H}$ we have

$$V(\sup_\phi f_\phi) = \sup_\phi V(f_\phi).$$

The idempotent measure μ is uniquely specified on $\mathcal{E}_u$. It is an idempotent probability if $\mathcal{H}$ contains the function identically equal to 1 and

(V0) $V(1) = 1.$

2. *Let $\mathcal{H}$, in addition, have either one of the following properties:*

   (a) *if $f \in \mathcal{H}$, then $(f - 1) \vee 0 \in \mathcal{H}$,*
   (b) *if $f \in \mathcal{H}$, then $f \wedge 1 \in \mathcal{H}$, and $\mathcal{H}$ is closed under multiplication.*

Then there exists an $\mathcal{E}$-idempotent measure μ on Ω such that

$$V(f) = \bigvee_{\Omega} f\, d\mu,\ f \in \mathcal{H},$$

if and only if the following condition holds:

(VC) *for every nets $\{f_\phi\}$ and $\{g_\psi\}$ of bounded functions from $\mathcal{H}$, which are increasing and decreasing, respectively, and such that $\sup_\phi f_\phi \geq \inf_\psi g_\psi$, we have*

$$\sup_\phi V(f_\phi) \geq \inf_\psi V(g_\psi).$$

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The idempotent measure μ *is also* τ*-smooth relative to* $\mathcal{E}_i$ *and is uniquely specified on* $\mathcal{E}_{iu}$.

If, in addition, for every $f, g \in \mathcal{H}$, *we have* $(f-g) \vee 0 \in \mathcal{H}$ *and* $V(f+g) \leq V(f) + V(g)$ *if* $f+g \in \mathcal{H}$, *then condition (VC) is equivalent to Daniell's condition:*

(VD) if $f_\phi \downarrow 0$, *where the* f_ϕ *are bounded functions from* $\mathcal{H}$, *then* $V(f_\phi) \downarrow 0$.

Proof. We first deal with the necessity parts. The condition of τ-smoothness of V along increasing nets is necessary for existence of μ in part 1 by Theorem 1.4.19. Let us show necessity of (VC) for μ being a τ-smooth idempotent measure. The condition follows by Theorem 1.4.19 (with $\mathcal{T} = \{\Omega\}$) if we note that every $f \in \mathcal{H}$ is $\mathcal{E}/\mathcal{U}$-measurable so that

$$\sup_\phi \bigvee_\Omega f_\phi \, d\mu = \bigvee_\Omega \big(\sup_\phi f_\phi\big)\, d\mu, \quad \inf_\psi \bigvee_\Omega f_\psi \, d\mu = \bigvee_\Omega \big(\inf_\psi f_\psi\big)\, d\mu.$$

We prove sufficiency in part 1. Let Ω_1 be the set whose elements are sets $[0,a) \times \{\omega\}$, $a \in \mathbb{R}_+, \omega \in \Omega$. For $\mathbb{R}_+$-valued functions f on Ω, let $W_f = \{[0,a) \times \{\omega\} : a \leq f(\omega)\}$. We define $\mathcal{E}_1 = \{W_f,\ f \in \mathcal{H}\}$. By the assumptions on $\mathcal{H}$ the collection $\mathcal{E}_1$ is a paving on Ω_1. We set $U(W_f) = V(f)$. Then the set function U is maxitive on $\mathcal{E}_1$ and satisfies the hypotheses of part 1 of Theorem 1.1.9 (in particular, since $V(0) = 0$ by $(V1)$, it follows that $U(\emptyset) = U(W_0) = V(0) = 0$). We define an extension of U to a set function on $\mathcal{P}(\Omega_1)$ as in the proof of Theorem 1.1.9 by

$$U^*([0,a) \times \{\omega\}) = \inf_{\substack{f \in \mathcal{H}:\\ f(\omega) \geq a}} U(W_f), \tag{1.4.1}$$

$$U^*(A_1) = \sup_{[0,a)\times\{\omega\} \in A_1} U^*\big([0,a) \times \{\omega\}\big),\ A_1 \subset \Omega_1.$$

By part 1 of Theorem 1.1.9 U^* is an idempotent measure on Ω_1, which extends U.

Since by $(V1)$, for $c \in \mathbb{R}_+$ and $f \in \mathcal{H}$, $U(W_{cf}) = V(cf) = cV(f) = cU(W_f)$, equality (1.4.1) implies that

$$U^*([0,ca) \times \{\omega\}) = cU^*([0,a) \times \{\omega\}). \tag{1.4.2}$$

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Now, for $A \subset \Omega$, we define $\mu(A) = U^*([0,1) \times A)$. Clearly, μ is an idempotent measure on Ω. Also, for arbitrary $f \in \mathcal{H}$, by τ-maxitivity of U^* and (1.4.2)

$$V(f) = U(W_f) = U\Big(\bigcup_{\substack{\omega\in\Omega:\\ a\le f(\omega)}} \big\{[0,a) \times \{\omega\}\big\}\Big)$$
$$= \sup_{\substack{\omega\in\Omega:\\ a\le f(\omega)}} U^*\big([0,a) \times \{\omega\}\big) = \sup_{\substack{\omega\in\Omega:\\ a\le f(\omega)}} aU^*\big([0,1) \times \{\omega\}\big)$$
$$= \sup_{\omega\in\Omega} f(\omega)\mu(\{\omega\}) = \bigvee_\Omega f\, d\mu.$$

Part 1 is proved.

We prove part 2. Note that under (VC) the set function U satisfies condition (S) of part 2 of Theorem 1.1.9. Therefore, U^* is an $\mathcal{E}_1$-idempotent measure. We check the τ-smoothness property for μ. Let condition (a) of part 2 hold. Since for $f : \Omega \to \mathbb{R}_+$ and $a > 0$

$$\mathbf{1}(f \ge a) = \inf_{x\in\mathbb{R}_+} \big(x\, \frac{f}{a} - x + 1\big)^+ \tag{1.4.3}$$

and the functions in the infimum belong to $\mathcal{H}$, the sets $W_{\mathbf{1}(f\ge a)}$ belong to $\mathcal{E}_{1,i}$ for $f \in \mathcal{H}$ and $a > 0$. Now, let $\{f_\phi \ge a_\phi\}, f_\phi \in \mathcal{H}, \phi \in \Phi$, be a decreasing net of sets from $\mathcal{E}$. We prove that

$$\inf_\phi \mu(f_\phi \ge a_\phi) = \mu\Big(\bigcap_\phi \{f_\phi \ge a_\phi\}\Big). \tag{1.4.4}$$

It is sufficient to consider the case $a_\phi > 0$. Then by (1.4.3) $W_{\mathbf{1}(f_\phi\ge a_\phi)} = \bigcap_{\psi\in\Psi} F_{\phi,\psi}$ for some $F_{\phi,\psi} \in \mathcal{E}_1$, where $\Psi = \mathbb{R}_+$. For finite subsets Φ_0 and Ψ_0 of Φ and Ψ, respectively, we define $G_{\Phi_0,\Psi_0} = \bigcap_{\substack{\phi\in\Phi_0\\ \psi\in\Psi_0}} F_{\phi,\psi}$. Then

$$G_{\Phi_0,\Psi_0} \supset \bigcap_{\phi\in\Phi_0} \bigcap_{\psi\in\Psi} F_{\phi,\psi} = \bigcap_{\phi\in\Phi_0} W_{\mathbf{1}(f_\phi\ge a_\phi)} \supset W_{\mathbf{1}(f_{\phi'}\ge a_{\phi'})}, \tag{1.4.5}$$

where ϕ' is such that $\phi \le \phi'$ for all $\phi \in \Phi_0$. Also $\bigcap_{\Phi_0,\Psi_0} G_{\Phi_0,\Psi_0} = \bigcap_\phi W_{\mathbf{1}(f_\phi\ge a_\phi)}$. Since $\{G_{\Phi_0,\Psi_0},\, (\Phi_0,\Psi_0) \in \mathcal{Q}(\Phi) \times \mathcal{Q}(\Psi)\}$ is a decreasing net of elements of $\mathcal{E}_1$ with respect to the partial order on the

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pairs (Φ_0, Ψ_0) by inclusion and U^* is τ-smooth relative to $\mathcal{E}_1$,

$$\inf_{\Phi_0,\Psi_0} U^*(G_{\Phi_0,\Psi_0}) = U^*\Big(\bigcap_{\Phi_0,\Psi_0} G_{\Phi_0,\Psi_0}\Big) = U^*\Big(\bigcap_\phi W_{\mathbf{1}(f_\phi \geq a_\phi)}\Big)$$
$$= U^*\Big(W_{\mathbf{1}(\bigcap_\phi \{f_\phi \geq a_\phi\})}\Big) = U^*\Big(\bigcup_{a\in[0,1)} \Big([0,a) \times \big(\bigcap_\phi \{f_\phi \geq a_\phi\}\big)\Big)\Big)$$
$$= \sup_{a\in[0,1)} aU^*\Big([0,1) \times \big(\bigcap_\phi \{f_\phi \geq a_\phi\}\big)\Big) = \mu\Big(\bigcap_\phi \{f_\phi \geq a_\phi\}\Big).$$

Also by (1.4.5) $\inf_{\Phi_0,\Psi_0} U^*(G_{\Phi_0,\Psi_0}) \geq \inf_\phi \mu(f_\phi \geq a_\phi)$. Therefore,

$$\mu\Big(\bigcap_\phi \{f_\phi \geq a_\phi\}\Big) \geq \inf_\phi \mu(f_\phi \geq a_\phi),$$

and (1.4.4) follows. Now, μ is an $\mathcal{E}_i$–idempotent measure by Lemma 1.1.8.

If the condition that $(f-1)\vee 0 \in \mathcal{H}$ for $f \in \mathcal{H}$ is replaced by the conditions that $f \wedge 1 \in \mathcal{H}$ for $f \in \mathcal{H}$ and $\mathcal{H}$ is closed under multiplication, the same proof applies except that (1.4.3) is replaced by the equality $\mathbf{1}(f \geq a) = \inf_{x\in\mathbb{N}} (f/a)^x \wedge 1$.

To end the proof, let us assume that $(f-g)\vee 0 \in \mathcal{H}$ if $f, g \in \mathcal{H}$, V is subadditive, i.e., $V(f+g) \leq V(f) + V(g)$ if $f, g, f+g \in \mathcal{H}$, and (VD) holds. We check that (VC) holds. Let $f_\phi \uparrow$, $g_\psi \downarrow$ and $\sup_\phi f_\phi \geq \inf_\psi g_\psi$, where the f_ϕ and g_ψ are bounded functions from $\mathcal{H}$. Then

$$V(g_\psi) \leq V(g_\psi \vee f_\phi) \leq V(f_\phi) + V((g_\psi - f_\phi)\vee 0).$$

Since $(g_\psi - f_\phi)\vee 0 \in \mathcal{H}$ and tends monotonically to zero, an application of (VD) yields (VC). □

Remark 1.4.23. *Under the hypotheses of part 1 (respectively, part 2) of Theorem 1.4.22 the functional V has a unique extension to a non-negative homogeneous maxitive functional on the set of $\mathcal{E}_u$-measurable (respectively, $\mathcal{E}_{iu}$-measurable) $\mathbb{R}_+$-valued functions on Ω.*

Remark 1.4.24. *If $\mathcal{H}$, in addition to the hypotheses of Theorem 1.4.22, is such that $(1-f)\vee 0 \in \mathcal{H}$ for every $f \in \mathcal{H}$, then $\mathcal{E}$ is a semi-τ-algebra, hence, $\mathcal{E}_{iu}$ is a τ-algebra, which is also generated by sets $\{f = a\}$, $f \in \mathcal{H}, a \in \mathbb{R}_+$. In particular, by the preceding*

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remark under the hypotheses of part 2 V has a unique extension on the set of $\mathbb{R}_+$*-valued functions that are measurable with respect to the* τ*-algebra generated by the elements of* $\mathcal{H}$*.*

Remark 1.4.25. *Theorem 1.4.22 implies Theorem 1.1.9 if we take* $\mathcal{H} = \{c\,\mathbf{1}(F),\ F \in \mathcal{E}, c \in \mathbb{R}_+\}$ *and* $V(c\,\mathbf{1}(F)) = c\,\mu(F)$ *(note that* $\big(c\,\mathbf{1}(F)\big) \wedge 1 = (c \wedge 1)\mathbf{1}(F)$ *and* $(c\,\mathbf{1}(F) - 1) \vee 0 = \big((c-1) \vee 0\big)\,\mathbf{1}(F)$ *so that* $f \wedge 1 \in \mathcal{H}$ *and* $(f-1) \vee 0 \in \mathcal{H}$ *if* $f \in \mathcal{H}$*).*

Remark 1.4.26. *Similarly to Theorem 1.1.9 (see Remark 1.1.11), Theorem 1.4.22 admits a version where the collection* $\mathcal{H}$ *need not be closed under the formation of maximums and minimums. Then the maxitivity condition* $(V2)$ *and the* τ*-smoothness condition along increasing nets should be replaced in part 1 by the following condition: if a collection* $\{f_j\}$ *of elements of* $\mathcal{H}$ *and* $f \in \mathcal{H}$ *are such that* $f \le \sup_j f_j$*, then* $V(f) \le \sup_j V(f_j)$*.*

In part 2, condition (VC) would have to be replaced by the following:

(VC') *If* $\{f_j\}$ *is a collection of bounded functions from* $\mathcal{H}$ *and* $\{g_\psi\}$ *is a decreasing net of bounded functions from* $\mathcal{H}$ *such that* $\sup_j f_j \ge \inf_\psi g_\psi$*, then* $\sup_j V(f_j) \ge \inf_\psi V(g_\psi)$*.*

Condition (VC') *is necessary and sufficient for* μ *to be* τ*-smooth relative to* $\mathcal{E}$*. If* $\mathcal{H}$ *is closed under the formation of minimums, then the extension is* τ*-smooth relative to* $\mathcal{E}_i$*.*

1.5 Product spaces

This section considers products of idempotent measure spaces.

Definition 1.5.1. *Let* $(\Omega, \mathcal{A})$ *and* $(\Omega', \mathcal{A}')$ *be* τ*-measurable spaces. We define the product* τ*-algebra as the* τ*-algebra on* $\Omega \times \Omega'$ *with the atoms* $[\omega]_{\mathcal{A}} \times [\omega']_{\mathcal{A}'}$*. It is denoted by* $\mathcal{A} \otimes \mathcal{A}'$*. The* τ*-measurable space* $(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}')$ *is called the product of* $(\Omega, \mathcal{A})$ *and* $(\Omega', \mathcal{A}')$*.*

Remark 1.5.2. *The product* τ*-algebra* $\mathcal{A} \otimes \mathcal{A}'$ *is generated by the semi-*τ*-algebra consisting of the rectangles* $A \times A'$*, where* $A \in \mathcal{A}$ *and*

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$A' \in \mathcal{A}'$. Since $\mathcal{A}\otimes\mathcal{A}'$ is also generated by the collection of sets $A\times\Omega'$ and $\Omega \times A'$, where $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$, the τ-algebra $\mathcal{A} \otimes \mathcal{A}'$ is the smallest τ-algebra $\tilde{\mathcal{A}}$ on $\Omega \times \Omega'$ such that the projections $(\omega, \omega') \to \omega$ and $(\omega, \omega') \to \omega'$ are $\tilde{\mathcal{A}}/\mathcal{A}$-measurable and $\tilde{\mathcal{A}}/\mathcal{A}'$-measurable, respectively.

Lemma 1.5.3. *Let $A \in \mathcal{A} \otimes \mathcal{A}'$. Then the projection $pr_{\Omega'} A = \{\omega' \in \Omega' : (\omega, \omega') \in A$ for some $\omega \in \Omega\}$ and cross-sections $A_\omega = \{\omega' \in \Omega' : (\omega, \omega') \in A\}$, where $\omega \in \Omega$, are elements of $\mathcal{A}'$. Also, if $f : \Omega \times \Omega' \to \mathbb{R}_+$ is $\mathcal{A} \otimes \mathcal{A}'$-measurable, then the cross-section $f_\omega(\omega') = f(\omega, \omega')$ is $\mathcal{A}'$-measurable as a function of ω'. Conversely, if $A_\omega \in \mathcal{A}'$ for every $\omega \in \Omega$ and $A_{\omega'} \in \mathcal{A}$ for every $\omega' \in \Omega'$, then $A \in \mathcal{A} \otimes \mathcal{A}'$.*

Proof. We start with cross-sections. Let $\omega' \in A_\omega$. Then $(\omega, \omega') \in A$ and by Corollary 1.1.16 $[\omega]_{\mathcal{A}} \times [\omega']_{\mathcal{A}'} = [(\omega, \omega')]_{\mathcal{A}\otimes\mathcal{A}'} \in A$. Thus, $[\omega']_{\mathcal{A}'} \in A_\omega$ and by Corollary 1.1.16 $A_\omega \in \mathcal{A}'$. The case of the projection is considered similarly. One could also use the representation $\mathrm{pr}_{\Omega'} A = \cup_{\omega\in\Omega} A_\omega$ and the definition of a τ-algebra. Measurability of f_ω follows since $f_\omega^{-1}(x) = \big(f^{-1}(x)\big)_\omega$ for $x \in \mathbb{R}_+$. For the final statement, we first note that the cross-sections A_ω depend only on the atom to which ω belongs for otherwise there would exist $\omega_1 \in \Omega$, $\omega_2 \in \Omega$ and $\omega' \in \Omega'$ such that $\omega_1 \overset{\mathcal{A}}{\sim} \omega_2$, $\omega' \in A_{\omega_1}$ and $\omega' \notin A_{\omega_2}$, which would imply that $\omega_1 \in A_{\omega'}$ but $\omega_2 \notin A_{\omega'}$ so that $A_{\omega'} \notin \mathcal{A}$. The required now follows by the equality $A = \cup_{\omega\in\Omega} [\omega]_{\mathcal{A}} \times A_\omega$ and the definition of $\mathcal{A} \otimes \mathcal{A}'$. ☐

Definition 1.5.4. *Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be τ-measurable spaces. A function $k : \Omega \times \mathcal{P}(\Omega') \to [0, 1]$ is called an idempotent transition kernel from $(\Omega, \mathcal{A})$ to $(\Omega', \mathcal{A}')$ if the following holds:*

- *(a) for every $\omega \in \Omega$, the function $k(\omega, A')$, $A' \subset \Omega'$, is an idempotent probability measure on Ω',*
- *(b) for every $A' \in \mathcal{A}'$, the function $k(\omega, A'), \omega \in \Omega$, is $\mathcal{A}$-measurable.*

Lemma 1.5.5. *Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be τ-measurable spaces.*

1. *Let μ be an idempotent measure on $(\Omega, \mathcal{A})$ and k be an idempotent transition kernel from $(\Omega, \mathcal{A})$ to $(\Omega', \mathcal{A}')$. Then there is a unique idempotent measure $\tilde{\mu}$ on $(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}')$ such that*

$$\tilde{\mu}(A\times A') = \bigvee_A k(\omega, A')\, d\mu(\omega),\ A \in \mathcal{A},\ A' \in \mathcal{A}'. \qquad (1.5.1)$$

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In particular, $\tilde{\mu}(A \times \Omega') = \mu(A)$.

If a function $f : \Omega \times \Omega' \to \mathbb{R}_+$ *is* $\mathcal{A} \otimes \mathcal{A}'$*-measurable, then the function*

$$g(\omega) = \bigvee_{\Omega'} f_\omega(\omega')\, k(\omega, d\omega'),\ \omega \in \Omega,$$

is $\mathcal{A}$*-measurable and*

$$\bigvee_{\Omega \times \Omega'} f\, d\tilde{\mu} = \bigvee_{\Omega} g\, d\mu.$$

2. *Conversely, if* $\tilde{\mu}$ *and* μ *are idempotent measures on respective* τ*-measurable spaces* $(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}')$ *and* $(\Omega, \mathcal{A})$ *such that* $\tilde{\mu}(A \times \Omega') = \mu(A)$ *for* $A \in \mathcal{A}$*, then there exists an idempotent transition kernel* k *from* $(\Omega, \mathcal{A})$ *to* $(\Omega', \mathcal{A}')$ *such that* (1.5.1) *holds. The idempotent probability* $(k(\omega, A'),\ A' \in \mathcal{A}')$ *is uniquely specified on* $(\Omega, \mathcal{A}')$ *for* μ*-almost all* ω*.*

Remark 1.5.6. *As we mentioned in Remark 1.1.29, saying "*μ *is an idempotent measure on* $(\Omega, \mathcal{A})$*" is to mean that* μ *is uniquely specified only for elements of* $\mathcal{A}$*. Likewise, uniqueness of* $\tilde{\mu}$ *is claimed on* $\mathcal{A} \otimes \mathcal{A}'$*.*

Proof of Lemma 1.5.5. We begin with part 1. Since necessarily $\tilde{\mu}([\omega]_{\mathcal{A}} \times [\omega']_{\mathcal{A}'}) = k(\omega, [\omega']_{\mathcal{A}'})\mu([\omega]_{\mathcal{A}})$, $\tilde{\mu}$ is uniquely specified on $\mathcal{A} \otimes \mathcal{A}'$. We can define $\tilde{\mu}$ on $\mathcal{P}(\Omega \times \Omega')$ by

$$\tilde{\mu}(\{(\omega, \omega')\}) = k(\omega, \{\omega'\})\mu(\{\omega\}),$$
$$\tilde{\mu}(A \times A') = \sup_{\omega \in A,\, \omega' \in A'} \tilde{\mu}\big(\{(\omega, \omega')\}\big).$$

The only thing that requires proof is that g is $\mathcal{A}$-measurable. This follows by the fact that $f_\omega(\omega')k(\omega, \{\omega'\}) = f_{\omega'}(\omega)k(\omega, \{\omega'\})$ is $\mathcal{A}$-measurable in ω for every $\omega' \in A'$ by Lemma 1.5.3 and Corollary 1.2.8.

In part 2 necessarily

$$k(\omega, \{\omega'\}) = \frac{\tilde{\mu}([\omega]_{\mathcal{A}} \times [\omega']_{\mathcal{A}'})}{\mu([\omega]_{\mathcal{A}})}$$

if $\mu([\omega]_{\mathcal{A}}) > 0$, which proves uniqueness. For existence, we define $k(\omega, \{\omega'\})$ by the latter formula if $\mu([\omega]_{\mathcal{A}}) > 0$ and let $k(\omega, A)$ be arbitrary idempotent probability measures on $(\Omega', \mathcal{A}')$ that are constant on the atoms of $\mathcal{A}$ on the rest of Ω. □

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Remark 1.5.7. *If $k(\omega, \{\omega'\}) = \mu'(\{\omega'\})$, where μ' is an idempotent measure on Ω', then $\tilde{\mu}\big(\{(\omega, \omega')\}\big) = \mu(\{\omega\})\mu'(\{\omega'\})$ and we obtain an analogue of Fubini's theorem. The idempotent measure $\tilde{\mu}$ is called the product of idempotent measures μ and μ' and denoted as $\mu \times \mu'$. The idempotent measure space $(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}', \mu \times \mu')$ is called the product of the idempotent measure spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$.*

We consider now τ-smoothness and tightness properties of idempotent measures on product spaces.

Theorem 1.5.8. *Let us assume that an idempotent measure μ on Ω is τ-smooth relative to a collection $\mathcal{E} \subset \mathcal{P}(\Omega)$ such that $\emptyset \in \mathcal{E}$ and has a tightening collection $\mathcal{T} \subset \mathcal{P}(\Omega)$. Let an idempotent transition kernel $k(\omega, A')$ from $(\Omega, \mathcal{P}(\Omega))$ to $(\Omega', \mathcal{P}(\Omega'))$ and collections $\mathcal{E}' \subset \mathcal{P}(\Omega')$ and $\mathcal{T}' \subset \mathcal{P}(\Omega')$, such that $\emptyset \in \mathcal{E}'$ and $T' \cap F' \in \mathcal{E}'$ for every $T' \in \mathcal{T}'$ and $F' \in \mathcal{E}'$, satisfy the following conditions:*

1. *$k(\omega, A')$ is Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable in ω for every $A' \in \mathcal{E}'$,*
2. *$k(\omega, A')$ is a τ-smooth idempotent measure in A' relative to $\mathcal{E}'$ for every $\omega \in \Omega$,*
3. *for every $\epsilon > 0$ and $T \in \mathcal{T}$ there exists $T' \in \mathcal{T}'$ such that $\sup_{\omega \in T} k(\omega, \Omega' \setminus T') \le \epsilon$.*

Then the idempotent measure $\tilde{\mu}$ on $\Omega \times \Omega'$ defined by $\tilde{\mu}\big(\{(\omega, \omega')\}\big) = k(\omega, \{\omega'\})\mu(\{\omega\})$ is τ-smooth relative to $\mathcal{E} \times \mathcal{E}' = \{F \times F', F \in \mathcal{E}, F' \in \mathcal{E}'\}$ and has the tightening collection $\mathcal{T} \times \mathcal{T}' = \{T \times T', T \in \mathcal{T}, T' \in \mathcal{T}'\}$.

Proof. We first check the τ-smoothness. Let $\{F_\psi \times F'_\psi,\ \psi \in \Psi\}$ be a decreasing net of elements of $\mathcal{E} \times \mathcal{E}'$. Let $F \times F' = \cap_{\psi \in \Psi}(F_\psi \times F'_\psi)$. We have that

$$\tilde{\mu}(F_\psi \times F'_\psi) = \bigvee_{\Omega} k(\omega, F'_\psi)\, \mathbf{1}(\omega \in F_\psi)\, d\mu(\omega). \tag{1.5.2}$$

The functions $k(\omega, F'_\psi)\, \mathbf{1}(\omega \in F_\psi)$ are bounded, Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable and monotonically converge as $\psi \in \Psi$ to $k(\omega, F')\, \mathbf{1}(\omega \in F)$. Therefore by Theorem 1.4.19 $\tilde{\mu}(F_\psi \times F'_\psi) \to \tilde{\mu}(F \times F')$ checking τ-smoothness of $\tilde{\mu}$. The fact that $\mathcal{T} \times \mathcal{T}'$ is a tightening collection for $\tilde{\mu}$ follows since by (1.5.2)

$$\tilde{\mu}\big((T \times T')^c\big) = \tilde{\mu}\big((\Omega \times T'^c) \cup (T^c \times \Omega')\big) \le \mu(T^c) \vee \sup_{\omega \in T} k\big(\omega, T'^c\big).$$

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□

For future use we also introduce the following notion.

Definition 1.5.9. *Let $\mathcal{A}$ be a τ-algebra on a set Ω and $\mathcal{B}$ be a σ-algebra on a set Υ. We define the product $\mathcal{B} \otimes \mathcal{A}$ as the collection of subsets of $\Upsilon \times \Omega$ that are expressed as unions of sets $B \times [z]_{\mathcal{A}}$, where $B \in \mathcal{B}$ and $[z]_{\mathcal{A}}$ are atoms of $\mathcal{A}$, such that each atom of $\mathcal{A}$ appears in a union only once.*

Remark 1.5.10. *Clearly, $\mathcal{B} \otimes \mathcal{A}$ is a σ-algebra but not a τ-algebra.*

1.6 Independence and conditioning

Let (Ω, Π) be an idempotent probability space.

Definition 1.6.1. *A finite collection $\{A_i,\, i = 1, \ldots, k\}$ of subsets of Ω is independent if*

$$\Pi\Bigl(\bigcap_{i=1}^{k} A_i\Bigr) = \prod_{i=1}^{k} \Pi(A_i).$$

A collection $\{A_\alpha\}$ of subsets of Ω is independent if every finite subcollection is independent.
A collection $\{\mathcal{E}_\alpha\}$ of families of subsets of Ω is independent if every collection of sets $\{A_\alpha\}$, where $A_\alpha \in \mathcal{E}_\alpha$, is independent.

Lemma 1.6.2. *A collection of families $\{\mathcal{E}_\alpha\}$ is independent if and only if the collection of families $\{(\mathcal{E}_\alpha)_u\}$ is independent.*

Definition 1.6.3. *Let a set Ω' be equipped with a τ-algebra $\mathcal{A}'$. A collection $\{f_\alpha\}$ of idempotent variables on Ω with values in Ω' is $\mathcal{A}'$-independent (or independent if $\mathcal{A}' = \mathcal{P}(\Omega')$) if the collection of the τ-algebras $f_\alpha^{-1}(\mathcal{A}')$ is independent. An idempotent variable $f : \Omega \to \Omega'$ and a collection $\mathcal{E}$ of subsets of Ω are $\mathcal{A}'$-independent (or independent if $\mathcal{A}' = \mathcal{P}(\Omega')$) if the τ-algebra $f^{-1}(\mathcal{A}')$ and $\mathcal{E}$ form an independent collection.*

Remark 1.6.4. *Loosely, we will often say that sets, τ-algebras or idempotent variables, respectively, are independent if the associated collections are independent.*

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Remark 1.6.5. *Clearly, sets are independent if and only if their indicator functions are independent.*

Lemma 1.6.6. *Let idempotent variables $f_\alpha : \Omega \to \Omega'$, where Ω' is equipped with a τ-algebra $\mathcal{A}'$, be measurable relative to respective τ-algebras $\mathcal{A}_\alpha$ on Ω. If the collection $\{\mathcal{A}_\alpha\}$ is independent, then the collection $\{f_\alpha\}$ is $\mathcal{A}'$-independent.*

We have the following consequences.

Lemma 1.6.7.

1. *Let $f_\alpha : \Omega \to \Omega'$ and Ω' be equipped with the discrete τ-algebra. The collection $\{f_\alpha\}$ is independent if and only if the collection of sets $\{f_\alpha^{-1}(\omega_\alpha)\}$ is independent for all $\omega_\alpha \in \Omega'$.*

2. *Let $(\Omega, \mathcal{A})$, $(\Omega', \mathcal{A}')$ and $(\Omega'', \mathcal{A}'')$ be τ-measurable spaces. Let $f : \Omega \to \Omega'$ and $F : \Omega' \to \Omega''$ be $\mathcal{A}/\mathcal{A}'$- and $\mathcal{A}'/\mathcal{A}''$-measurable, respectively. If f is independent of a τ-algebra $\mathcal{B}$ on Ω, then $F \circ f$ is also independent of $\mathcal{B}$.*

In the rest of the section we consider $\mathbb{R}_+$-valued functions on Ω unless specified otherwise. We assume that $\mathbb{R}_+$ is equipped with the discrete τ-algebra $\mathcal{P}(\mathbb{R}_+)$ and, according to the convention adopted in Section 1.2, refer to $\mathcal{A}/\mathcal{P}(\mathbb{R}_+)$-measurable functions $f : \Omega \to \mathbb{R}_+$ as $\mathcal{A}$-measurable. We recall that if $\mathcal{A}$ is a τ-algebra on Ω, then f is $\mathcal{A}$-measurable if and only if the inverse images of one-element subsets of $\mathbb{R}_+$ belong to $\mathcal{A}$.

Lemma 1.6.8. *If $f : \Omega \to \mathbb{R}_+$ and $g : \Omega \to \mathbb{R}_+$ are independent, then $S(fg) = S(f)S(g)$.*

Proof. The result follows by the representations $f(\omega) = \sup_{x\in\mathbb{R}_+} x\, \mathbf{1}(f(\omega) = x)$, $g(\omega) = \sup_{x\in\mathbb{R}_+} x\, \mathbf{1}(g(\omega) = x)$ and $(fg)(\omega) = \sup_{x,y\in\mathbb{R}_+}(xy)\, \mathbf{1}(f(\omega) = x)\, \mathbf{1}(g(\omega) = y)$, and properties of idempotent expectations. □

We now define conditional idempotent probabilities and conditional idempotent expectations. Let $\mathcal{A}$ be a τ-algebra on Ω, $\mathcal{E}$ be a π-system of subsets of Ω containing $\emptyset$ and $\mathcal{T}$ be a collection of subsets of Ω such that $T \cap F \in \mathcal{E}$ for $T \in \mathcal{T}$ and $F \in \mathcal{E}$. For economy of notation we denote $\Pi(\{\omega\})$ as $\Pi(\omega)$.

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Definition 1.6.9. *The conditional idempotent probability $\Pi(\omega'|\mathcal{A})(\omega)$ of ω' given $\mathcal{A}$ is defined by*

$$\Pi(\omega'|\mathcal{A})(\omega) = \begin{cases} \dfrac{\Pi(\omega')}{\Pi([\omega]_{\mathcal{A}})}\,\mathbf{1}(\omega' \stackrel{\mathcal{A}}{\sim} \omega), & \text{if } \Pi([\omega]_{\mathcal{A}}) > 0, \\ \tilde{\Pi}(\omega'), & \text{if } \Pi([\omega]_{\mathcal{A}}) = 0, \end{cases}$$

where $\tilde{\Pi}$ is some idempotent probability on Ω.

For $B \subset \Omega$, we define $\Pi(\omega'|B) = \Pi(\omega'|\mathcal{A}_B)(\omega)$, where $\omega \in B$ and $\mathcal{A}_B$ is a τ-algebra, which has B as an atom (note that the right-hand side does not depend on the particular choice of $\omega \in B$ and $\mathcal{A}_B$).

If $A \subset \Omega$, then the conditional idempotent probability of A given $\mathcal{A}$ is defined by

$$\Pi(A|\mathcal{A})(\omega) = \sup_{\omega' \in A} \Pi(\omega'|\mathcal{A})(\omega),\ \omega \in \Omega.$$

Similarly, $\Pi(A|B) = \sup_{\omega' \in A} \Pi(\omega'|\mathcal{A}_B)(\omega)$, $\omega \in B$.

If Π is an $\mathcal{E}$-idempotent probability, then the conditional $\mathcal{E}$-idempotent probability given $\mathcal{A}$ is defined in an analogous manner except that $\tilde{\Pi}$ is required to be an $\mathcal{E}$-idempotent probability on Ω. Likewise, if Π is in addition $\mathcal{T}$-tight, we require $\tilde{\Pi}$ to be $\mathcal{T}$-tight.

If $f : \Omega \to \Omega'$, where Ω' is equipped with a τ-algebra $\mathcal{A}'$, we define $\Pi(A|f) = \Pi\big(A|f^{-1}(\mathcal{A}')\big)$.

Remark 1.6.10. *According to the definition, conditional idempotent probability is uniquely specified Π-a.e. More precisely, if $N = \{\omega \in \Omega :\ \Pi([\omega]_{\mathcal{A}}) = 0\}$, then $N \in \mathcal{A}$, $\Pi(N) = 0$ and*

$$\Pi(A|\mathcal{A})(\omega) = \frac{\Pi(A \cap [\omega]_{\mathcal{A}})}{\Pi([\omega]_{\mathcal{A}})} \text{ for all } A \subset \Omega \text{ and } \omega \in N^c.$$

Also, if $B \subset \Omega$ is such that $\Pi(B) > 0$, then our definition of $\Pi(A|B)$ agrees with the "standard" one in that $\Pi(A|B) = \Pi(A \cap B)/\Pi(B)$.

Remark 1.6.11. *Let us assume that Π is uniquely specified only on a τ-algebra $\mathcal{A}' \supset \mathcal{A}$. Then, recalling that the atoms of $\mathcal{A}$ are unions of the atoms of $\mathcal{A}'$, we have for $A' \in \mathcal{A}'$ by the above definition that*

$$\Pi([\omega']_{\mathcal{A}'}|\mathcal{A})(\omega) = \begin{cases} \dfrac{\Pi([\omega']_{\mathcal{A}'})}{\Pi([\omega]_{\mathcal{A}})}\,\mathbf{1}(\omega' \stackrel{\mathcal{A}}{\sim} \omega), & \text{if } \Pi([\omega]_{\mathcal{A}}) > 0, \\ \tilde{\Pi}([\omega']_{\mathcal{A}'}), & \text{if } \Pi([\omega]_{\mathcal{A}}) = 0, \end{cases}$$

so the values of the conditional idempotent probability on the elements of $\mathcal{A}'$ do not depend on the extension of Π to $\mathcal{P}(\Omega)$.

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We now list properties of conditional idempotent probabilities.

Theorem 1.6.12. *The function* $(\Pi(A|\mathcal{A})(\omega), A \subset \Omega, \omega \in \Omega)$ *has the following properties:*

1. *it is* $\mathcal{A}$*–measurable in* ω *for all* $A \subset \Omega$*,*
2. *it is an idempotent probability in* $A \subset \Omega$ *for every* $\omega \in \Omega$*,*
3. *for all* $A \subset \Omega$ *and* $B \in \mathcal{A}$*,*
$$\Pi\big(A \cap B\big) = S[\Pi(A|\mathcal{A})(\omega)\,\mathbf{1}(\omega \in B)].$$
4. *If* Π *is an* $\mathcal{E}$*-idempotent probability and* $[\omega]_{\mathcal{A}} \in \mathcal{E}$ *(respectively,* $B \in \mathcal{E}$*), then* $(\Pi(A|\mathcal{A})(\omega), A \subset \Omega)$ *(respectively,* $(\Pi(A|B), A \subset \Omega)$*) is an* $\mathcal{E}$*-idempotent probability in* A*.*
5. *If* Π *is* $\mathcal{T}$*-tight, then* $(\Pi(A|\mathcal{A})(\omega), A \subset \Omega)$ *is* $\mathcal{T}$*-tight for all* $\omega \in \Omega$ *and* $(\Pi(A|B), A \subset \Omega)$ *is* $\mathcal{T}$*-tight for all* $B \subset \Omega$*.*

Proof. We begin with property 1. The function $\omega \to \Pi([\omega]_{\mathcal{A}})$ is $\mathcal{A}$-measurable since it is constant on the atoms of $\mathcal{A}$. Obviously, $\mathbf{1}(\omega' \sim \omega)$ is also $\mathcal{A}$-measurable in ω. By Lemma 1.2.8 we conclude that $\omega \to \Pi(\omega'|\mathcal{A})(\omega)$ is $\mathcal{A}$-measurable; hence, $\Pi(A|\mathcal{A})(\omega)$ is also $\mathcal{A}$-measurable.

Property 2 follows by the definition. We prove property 3. Assume, first, that $B = [\hat{\omega}]_{\mathcal{A}}$ for some $\hat{\omega} \in \Omega$. Then, adopting the convention that $\Pi(\omega)/\Pi([\omega]_{\mathcal{A}}) = 0$ if $\Pi([\omega]_{\mathcal{A}}) = 0$,

$$\begin{aligned} S\big[\Pi(A|\mathcal{A})(\omega)\,\mathbf{1}(\omega \sim \hat{\omega})\big] &= \sup_{\omega \in \Omega}\sup_{\omega' \in A} \Pi(\omega'|\mathcal{A})(\omega)\,\mathbf{1}(\omega \sim \hat{\omega})\Pi(\omega) \\ &= \sup_{\omega \in \Omega}\sup_{\omega' \in A} \frac{\Pi(\omega')}{\Pi([\omega]_{\mathcal{A}})}\,\mathbf{1}(\omega' \sim \omega)\,\mathbf{1}(\omega \sim \hat{\omega})\Pi(\omega) \\ &= \sup_{\omega' \in A} \frac{\Pi(\omega')}{\Pi([\hat{\omega}]_{\mathcal{A}})}\,\mathbf{1}(\omega' \sim \hat{\omega})\Pi([\hat{\omega}]_{\mathcal{A}}) = \sup_{\omega' \in A}\Pi(\omega')\,\mathbf{1}(\omega' \sim \hat{\omega}) \\ &= \Pi(A \cap [\hat{\omega}]_{\mathcal{A}}). \end{aligned}$$

In general, if $B \in \mathcal{A}$, then $B = \bigcup_{\omega \in B}[\omega]_{\mathcal{A}}$, and this case reduces to the preceding one.

Part 4 of the lemma concerning ω is a consequence of the definition when $\Pi([\omega]_{\mathcal{A}}) = 0$. If $\Pi([\omega]_{\mathcal{A}}) > 0$, then by Remark 1.6.10

$$\Pi(A|\mathcal{A})(\omega) = \frac{\Pi(A \cap [\omega]_{\mathcal{A}})}{\Pi([\omega]_{\mathcal{A}})},$$

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which is a τ-smooth idempotent probability relative to $\mathcal{E}$ provided so is Π since $[\omega]_{\mathcal{A}} \in \mathcal{E}$ and $\mathcal{E}$ is a π-system. The proof for B is similar.

In part 5 we also can assume that $\Pi(\omega) > 0$. Since Π is $\mathcal{T}$-tight, given $\epsilon > 0$, there exists $T \in \mathcal{T}$ such that $\Pi(T^c) \leq \epsilon\Pi([\omega]_{\mathcal{A}})$, which implies that $\Pi(T^c|\mathcal{A})(\omega) \leq \epsilon$. The proof for B is similar. □

Remark 1.6.13. *According to the lemma* $\Pi(A|\mathcal{A})(\omega)$ *is an analogue of regular conditional probability (cf., e.g., Ikeda and Watanabe [66, definition 3.2]).*

Remark 1.6.14. *Let* Ω *and* Ω' *be* τ*-measurable sets with respective* τ*-algebras* $\mathcal{A}$ *and* $\mathcal{A}'$*, and let* Π *be an idempotent probability on* $\Omega\times\Omega'$*. Then* $k(\omega, A') = \Pi([\omega]_{\mathcal{A}} \times A'|[\omega]_{\mathcal{A}} \times \Omega')$ *is an idempotent transition kernel from* $(\Omega,\mathcal{A})$ *to* $(\Omega',\mathcal{A}')$*.*

We now define conditional idempotent expectations.

Definition 1.6.15. *If* f *is an* $\mathbb{R}_+$*-valued function on* Ω*, then the conditional idempotent expectation of* f *given* $\mathcal{A}$ *is defined as*

$$S(f|\mathcal{A})(\omega) = \sup_{\omega'\in\Omega} f(\omega')\Pi(\omega'|\mathcal{A})(\omega).$$

If $g : \Omega \to \hat{\Omega}$*, where* $\hat{\Omega}$ *is equipped with a* τ*-algebra* $\hat{\mathcal{A}}$*, we define* $S(f|g) = S\big(f|g^{-1}(\hat{\mathcal{A}})\big)$*. We also let*

$$S(f|g = \hat{\omega}) = \sup_{\omega'\in\Omega} f(\omega')\Pi(\omega'|g = \hat{\omega}).$$

Remark 1.6.16. *Note that* $S(f|\mathcal{A})(\omega) < \infty$ Π*-a.e. if* $Sf < \infty$ *and* $\Pi(A|\mathcal{A})(\omega) = S(\mathbf{1}(A)|\mathcal{A})(\omega)$ Π*-a.e.*

Remark 1.6.17. *We will often use the following form of the definition of conditional idempotent expectation:*

$$S(f|\mathcal{A})(\omega) = \frac{\sup_{\omega'\in\Omega} f(\omega')\,\mathbf{1}\big(\omega' \in [\omega]_{\mathcal{A}}\big)\Pi(\omega')}{\Pi\big([\omega]_{\mathcal{A}}\big)} \quad \Pi\text{-}a.e. \tag{1.6.1}$$

Remark 1.6.18. *If* $\mathcal{A} \subset \mathcal{A}'$ *and* f *is* $\mathcal{A}'$*-measurable, then*

$$S(f|\mathcal{A})(\omega) = \sup_{\omega'\in\Omega} f(\omega')\Pi([\omega']_{\mathcal{A}'}|\mathcal{A})(\omega)$$

so that $S(f|\mathcal{A})$ *depends only on the values of* Π *on* $\mathcal{A}'$*.*

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Remark 1.6.19. *In order to refer explicitly to the idempotent probability* Π, *we denote the conditional idempotent expectation as* $S_\Pi(f|\mathcal{A})$.

Since $\Pi(\omega'|\mathcal{A})(\omega)$ is specified Π–a.e. in ω, $S(f|\mathcal{A})(\omega)$ is also specified up to a set of zero idempotent probability. We call any such function a version of the conditional idempotent expectation.

The following result is a consequence of the definitions.

Lemma 1.6.20. *Let* $f : \Omega \to \mathbb{R}_+$ *and* $g : \Omega \to \Omega'$, *where* Ω' *is equipped with the discrete* τ*-algebra. Then* Π*-a.e.*

$$S(f|g)(\omega) = \sup_{\omega' \in \Omega'} S(f|g = \omega')\,\mathbf{1}(g(\omega) = \omega'),$$

in particular, Π*-a.e.*

$$\Pi(A|g)(\omega) = \sup_{\omega' \in \Omega'} \Pi(A|g = \omega')\,\mathbf{1}(g(\omega) = \omega'),\ A \subset \Omega.$$

Conditional idempotent expectations have properties similar to the properties of conditional expectations in probability theory. They are summarised in the next lemma. All the equalities and inequalities involving conditional idempotent expectations are understood to hold Π-a.e. Following a convention of probability theory, we routinely omit argument ω in conditional idempotent probabilities and idempotent expectations.

Lemma 1.6.21. *Let* f, f_j, *and* g *denote* $\mathbb{R}_+$*-valued functions on* Ω, *and* $\mathcal{A}$ *and* $\mathcal{B}$ *denote* τ*-algebras on* Ω.

1. $S(f|\mathcal{A})$ *is* $\mathcal{A}$*–measurable.*
2. *If* $f = g$ Π*-a.e., then* $S(f|\mathcal{A}) = S(g|\mathcal{A})$.
3. $S(0|\mathcal{A}) = 0$, $S(1|\mathcal{A}) = 1$.
4. $S(cf|\mathcal{A}) = cS(f|\mathcal{A})$, $c \in \mathbb{R}_+$.
5. $S\big(\sup_{j\in J} f_j|\mathcal{A}\big) = \sup_{j\in J} S(f_j|\mathcal{A})$.
6. $|S(f|\mathcal{A}) - S(g|\mathcal{A})| \le S(|f-g||\mathcal{A})$, *if* $Sf < \infty$ *and* $Sg < \infty$.

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7. $S\big(S(f|\mathcal{A})\big) = Sf$.

8 If f *is independent of* $\mathcal{A}$, *then* $S(f|\mathcal{A}) = Sf$.

9. Let $(\Omega', \mathcal{A}')$ *be a* τ*-measurable space,* $F : \Omega \times \Omega' \to \mathbb{R}_+$ *be such that the cross-sections* $F_\omega : \Omega' \to \mathbb{R}_+$ *are* $\mathcal{A}'$*-measurable for every* $\omega \in \Omega$, *and* $h : \Omega \to \Omega'$ *be* $\mathcal{A}/\mathcal{A}'$*-measurable. Then* $S\big(F(\cdot, h)|\mathcal{A}\big) = S\big(F(\cdot, x)|\mathcal{A}\big)|_{x=h}$. *In particular, if* $g : \Omega \to \mathbb{R}_+$ *is* $\mathcal{A}$*-measurable, then* $S(fg|\mathcal{A}) = gS(f|\mathcal{A})$ *and* $S(g|\mathcal{A}) = g$.

10. If $\mathcal{B} \subset \mathcal{A}$, *then* $S\big(S(f|\mathcal{A})|\mathcal{B}\big) = S(f|\mathcal{B})$.

11. If $\mathcal{A}$ *and* $\mathcal{B}$ *are independent, and* f *is independent of* $\mathcal{B}$, *then* $S(f|\tau(\mathcal{A}, \mathcal{B})) = S(f|\mathcal{A})$.

12. If f *is maximable, then the family* $\{S(f|\mathcal{B}), \mathcal{B} \subset \mathcal{A}\}$ *is uniformly maximable.*

13. If $0 < p \le q$, *then* $\big(S(f^p|\mathcal{A})\big)^{1/p} \le \big(S(f^q|\mathcal{A})\big)^{1/q}$.

14. If $p \ge 1$ *and* $q \ge 1$ *are such that* $1/p + 1/q = 1$, *then* $S(fg|\mathcal{A}) \le \big(S(f^p|\mathcal{A})\big)^{1/p}\big(S(f^q|\mathcal{A})\big)^{1/q}$.

15. $\Pi(f \ge a|\mathcal{A}) \le S(f|\mathcal{A})/a$, $a > 0$.

Proof. Properties 1–6 follow by definition.

Consider property 7. By the definition of conditional idempotent expectation, properties of idempotent integrals, and part 3 of Theorem 1.6.12,

$$S\big(S(f|\mathcal{A})\big) = S\big[\sup_{\omega' \in \Omega} f(\omega')\Pi(\omega'|\mathcal{A})(\omega)\big]$$
$$= \sup_{\omega' \in \Omega} f(\omega') S[\Pi(\omega'|\mathcal{A})(\omega)] = \sup_{\omega' \in \Omega} f(\omega')\Pi(\omega') = Sf.$$

Consider property 8. According to the definitions Π-a.e. in ω

$$S(f|\mathcal{A})(\omega) = \sup_{\omega' \in \Omega} f(\omega') \frac{\mathbf{1}(\omega' \sim \omega)}{\Pi([\omega]_\mathcal{A})}\,\Pi(\omega').$$

Since $\mathbf{1}(\omega' \sim \omega)/\Pi([\omega]_\mathcal{A})$ is $\mathcal{A}$–measurable in ω' and $f(\omega')$ is independent of $\mathcal{A}$,

$$\sup_{\omega' \in \Omega} f(\omega') \frac{\mathbf{1}(\omega' \sim \omega)}{\Pi([\omega]_\mathcal{A})}\Pi(\omega')$$
$$= \big[\sup_{\omega' \in \Omega} f(\omega')\Pi(\omega')\big]\big[\sup_{\omega' \in \Omega} \frac{\mathbf{1}(\omega' \sim \omega)}{\Pi([\omega]_\mathcal{A})}\Pi(\omega')\big] = Sf.$$

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Property 8 is proved.

We prove property 9. We have that Π-a.e.

$$S(F(\cdot,h)|\mathcal{A})(\omega) = \sup_{\omega'\in\Omega} F(\omega',h(\omega'))\Pi(\omega'|\mathcal{A})(\omega)$$
$$= \sup_{\omega'\in\Omega} F(\omega',h(\omega'))\frac{\Pi(\omega')}{\Pi([\omega]_{\mathcal{A}})}\mathbf{1}(\omega'\sim\omega).$$

By $\mathcal{A}/\mathcal{A}'$–measurability of h and $\mathcal{A}'$-measurability of $F_{\omega'}$ we have that $F(\omega',h(\omega')) = F(\omega',h(\omega))$ if $\omega'\sim\omega$, so we conclude that

$$S(F(\cdot,h)|\mathcal{A})(\omega) = \sup_{\omega'\in\Omega} F(\omega',h(\omega))\frac{\Pi(\omega')}{\Pi([\omega]_{\mathcal{A}})}\mathbf{1}(\omega'\sim\omega)$$
$$= S(F(\cdot,x)|\mathcal{A})(\omega)|_{x=h(\omega)}.$$

Property 9 is proved.

Proof of property 10. It is easy to derive from the definition of conditional idempotent expectation that for Π–almost all ω

$$S\big(S(f|\mathcal{A})|\mathcal{B}\big)(\omega) = \sup_{\omega''\in\Omega} f(\omega'')\sup_{\omega'\in\Omega}\Pi(\omega'|\mathcal{B})(\omega)\Pi(\omega''|\mathcal{A})(\omega').$$

Hence, it suffices to prove that for Π–almost all ω and all ω''

$$\sup_{\omega'\in\Omega}\Pi(\omega'|\mathcal{B})(\omega)\Pi(\omega''|\mathcal{A})(\omega') = \Pi(\omega''|\mathcal{B})(\omega). \tag{1.6.2}$$

If $\Pi(\omega'')=0$, then $\Pi(\omega''|\mathcal{B})(\omega)=0$ Π–a.e. in ω, so the right-hand side of (1.6.2) is equal to 0 Π-a.e. As for the left-hand side, if $\Pi(\omega'')=0$ and $\Pi(\omega''|\mathcal{A})(\omega')>0$, then $\Pi(\omega')=0$. The latter implies that if, in addition $\Pi(\omega'|\mathcal{B})(\omega)>0$, then $\Pi(\omega)=0$. Thus, we conclude that if $\Pi(\omega'')=0$ and $\Pi(\omega'|\mathcal{B})(\omega)\Pi(\omega''|\mathcal{A})(\omega')>0$, then $\Pi(\omega)=0$. Hence, the left-hand side of (1.6.2) also is Π–a.e. equal to 0 when $\Pi(\omega'')=0$. This ends the proof of (1.6.2) when $\Pi(\omega'')=0$.

Let $\Pi(\omega'')>0$. We can assume that $\Pi(\omega)>0$ so that $\Pi([\omega]_{\mathcal{A}})>0$ and $\Pi([\omega]_{\mathcal{B}})>0$. Then by the definition of conditional idempotent probability the right-hand side of (1.6.2) takes the form

$$\Pi(\omega''|\mathcal{B})(\omega) = \frac{\Pi(\omega'')}{\Pi([\omega]_{\mathcal{B}})}\mathbf{1}(\omega''\overset{\mathcal{B}}{\sim}\omega). \tag{1.6.3}$$

On the other hand, $\Pi(\omega'')>0$ implies that $\Pi([\omega'']_{\mathcal{A}})>0$, hence, $\Pi([\omega']_{\mathcal{A}})>0$ when $\omega'\overset{\mathcal{A}}{\sim}\omega''$, and by the definition of conditional

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idempotent probability we have for the left-hand side of (1.6.2)

$$\sup_{\omega'\in\Omega} \Pi(\omega'|\mathcal{B})(\omega)\Pi(\omega''|\mathcal{A})(\omega')$$
$$= \sup_{\omega':\,\omega'\overset{\mathcal{A}}{\sim}\omega''} \frac{\Pi(\omega')}{\Pi([\omega]_{\mathcal{B}})}\,\mathbf{1}(\omega' \overset{\mathcal{B}}{\sim} \omega)\frac{\Pi(\omega'')}{\Pi([\omega']_{\mathcal{A}})}.$$

The equivalence $\omega' \overset{\mathcal{A}}{\sim} \omega''$ implies that $[\omega']_{\mathcal{A}} = [\omega'']_{\mathcal{A}}$. It also implies, since $\mathcal{B} \subset \mathcal{A}$, that $\omega' \overset{\mathcal{B}}{\sim} \omega''$, and hence $\mathbf{1}(\omega' \overset{\mathcal{B}}{\sim} \omega) = \mathbf{1}(\omega'' \overset{\mathcal{B}}{\sim} \omega)$ if $\omega' \overset{\mathcal{A}}{\sim} \omega''$. Thus, on replacing in the latter supremum $[\omega']_{\mathcal{A}}$ with $[\omega'']_{\mathcal{A}}$ and $\mathbf{1}(\omega' \overset{\mathcal{B}}{\sim} \omega)$ with $\mathbf{1}(\omega'' \overset{\mathcal{B}}{\sim} \omega)$, we obtain that the supremum coincides with the right-hand side of (1.6.3). Equality (1.6.2) is proved. Property 10 is proved.

We prove property 11. Let $\mathcal{C} = \tau(\mathcal{A},\mathcal{B})$. Obviously, $[\omega]_{\mathcal{C}} = [\omega]_{\mathcal{A}} \cap [\omega]_{\mathcal{B}}$. Then by definition Π-a.e.

$$S(f|\mathcal{C})(\omega) = \sup_{\omega'} f(\omega')\frac{\mathbf{1}(\omega' \in [\omega]_{\mathcal{C}})}{\Pi([\omega]_{\mathcal{C}})}\Pi(\omega')$$
$$= \sup_{\omega'} f(\omega')\frac{\mathbf{1}(\omega' \in [\omega]_{\mathcal{A}} \cap [\omega]_{\mathcal{B}})}{\Pi([\omega]_{\mathcal{A}} \cap [\omega]_{\mathcal{B}})}\Pi(\omega')$$
$$= \left[\sup_{\omega'} f(\omega')\frac{\mathbf{1}(\omega' \in [\omega]_{\mathcal{A}})}{\Pi([\omega]_{\mathcal{A}})}\Pi(\omega')\right]\left[\frac{\mathbf{1}(\omega' \in [\omega]_{\mathcal{B}})}{\Pi([\omega]_{\mathcal{B}})}\Pi(\omega')\right]$$
$$= S(f|\mathcal{A})(\omega),$$

where the equality before the last one is obtained by using the fact that f and $\mathcal{A}$ are independent of $\mathcal{B}$.

We prove property 12. By property 1 $S(f|\mathcal{B})$ is $\mathcal{B}$–measurable, so, successively applying properties 9 and 7, we have for $a > 0$

$$S\big(S(f|\mathcal{B})\,\mathbf{1}\big(S(f|\mathcal{B}) > a\big)\big) = S\big(S(f\,\mathbf{1}\big(S(f|\mathcal{B}) > a\big))|\mathcal{B}\big)$$
$$= S\big(f\,\mathbf{1}(S(f|\mathcal{B}) > a)\big).$$

Now, by properties of idempotent expectations, for $b > 0$,

$$S\big(f\,\mathbf{1}(S(f|\mathcal{B}) > a)\big)$$
$$= S\big(f\,\mathbf{1}(f > b)\,\mathbf{1}(S(f|\mathcal{B}) > a)\big) \vee S\big(f\,\mathbf{1}(f \le b)\,\mathbf{1}(S(f|\mathcal{B}) > a)\big)$$
$$\le S\big(f\,\mathbf{1}(f > b)\big) \vee \big(b\,\Pi(S(f|\mathcal{B}) > a)\big)$$
$$\le S\big(f\,\mathbf{1}(f > b)\big) \vee \Big(\frac{b}{a}\,Sf\Big),$$

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where the last inequality is by "the Chebyshev inequality" and property 7. Hence,

$$\limsup_{a\to\infty}\sup_{\mathcal{B}\subset\mathcal{A}} S\big(S(f|\mathcal{B})\mathbf{1}(S(f|\mathcal{B})>a)\big)\le S\big(f\,\mathbf{1}(f>b)\big),$$

which goes to 0 as $b\to\infty$ by maximability of f.

Inequalities 13 and 14 follow from the definitions. Property 15, similarly to the probability theory counterpart, follows by the inequality $\mathbf{1}(f\ge a)\le f/a$. □

The following lemma, which contains facts on convergence of conditional idempotent expectations analogous to facts from probability theory, is a consequence of the definition of conditional idempotent expectation, Theorem 1.4.19, Theorem 1.6.12, and Lemma 1.6.21.

Lemma 1.6.22. *Let $\mathcal{A}$ be a τ-algebra on Ω. Let $\{f_\psi,\psi\in\Psi\}$ be a net of $\mathbb{R}_+$-valued functions and f be an $\mathbb{R}_+$-valued function on Ω.*

1. *If $\liminf_{\psi\in\Psi} f_\psi\ge f$ Π–a.e., then*

$$\liminf_{\psi\in\Psi} S(f_\psi|\mathcal{A})\ge S(f|\mathcal{A})\ \ \Pi\text{–a.e.}$$

2. *If $S|f_\psi-f|\to 0$, $\psi\in\Psi$, then*

$$S|S(f_\psi|\mathcal{A})-S(f|\mathcal{A})|\to 0.$$

3. *If $f_\psi\overset{\Pi}{\to}f$, $\psi\in\Psi$, and $\{f_\psi\}$ is uniformly maximable, then*

$$S(f_\psi|\mathcal{A})\overset{\Pi}{\to}S(f|\mathcal{A}).$$

4. *If $f_\psi\uparrow f$, $\psi\in\Psi$, Π–a.e., then*

$$\lim_{\psi\in\Psi} S(f_\psi|\mathcal{A})=S(f|\mathcal{A})\ \ \Pi\text{–a.e.}$$

5. *Let Π be an $\mathcal{E}$-idempotent probability, $\mathcal{T}$ be a tightening collection for Π, and the f_ψ be Luzin $(\mathcal{E},\mathcal{T})/\mathcal{U}$-measurable and maximable. Let the atoms of $\mathcal{A}$ belong to $\mathcal{E}$. If $f_\psi\downarrow f$ Π-a.e., then*

$$S(f_\psi|\mathcal{A})\downarrow S(f|\mathcal{A})\ \ \Pi\text{-a.e.}$$

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We now give versions of Lévy's upward and downward theorems.

Lemma 1.6.23. *Let $\mathcal{A}$ be a τ-algebra on Ω and $f : \Omega \to \mathbb{R}_+$. Let either one of the following conditions hold:*

1. *$\{\mathcal{A}_\psi, \psi \in \Psi\}$ is a decreasing net of τ-algebras and $\mathcal{A} = \bigcap_{\psi \in \Psi} \mathcal{A}_\psi$,*
2. *$\{\mathcal{A}_\psi, \psi \in \Psi\}$ is an increasing net of τ-algebras, $\mathcal{E}$ includes the atoms of the $\mathcal{A}_\psi$, $\mathcal{A} = \tau\Big(\bigcup_{\psi \in \Psi} \mathcal{A}_\psi\Big)$, Π is a $\mathcal{T}$-tight $\mathcal{E}$-idempotent probability, and f is Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable and maximable.*

Then

$$S(f|\mathcal{A}) = \lim_{\psi \in \Psi} S(f|\mathcal{A}_\psi) \quad \Pi\text{-}a.e.$$

Proof. We prove part 1. Note that the net $\{[\omega]_{\mathcal{A}_\psi}, \psi \in \Psi\}$ is increasing for every $\omega \in \Omega$ and $[\omega]_{\mathcal{A}} = \cup_{\psi \in \Psi} [\omega]_{\mathcal{A}_\psi}$. Equality (1.6.1) implies the claim by τ-maxitivity of Π and part 3 of Theorem 1.4.19.

The proof of part 2 is similar: we note that the net $\{[\omega]_{\mathcal{A}_\psi}, \psi \in \Psi\}$ is decreasing for every $\omega \in \Omega$ and $[\omega]_{\mathcal{A}} = \cap_{\psi \in \Psi} [\omega]_{\mathcal{A}_\psi}$, invoke (1.6.1), part 4 of Theorem 1.4.19, and the τ-smoothness property of Π. ☐

In the sequel we will need the following characterisation of conditional idempotent expectations, which is a straightforward consequence of (1.6.1).

Lemma 1.6.24. *Let $f : \Omega \to \mathbb{R}_+$, $g : \Omega \to \mathbb{R}_+$ and $\mathcal{A}$ be a τ-algebra on Ω. Then $S(f|\mathcal{A}) \le S(g|\mathcal{A})$ Π-a.e. if and only if $S(f\,\mathbf{1}(A)) \le S(g\,\mathbf{1}(A))$ for every $A \in \mathcal{A}$.*

The next implication of the lemma characterises conditional idempotent expectation in a manner similar to the definition of conditional expectation in probability theory. We say that functions f and g on (Ω, Π) are indistinguishable if $\Pi(f \ne g) = 0$.

Lemma 1.6.25. *Let $\mathcal{A}$ be a τ-algebra and $f : \Omega \to \mathbb{R}_+$. Then $S(f|\mathcal{A})$ is the only up to indistinguishability function $g : \Omega \to \mathbb{R}_+$, which is $\mathcal{A}$-measurable and satisfies the equality $S(fh) = S(gh)$ for all $\mathcal{A}$-measurable functions $h : \Omega \to \mathbb{R}_+$.*

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Remark 1.6.26. *If we defined conditional idempotent expectation by the property in the lemma, then Definition 1.6.9 would prove existence. In general, one cannot replace in Lemma 1.6.25 τ-algebras with σ-algebras. Let us consider the following example. Let $\Omega = [0,1]$, $\Pi(\omega) = 1$ for every $\omega \in [0,1]$, $\mathcal{B}\big([0,1]\big)$ be the Borel σ-algebra on $[0,1]$, and f be an $\mathbb{R}_+$-valued function on Ω that is not Borel measurable. Suppose there exists conditional idempotent expectation g of f given $\mathcal{B}\big([0,1]\big)$ satisfying the requirements of Lemma 1.6.25. Since, given $x \in [0,1]$, the function $\mathbf{1}(\omega = x)$ is Borel, we have $\sup_{\omega\in[0,1]} \mathbf{1}(\omega = x)f(\omega) = \sup_{\omega\in[0,1]} \mathbf{1}(\omega = x)g(\omega)$, so that $g(x) = f(x)$ for every $x \in [0,1]$, which contradicts the requirement that g be Borel measurable.*

We give "a transitivity law" for conditional idempotent expectations.

Lemma 1.6.27. *Let $f : \Omega \to \Omega'$, $g : \Omega' \to \mathbb{R}_+$ and $\mathcal{A}'$ be a τ-algebra on Ω'. Then*

$$S_\Pi(g\circ f|f^{-1}(\mathcal{A}'))(\omega) = S_{\Pi\circ f^{-1}}(g|\mathcal{A}')(f(\omega)) \qquad \Pi\text{-}a.e.$$

Proof. Let $h(\omega') = S_{\Pi\circ f^{-1}}(g|\mathcal{A}')(\omega')$, $\omega' \in \Omega'$. By Lemma 1.6.25 for every $\mathcal{A}'$-measurable function $v : \Omega' \to \mathbb{R}_+$ we have $S_{\Pi\circ f^{-1}}(gv) = S_{\Pi\circ f^{-1}}(hv)$. By Theorem 1.4.6 this implies the equality $S_\Pi(g\circ f\, v\circ f) = S_\Pi(h\circ f\, v\circ f)$. Since by Lemma 1.2.7 an arbitrary $f^{-1}(\mathcal{A}')$-measurable $\mathbb{R}_+$-valued function on Ω is of the form $v\circ f$ for a suitable $\mathcal{A}'$-measurable function v and $h\circ f$ is $f^{-1}(\mathcal{A}')$-measurable, we conclude by Lemma 1.6.25 again that $h\circ f = S_\Pi(g\circ f|f^{-1}(\mathcal{A}'))$ Π-a.e. □

The next lemma concerns evaluating conditional idempotent expectations for product idempotent probabilities.

Lemma 1.6.28. *Let $(\Omega, \mathcal{A}, \Pi)$ and $(\Omega', \mathcal{A}', \Pi')$ be idempotent probability spaces and $\Omega \times \Omega'$ be equipped with τ-algebra $\mathcal{A} \otimes \mathcal{A}'$ and idempotent probability $\Pi \times \Pi'$. Let $f : \Omega \times \Omega' \to \mathbb{R}_+$. Then $S_{\Pi\times\Pi'}(f|\mathcal{A} \otimes \mathcal{A}') = S_\Pi(\tilde{f}|\mathcal{A})$, where $\tilde{f}(\omega) = S_{\Pi'}(f_\omega(\omega')|\mathcal{A}')$, $\omega \in \Omega$. In particular, if $g : \Omega \to \mathbb{R}_+$ and $g' : \Omega' \to \mathbb{R}_+$, then $S_{\Pi\times\Pi'}(gg'|\mathcal{A} \otimes \mathcal{A}') = S_\Pi(g|\mathcal{A})S_{\Pi}(g'|\mathcal{A}')$.*

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Proof. The required follows since

$$S_{\Pi\times\Pi'}(f|\mathcal{A}\otimes\mathcal{A}') = \sup_{(\omega,\omega')\in\Omega\times\Omega'} f(\omega,\omega')\Pi\times\Pi'\big((\omega,\omega')|\mathcal{A}\otimes\mathcal{A}'\big)$$
$$= \sup_{\omega\in\Omega}\big(\sup_{\omega'\in\Omega'} f(\omega,\omega')\Pi'(\omega'|\mathcal{A}')\big)\Pi(\omega|\mathcal{A}).$$

□

In probability theory existence of conditional expectation is proved by means of the Radon-Nikodym theorem. We can do without an analogue of the latter; however, it comes in useful below, so we state and prove it.

Definition 1.6.29. *Let $\mathcal{A}$ be a τ-algebra on Ω. Let Π and Π' be idempotent probabilities on Ω. We say that Π' is absolutely continuous with respect to Π on $\mathcal{A}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\Pi(A) < \delta$ implies that $\Pi'(A) < \epsilon$ for all $A \in \mathcal{A}$.*

Remark 1.6.30. *Equivalently, we may require the above condition to hold only for the atoms of $\mathcal{A}$. Note also that our definition implies that if Π' is absolutely continuous with respect to Π, then $\Pi'(A) = 0$ whenever $A \in \mathcal{A}$ and $\Pi(A) = 0$; by contrast with the situation in measure theory, the converse is not true. We have chosen the "strong" version as a definition since it implies maximability of the Radon-Nikodym derivative (see Theorem 1.6.34 below).*

Remark 1.6.31. *If Π' and Π are restricted to $\mathcal{A}$, we simply say that Π' is absolutely continuous with respect to Π.*

Definition 1.6.32. *We say that a function $f : \Omega \to \mathbb{R}_+$ is a Radon-Nikodym derivative of an idempotent probability Π' with respect to an idempotent probability Π on a τ-algebra $\mathcal{A}$ if f is $\mathcal{A}$-measurable, Π-maximable and $\Pi'(A) = \bigvee_A f\,d\Pi$ for all $A \in \mathcal{A}$. We then denote $f = d\Pi/d\Pi'$. We also write $d\Pi = f\,d\Pi'$.*

Lemma 1.6.33. *If $f = d\Pi'/d\Pi$, then $f(\omega) = \Pi'([\omega]_{\mathcal{A}})/\Pi([\omega]_{\mathcal{A}})$ Π-a.e.*

Proof. Let $\Pi(\omega) > 0$. By $\mathcal{A}$-measurability of f it is constant on $[\omega]_{\mathcal{A}}$ so that

$$\Pi'([\omega]_{\mathcal{A}}) = \bigvee_{[\omega]_{\mathcal{A}}} f\,d\Pi = f(\omega)\Pi([\omega]_{\mathcal{A}}).$$

□

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Thus, a Radon-Nikodym derivative is unique Π-a.e.

Theorem 1.6.34. *An idempotent probability Π' is absolutely continuous with respect to an idempotent probability Π on $\mathcal{A}$ if and only if there exists a Radon-Nikodym derivative of Π with respect to Π' on $\mathcal{A}$.*

Proof. Existence of the derivative implies the absolute continuity by Corollary 1.4.12. For the converse, we define the derivative as in Lemma 1.6.33. Then f is $\mathcal{A}$-measurable and $\Pi'(A) = \bigvee_A f\, d\Pi$ for all $A \in \mathcal{A}$. To show f is Π-maximable, we write for $a > 0$

$$\bigvee_{\Omega} f\, \mathbf{1}(f > a)\, d\Pi = \Pi'(f > a) \le \Pi'(\omega : \Pi(\omega) \le 1/a).$$

Since Π' is absolutely continuous with respect to Π, the latter idempotent probability can be made less than arbitrary $\epsilon > 0$ by choosing a large enough. ☐

Lemma 1.6.35. *Let an idempotent probability Π' on Ω be absolutely continuous with respect to Π on a τ-algebra $\mathcal{A}$. Let $f = d\Pi'/d\Pi > 0$ Π-a.e. Then, for a function $g : \Omega \to \mathbb{R}_+$ and τ-algebra $\mathcal{B} \subset \mathcal{A}$, Π-a.e.*

$$S_{\Pi'}(g|\mathcal{B}) = \frac{S_\Pi(fg|\mathcal{B})}{S_\Pi(f|\mathcal{B})}.$$

Proof. Since $0 = S_\Pi\big(S_\Pi(f|\mathcal{B})\, \mathbf{1}(S_\Pi(f|\mathcal{B}) = 0)\big) = S_\Pi\big(f\, \mathbf{1}(S_\Pi(f|\mathcal{B}) = 0)\big)$ and $f > 0$ Π-a.e., it follows that $S_\Pi(f|\mathcal{B}) > 0$ Π-a.e., so the right-hand side in the above equality is well defined Π-a.e. We next have by properties of conditional idempotent expectations that for $B \in \mathcal{B}$

$$\begin{aligned} S_\Pi\big(S_{\Pi'}(g|\mathcal{B}) S_\Pi(f|\mathcal{B})\, \mathbf{1}(B)\big) &= S_\Pi\Big(S_\Pi\big(S_{\Pi'}(g|\mathcal{B}) f\, \mathbf{1}(B)|\mathcal{B}\big)\Big) \\ = S_\Pi\big(S_{\Pi'}(g|\mathcal{B}) f\, \mathbf{1}(B)\big) &= S_{\Pi'}\big(S_{\Pi'}(g|\mathcal{B})\, \mathbf{1}(B)\big) \\ &= S_{\Pi'}(g\, \mathbf{1}(B)) = S_\Pi(fg\, \mathbf{1}(B)). \end{aligned}$$

Thus, by Lemma 1.6.25 $S_{\Pi'}(g|\mathcal{B}) S_\Pi(f|\mathcal{B}) = S_\Pi(fg|\mathcal{B})$. ☐

We end the section by giving versions of Lemma 1.6.24 and Lemma 1.6.25 for τ-smooth tight idempotent probabilities.

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Lemma 1.6.36. *Let $\mathcal{E}$ be a semi-τ-algebra, Π be τ-smooth relative to $\mathcal{E}$ and $\mathcal{T}$ be a tightening collection for Π. Let $f : \Omega \to \mathbb{R}_+$ and $g : \Omega \to \mathbb{R}_+$ be Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable and maximable functions. Then the following holds.*

1. *$S(f|\tau(\mathcal{E})) \leq S(g|\tau(\mathcal{E}))$ Π-a.e. if and only if $S(f\,\mathbf{1}(A)) \leq S(g\,\mathbf{1}(A))$ for every $A \in \mathcal{E}$.*

2. *$S(f|\tau(\mathcal{E}))$ is the only up to indistinguishability Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable maximable function $f' : \Omega \to \mathbb{R}_+$ that is $\tau(\mathcal{E})$-measurable and satisfies the equality $S(fh) = S(f'h)$ for all $\mathcal{E}$-measurable functions $h : \Omega \to \mathbb{R}_+$.*

Proof. Necessity of the condition in part 1 is obvious. We prove sufficiency. Let $\mu(A) = S(f\,\mathbf{1}(A))$ and $\nu(A) = S(g\,\mathbf{1}(A))$. By Corollary 1.4.20 μ and ν are $\mathcal{E}$-idempotent measures on Ω such that $\mu \leq \nu$ on $\mathcal{E}$. Theorem 1.1.7 implies that $\mu \leq \nu$ on $\mathcal{E}_{iu}$. Since $\mathcal{E}$ is a semi-τ-algebra, $\mathcal{E}_{iu} = \tau(\mathcal{E})$, completing the proof by Lemma 1.6.24. Part 2 is a consequence of part 1. ☐

Lemma 1.6.37. *Let Π be τ-smooth relative to a collection $\mathcal{E}$ and has a tightening collection $\mathcal{T}$. Let $\mathcal{H}$ be a collection of Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable $\mathbb{R}_+$-valued functions on Ω that contains the zero function, is closed under multiplication by non-negative scalars and the formation of maximums and minimums, and is such that if $h \in \mathcal{H}$, then $(h-1)\vee 0 \in \mathcal{H}$ and $(1-h)\vee 0 \in \mathcal{H}$. Let $f : \Omega \to \mathbb{R}_+$ be maximable and Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable and let $\mathcal{A}$ denote the τ-algebra generated by the elements of $\mathcal{H}$. If a Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable maximable function $g : \Omega \to \mathbb{R}_+$ is $\mathcal{A}$-measurable and such that $S(fh) = S(gh)$ for all $h \in \mathcal{H}$, then $g = S(f|\mathcal{A})$ Π-a.e.*

Proof. Let $V(h) = S(fh)$, $h \in \mathcal{H}$. By Theorem 1.4.19 and the hypotheses the functional V satisfies condition (VC) of Theorem 1.4.22. Therefore, V has a unique extension to a non-negative homogeneous maxitive functional on the set of $\mathcal{A}$-measurable $\mathbb{R}_+$-valued functions on Ω (see Remark 1.4.24). The same fact is true for the functional $V'(h) = S(gh)$. Hence, $S(fh) = S(gh)$ for every $\mathcal{A}$-measurable $\mathbb{R}_+$-valued function $h : \Omega \to \mathbb{R}_+$ and the required follows by Lemma 1.6.25. ☐

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1.7 Idempotent measures on topological spaces

In the next three sections we consider τ-smooth idempotent measures on topological spaces. Let E be a Hausdorff topological space. It will play the part of the set Ω. The part of the collection $\mathcal{E}$ will be played either by the collection $\mathcal{F}$ of closed subsets of E or the collection $\mathcal{K}$ of compact subsets of E. We consider only finite idempotent measures throughout.

Definition 1.7.1. *We say that an idempotent measure μ on E is tight if it has $\mathcal{K}$ as a tightening collection, i.e., for every $\varepsilon > 0$ there exists compact $K \subset E$ such that $\mu(K^c) \leq \varepsilon$.*

Note that since E is Hausdorff, $\mathcal{F}$-idempotent measures are $\mathcal{K}$-idempotent measures; also the classes of tight $\mathcal{F}$-idempotent measures and tight $\mathcal{K}$-idempotent measures coincide. Therefore, we occasionally refer to tight $\mathcal{F}$-idempotent measures on Hausdorff spaces as tight τ-smooth idempotent measures.

We denote $\mu(z) = \mu(\{z\})$ so that symbol μ will alternatingly be used for an idempotent measure and its density. To avoid confusion we will denote the density by $\mu(z)$ and the idempotent measure by μ. We recall that according to Lemma 1.1.4

$$\mu(A) = \sup_{z \in A} \mu(z), \ A \subset E. \tag{1.7.1}$$

The next lemma relates properties of $\mu(z)$ to properties of μ. Recall that a function $f : E \to \mathbb{R}_+$ is upper semi-continuous if the sets $\{z \in E : \ f(z) \geq a\}$ are closed.

Definition 1.7.2. *A function $f : E \to \mathbb{R}_+$ is said to be upper compact if the sets $\{z \in E : \ f(z) \geq a\}$ are compact for $a > 0$.*

Remark 1.7.3. *Upper compact functions attain suprema on closed sets.*

Lemma 1.7.4. *Let a function $\mu(z) : E \to \mathbb{R}_+$ and set function $\mu : \mathcal{P}(E) \to \mathbb{R}_+$ be related by (1.7.1). Then the following holds.*

1. *If the set function μ is an $\mathcal{F}$–idempotent measure on E, then the function $\mu(z)$ is upper semi-continuous.*

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2. *If the function $\mu(z)$ is upper semi-continuous, then the set function μ is a $\mathcal{K}$–idempotent measure. If E is either first countable or locally compact, then the converse is also true.*

3. *The set function μ is a tight $\mathcal{F}$–idempotent measure on E if and only if the function $\mu(z)$ is upper compact.*

Proof. Let μ be an $\mathcal{F}$–idempotent measure and let $z_\phi \to z, \phi \in \Phi$. Then applying the τ-smoothness property of μ to the sets $F_\phi = \text{cl}\{z_{\phi'}, \phi' \geq \phi\}$ and noting that $\cap_\phi F_\phi = \{z\}$, we conclude that $\limsup_\phi \mu(z_\phi) \leq \mu(z)$, i.e., μ is upper semi-continuous.

The first assertion in part 2 follows from Lemma 1.7.6 below with $f_\phi(z) = \mu(z)\,\mathbf{1}(z \in K_\phi)$, where $\{K_\phi\}$ is a decreasing net of compacts. The second assertion in the case E is first countable is proved by the argument of the proof of part 1 since we can assume that $\{z_\phi\}$ is countable so that the F_ϕ are compact. If E is locally compact, then, given $z_\phi \to z$ and $\epsilon > 0$, by τ-smoothness there exists a compact K such that $z \in K$, $z_\phi \in K$ for "large" ϕ and $\mu(K) \leq \mu(z) + \epsilon$.

If μ is a tight $\mathcal{F}$-idempotent measure, then $\mu(z)$ is upper semi-continuous by part 1; besides, if K is a compact such that $\mu(K^c) < a$, then $\{z : \mu(z) \geq a\} \subset K$, so $\mu(z)$ is upper compact. If $\mu(z)$ is upper compact, then μ is a $\mathcal{K}$-idempotent measure by part 2. It is tight since, given $\epsilon > 0$, one can take $K = \{z : \mu(z) \geq \varepsilon\}$. ☐

In the sequel, we denote

$$K_\mu(a) = \{z \in E : \mu(z) \geq a\},\ a > 0. \tag{1.7.2}$$

Remark 1.7.5. *According to the lemma, if μ is a tight $\mathcal{F}$-idempotent measure, then the sets $K_\mu(a)$, $a > 0$, make up a tightening collection for μ.*

We give the lemma used in the above proof.

Lemma 1.7.6. *Let f_ϕ be a net of $\mathbb{R}_+$-valued upper compact functions on E monotonically decreasing and converging pointwise to function f. Then*

$$\sup_{z \in E} f_\phi(z) \downarrow \sup_{z \in E} f(z).$$

Proof. For $\varepsilon > 0$, let $B_\phi = \{z \in E : f_\phi(z) \geq \sup_{z' \in E} f(z') + \varepsilon\}$. The sets B_ϕ are compact, decreasing and $\cap_\phi B_\phi = \emptyset$. Hence, $B_{\phi_0} = \emptyset$ for some ϕ_0. ☐

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The following useful fact is in the same theme.

Theorem 1.7.7. *Let μ be an $\mathcal{F}$-idempotent measure on E. Let $\{f_j, j \in J\}$ be a collection of $\mathbb{R}_+$-valued bounded upper semi-continuous functions closed under the formation of minimums. Then*

$$\inf_{j\in J} \bigvee_E f_j \, d\mu = \bigvee_E \inf_{j\in J} f_j \, d\mu.$$

Proof. The claim follows by the last assertion of Theorem 1.4.19 if we observe that $\mathcal{T} = \{E\}$ is a tightening collection for μ and upper semi-continuous functions are Luzin $(\mathcal{F}, \mathcal{T})/\mathcal{U}$-measurable. □

The next result is an analogue of Ulam's theorem, see, e.g., Billingsley [11].

Theorem 1.7.8. *Let E be either homeomorphic to a complete metric space or locally compact. If μ is an $\mathcal{F}$-idempotent measure on E, then μ is tight.*

Proof. Let E be metrised by a complete metric. Given $\delta > 0$, let $\mathcal{O}_\delta$ denote the collection of finite unions of open δ-balls in E ordered by inclusion. It follows by τ-smoothness of μ relative to $\mathcal{F}$ that $\lim_{O\in\mathcal{O}_\delta} \mu(E \setminus O) = 0$. Therefore, for arbitrary $\epsilon > 0$ and $n \in \mathbb{N}$ there exist open $1/n$-balls $A_{n,1}, \dots, A_{n,k_n}$ such that $\mu\big(E \setminus \cup_{i=1}^{k_n} A_{n,i}\big) < \epsilon$. The set $A = \cap_{n=1}^{\infty} \cup_{i=1}^{k_n} A_{n,i}$ is totally bounded so that, since E is complete, the set $K = \operatorname{cl} A$ is compact. Also $\mu(E\setminus K) \le \sup_{n\in\mathbb{N}} \mu\big(E\setminus \cup_{i=1}^{k_n} A_{n,i}\big) \le \epsilon$.

If E is locally compact, then the τ-smoothness property of μ implies that for every $\epsilon > 0$ there exist open sets $A_1, \dots, A_k$ with compact closures such that $\mu(E \setminus \cup_{i=1}^{k} A_i) < \epsilon$. □

Let E' be a Hausdorff topological space and $\mathcal{F}'$ denote the collection of closed subsets of E'. We introduce a class of maps $f : E \to E'$ that preserve the property of an idempotent measure being tight and τ-smooth relative to the collection of closed sets.

Definition 1.7.9. *Let μ be a tight $\mathcal{F}$-idempotent measure on E. A function $f : E \to E'$ is said to be μ-Luzin measurable (or simply Luzin measurable if μ is understood) if it is continuous when restricted to the compacts $K_\mu(a), a > 0$.*

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Remark 1.7.10. *The definition adapts the abstract notion of Luzin* $(\mathcal{E}, \mathcal{T})/\mathcal{E}'$*-measurable functions in that* f *is* μ*-Luzin measurable if and only if it is* $(\mathcal{F}, \mathcal{K}_\mu)/\mathcal{F}'$*-measurable, where* $\mathcal{K}_\mu = \{K_\mu(a),\, a > 0\}$.

The equality $f\big(K_\mu(a)\big) = K_{\mu\circ f^{-1}}(a)$, which is valid for arbitrary $f : E \to E'$, and Theorem 1.2.14 yield the following result.

Theorem 1.7.11. *If* μ *is a tight* $\mathcal{F}$*-idempotent measure on* E *and* f *is a* μ*-Luzin measurable mapping from* E *to* E'*, then* $\mu' = \mu \circ f^{-1}$ *is a tight* $\mathcal{F}'$*-idempotent measure on* E'.

An application to f being an embedding provides the following extension result.

Corollary 1.7.12. *Let* $E \subset E'$ *and the topology on* E *be finer than the topology induced by the topology on* E'*. Let* μ *be a tight* $\mathcal{F}$*-idempotent measure on* E*. Then the set function* μ' *on* E' *defined by* $\mu'(A') = \mu(A' \cap E)$ *is a tight* $\mathcal{F}'$*-idempotent measure on* E'.

In view of an important role played by tight $\mathcal{F}$-idempotent probabilities in large deviation theory, we give them a special name.

Definition 1.7.13. *A tight* $\mathcal{F}$*-idempotent probability is called a deviability. If the idempotent distribution of an idempotent variable is a deviability, we also refer to it as a deviability distribution.*

Remark 1.7.14. *By Theorem 1.7.8 on complete metric spaces and on locally compact spaces* $\mathcal{F}$*-idempotent probabilities are deviabilities.*

The following is a corollary of Lemma 1.7.4 that will be used frequently below.

Corollary 1.7.15. *An idempotent measure* Π *is a deviability if and only if the density* $\Pi(z)$ *is an upper compact function and* $\sup_{z\in E} \Pi(z) = 1$.

Remark 1.7.16. *Since upper compact functions attain suprema on closed sets, for every deviability* Π *there exists* $z \in E$ *such that* $\Pi(z) = 1$.

Remark 1.7.17. *We note that* Π *is a deviability if and only if the function* $I(z) = -\ln \Pi(z)$ *is a tight probability rate function in the sense that the sets* $\{z \in E : I(z) \le a\}$ *are compact for all* $a \in \mathbb{R}_+$ *and* $\inf_{z\in E} I(z) = 0$.

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Definition 1.7.18. *1. Let (Ω, Π) be an idempotent probability space. An idempotent variable $f : \Omega \to E$ is called a Luzin idempotent variable on (Ω, Π) if its idempotent distribution $\Pi \circ f^{-1}$ is a deviability on E.*

2. Let, in addition, Ω be a Hausdorff topological space and Π be a deviability. An idempotent variable $f : \Omega \to E$ is called a strictly Luzin variable on (Ω, Π) if it is Π-Luzin measurable.

Remark 1.7.19. *By Theorem 1.7.11 strictly Luzin idempotent variables are Luzin idempotent variables.*

Strictly Luzin idempotent variables have another useful property. It is a topological version of Corollary 1.4.20.

Lemma 1.7.20. *Let Ω be a Hausdorff topological space and Π be a deviability on Ω. Let f be a maximable $\mathbb{R}_+$-valued strictly Luzin idempotent variable on (Ω, Π) such that $S_\Pi f = 1$. Then the set function $\Pi'(A) = S_\Pi f\, \mathbf{1}(A)$, $A \subset E$, is a deviability on E.*

We now consider topological versions of Theorem 1.4.22. They are analogues of Riesz' representation theorem. Let $C^+_{\mathcal{K}}(E)$ denote the set of $\mathbb{R}_+$-valued continuous functions on E with compact support.

Theorem 1.7.21. *Let E be a locally compact Hausdorff topological space and $V : C^+_{\mathcal{K}}(E) \to \mathbb{R}_+$ be a functional with properties $(V1)$ and $(V2)$ from Theorem 1.4.22, i.e.,*

$(V1)$ $V(cf) = cV(f),\ c \in \mathbb{R}_+,$

$(V2)$ $V(f \vee g) = V(f) \vee V(g).$

Then there exists a $\mathcal{K}$–idempotent measure μ on E such that

$$V(f) = \bigvee_E f\, d\mu,\ f \in C^+_{\mathcal{K}}(E).$$

The idempotent measure μ is uniquely specified on $\mathcal{P}(E)$.

Proof. By Theorem 1.4.22 in order to prove existence of μ we need to check that if $\{f_\varphi, \varphi \in \Phi\}$ and $\{g_\psi, \psi \in \Psi\}$ are, respectively, increasing and decreasing nets of elements of $C^+_{\mathcal{K}}(E)$ such that

$$\sup_{\varphi \in \Phi} f_\varphi(z) \geq \inf_{\psi \in \Psi} g_\psi(z),\ z \in E,$$

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then

$$\sup_{\varphi \in \Phi} V(f_\varphi) \geq \inf_{\psi \in \Psi} V(g_\psi). \tag{1.7.3}$$

Replacing if necessary the net $\{f_\varphi\}$ by the net $\{f_\varphi \wedge g_{\hat{\psi}}\}$, where $\hat{\psi} \in \Psi$ is picked arbitrarily, we can assume that all the above functions are supported by a compact K. Let $h_K \in C_{\mathcal{K}}^+(E)$ be such that $h_K = 1$ on K. Given $\varepsilon > 0$, the net $\{(g_\psi/(f_\varphi \vee (\varepsilon h_K)) - 1)^+\}$ indexed by (Φ, Ψ) monotonically converges to 0 on K. By Dini's theorem the convergence is uniform, so there exist φ_0 and ψ_0 such that

$$g_\psi(z) \leq (1+\varepsilon)(f_\varphi(z) \vee (\varepsilon h_K(z))), \ z \in E, \psi \geq \psi_0, \varphi \geq \varphi_0.$$

Therefore, by the properties of V

$$V(g_\psi) \leq (1+\varepsilon)(V(f_\varphi) \vee (\varepsilon V(h_K))), \ \psi \geq \psi_0, \varphi \geq \varphi_0,$$

which implies (1.7.3) since $\varepsilon > 0$ is arbitrary.

Since $\mathcal{K}_{iu} = \mathcal{P}(E)$, by Theorem 1.4.22 μ is uniquely specified on $\mathcal{P}(E)$. □

Corollary 1.7.22. *Let μ and μ' be $\mathcal{K}$-idempotent measures on a locally compact Hausdorff topological space E. If*

$$\bigvee_E f \, d\mu = \bigvee_E f \, d\mu', \ f \in C_{\mathcal{K}}^+(E),$$

then $\mu = \mu'$ on $\mathcal{P}(E)$.

The following version for compact spaces has an analogous proof.

Theorem 1.7.23. *Let E be a compact Hausdorff topological space and $\mathcal{H}$ be a set of $\mathbb{R}_+$-valued continuous functions on E that contains the zero function, is closed under the multiplication by non-negative scalars and formation of maximums and minimums, and is such that if $f \in \mathcal{H}$, then $(f-1) \vee 0 \in \mathcal{H}$. If $V : \mathcal{H} \to \mathbb{R}_+$ is a functional with properties $(V1)$ and $(V2)$, then there exists a $(\mathcal{K}_{\mathcal{H}})_i$-idempotent measure μ on E such that*

$$V(f) = \bigvee_E f \, d\mu, \ f \in \mathcal{H},$$

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where $\mathcal{K}_{\mathcal{H}}$ is the collection of compacts $\{z \in E : f(z) \geq a\}, a \in \mathbb{R}_+, f \in \mathcal{H}$. The idempotent measure μ is uniquely specified on $(\mathcal{K}_{\mathcal{H}})_{iu}$.

If, in addition, $\mathcal{H}$ contains constants and

$(V0)$ $V(1) = 1$,

then μ is a $(\mathcal{K}_{\mathcal{H}})_i$-idempotent probability.

Remark 1.7.24. *Theorem 1.7.21 can be derived from Theorem 1.7.23 if one recalls that a locally compact Hausdorff space is homeomorphic to an open subset of a compact Hausdorff space.*

For an $\mathbb{R}_+$-valued function f on E, let $\|f\| = \sup_{z \in E} f(z)$. Let $C_b^+(E)$ denote the set of $\mathbb{R}_+$-valued bounded continuous functions on E. We recall that Tihonov spaces are completely regular T_1-spaces, see, e.g., Kelley [71].

Theorem 1.7.25. *Let E be a Tihonov topological space and $V : C_b^+(E) \to \mathbb{R}_+$ be a functional with properties $(V1)$ and $(V2)$, which is, in addition, tight in the sense that for arbitrary $\varepsilon > 0$ there exists a compact $K \subset E$ such that $V(f) \leq \varepsilon \|f\|$ for every $f \in C_b^+(E)$ that equals 0 on K.*

Then there exists a tight $\mathcal{F}$–idempotent measure μ on E such that

$$V(f) = \bigvee_E f\, d\mu, \; f \in C_b^+(E).$$

The idempotent measure μ is specified uniquely on $\mathcal{P}(E)$.
If, in addition, condition $(V0)$ holds, then μ is a tight $\mathcal{F}$-idempotent probability.

Proof. Let $\overline{E}$ be the Stone-Čzech compactification of E, see, e.g., Engelking [47]. We define a functional $\overline{V}$ on $C_b^+(\overline{E})$ by $\overline{V}(\overline{f}) = V(f)$, where f denotes the restriction of $\overline{f} \in C_b^+(\overline{E})$ to E. It is obvious that $\overline{V}$ satisfies the conditions of Theorem 1.7.21. By Theorem 1.7.21 there exists a $\overline{\mathcal{K}}$–idempotent measure $\overline{\mu}$ on $\overline{E}$, where $\overline{\mathcal{K}}$ is the collection of compact subsets of $\overline{E}$, such that

$$\overline{V}(\overline{f}) = \bigvee_{\overline{E}} \overline{f}\, d\overline{\mu}, \; \overline{f} \in C_b^+(\overline{E}). \tag{1.7.4}$$

We show that $\overline{\mu}$ is $\mathcal{K}$-tight, i.e., for every $\varepsilon > 0$ there exists a compact $K \subset E$ such that $\overline{\mu}(\overline{E} \setminus K) \leq \varepsilon$. Let K be as in the hypotheses.

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The set $\overline{E} \setminus K$ is open in $\overline{E}$ so, since $\overline{E}$ is Tihonov, $\mathbf{1}(\overline{E} \setminus K) = \sup \overline{f}$ over $\overline{f} \in C_b^+(\overline{E})$ such that $\overline{f} \le \mathbf{1}(\overline{E} \setminus K)$. Therefore, since all these $\overline{f}$ are equal to 0 on K and $\overline{\mu}$ is an idempotent measure, $\overline{\mu}(\overline{E} \setminus K) = \sup_{\overline{f}} \overline{V}(\overline{f}) = \sup_f V(f) \le \varepsilon$. Since the embedding $E \to \overline{E}$ is continuous, the restriction of $\overline{\mu}$ to E, defined by $\mu(A) = \overline{\mu}(A)$ for $A \subset E$, is a $\mathcal{K}$-idempotent measure. It is tight since $\overline{\mu}$ is $\mathcal{K}$-tight. Moreover, $\mathcal{K}$-tightness of $\overline{\mu}$ implies that $\overline{\mu}(\overline{E} \setminus E) = 0$, so $\bigvee_{\overline{E}} \overline{f}\, d\overline{\mu} = \bigvee_E f\, d\mu$. Thus, by (1.7.4) we have for $f \in C_b^+(E)$, denoting by $\overline{f}$ the continuous extension of f to $\overline{E}$, that

$$V(f) = \overline{V}(\overline{f}) = \bigvee_{\overline{E}} \overline{f}\, d\overline{\mu} = \bigvee_E f\, d\mu.$$

□

Remark 1.7.26. *According to the theorem and part* $(JS5)$ *of Theorem 1.4.4, the functional* $V : C_b^+(E) \to \mathbb{R}_+$ *admits a continuous extension to a functional on the space of bounded* $\mathbb{R}_+$*-valued functions on* E *with sup-norm.*

Theorem 1.7.27. *Let* μ *and* μ' *be* $\mathcal{F}$*-idempotent measures on a Tihonov topological space* E*. If*

$$\bigvee_E f\, d\mu = \bigvee_E f\, d\mu', \ f \in C_b^+(E),$$

then $\mu = \mu'$ *on* $\mathcal{P}(E)$*.*

Proof. Given $z \in E$, we have, since E is Tihonov, that $\mathbf{1}(\{z\}) = \inf\{f \in C_b^+(E) : f(z) = 1\}$. Therefore, by Theorem 1.7.7

$$\mu(z) = \inf_{\substack{f \in C_b^+(E):\\ f(z)=1}} \bigvee_E f\, d\mu, \qquad \mu'(z) = \inf_{\substack{f \in C_b^+(E):\\ f(z)=1}} \bigvee_E f\, d\mu'.$$

□

1.8 Idempotent measures on projective limits

Our purpose here is to prove analogues of extension theorems for projective systems in measure theory. We are only able to get nice results for projective systems of tight τ-smooth idempotent measures.

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We actually formulate the results for idempotent probabilities, which are our main concern in this book.

Let $(E_\psi)_{\psi\in\Psi}$ be a net of Hausdorff topological spaces indexed by a directed set Ψ. We assume that for all $\psi \le \phi$, $\psi \in \Psi$, $\phi \in \Psi$, there are maps $\pi_{\psi\phi} : E_\phi \to E_\psi$ such that $\pi_{\psi\phi} = \pi_{\psi\chi} \circ \pi_{\chi\phi}$ for $\psi \le \chi \le \phi$. We denote by $\mathcal{F}_\psi$ the collections of closed subsets of the E_ψ. Let the E_ψ be equipped with deviabilities Π_ψ (i.e., tight $\mathcal{F}_\psi$-idempotent probabilities). For $\varepsilon > 0$, we denote $K_{\varepsilon,\psi} = \{z_\psi \in E_\psi : \Pi_\psi(z_\psi) \ge \varepsilon\}$. We assume that the maps $\pi_{\psi\phi}$ are Luzin measurable, i.e., their restrictions to the $K_{\varepsilon,\phi}$ are continuous. The deviabilities Π_ψ are assumed to form a projective system in that $\Pi_\psi = \Pi_\phi \circ \pi_{\psi\phi}^{-1}$ for $\psi \le \phi$. We note that this implies that $\pi_{\psi\phi} K_{\epsilon,\phi} = K_{\epsilon,\psi}$. Let E be a Hausdorff topological space and maps $\pi_\psi : E \to E_\psi$ be such that $\pi_\psi = \pi_{\psi\phi} \circ \pi_\phi$ for $\psi \le \phi$. Let $\mathcal{F}$ denote the collection of closed subsets of E.

Theorem 1.8.1. *Let the maps $\pi_\psi, \psi \in \Psi$, separate points in E. Let for every $\varepsilon > 0$ there exist a compact $K_\varepsilon \subset E$ such that the restrictions of the maps π_ψ to K_ε are continuous and*

$$(\epsilon, K) \qquad \Pi_\psi\big((\pi_\psi K_\varepsilon)^c\big) \le \varepsilon.$$

Then there exists a deviability Π on E such that $\Pi_\psi = \Pi \circ \pi_\psi^{-1}$. Deviability Π is uniquely specified by $\Pi(z) = \inf_{\psi\in\Psi} \Pi_\psi(\pi_\psi z)$.

Remark 1.8.2. *Condition (ϵ, K), being an analogue of the (ε, K)-condition for Radon measures (Schwartz [118]) is referred to below as such.*

We first consider the following special case.

Lemma 1.8.3. *Let E be the projective limit of the system (E_ψ) and the π_ψ be the canonical projections from E to E_ψ. Then the assertion of Theorem 1.8.1 holds.*

Proof. We define

$$\Pi(z) = \inf_{\psi\in\Psi} \Pi_\psi(\pi_\psi z), \; z \in E, \tag{1.8.1}$$

and $\Pi(A) = \sup_{z\in A} \Pi(z), \; A \subset E$. Let $K_\varepsilon = \{z \in E : \Pi(z) \ge \varepsilon\}$. Then

$$K_\epsilon = \bigcap_\psi \pi_\psi^{-1} K_{\epsilon,\psi}, \tag{1.8.2}$$

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so K_ϵ is the projective limit of the $(K_{\epsilon,\psi},\ \psi \in \Psi)$. Therefore, K_ϵ is compact for $\epsilon \in (0,1]$. It is also empty for $\epsilon > 1$. Thus, Π is a deviability on E by Corollary 1.7.15. Since the "bonding" maps $\pi_{\psi\phi} : K_{\epsilon,\phi} \to K_{\epsilon,\psi}$ are onto and continuous, the maps $\pi_\psi : K_\epsilon \to K_{\epsilon,\psi}$ are also onto, see Engelking [47, Corollary 3.2.15]. Therefore,

$$K_{\epsilon,\psi} = \pi_\psi K_\epsilon, \tag{1.8.3}$$

which is equivalent to the equality

$$\Pi_\psi(z_\psi) = \sup_{z \in \pi_\psi^{-1} z_\psi} \Pi(z), \tag{1.8.4}$$

which proves that $\Pi_\psi = \Pi \circ \pi_\psi^{-1}$.

Conversely, if Π is a deviability on E such that $\Pi_\psi = \Pi \circ \pi_\psi^{-1}$, $\psi \in \Psi$, then (1.8.4) holds, which is equivalent to (1.8.3), which implies, since K_ϵ is compact, that K_ϵ is the projective limit of the $(K_{\epsilon,\psi},\ \psi \in \Psi)$ so that (1.8.2) holds, Engelking [47, Proposition 2.5.6]. The latter is equivalent to (1.8.1). □

Remark 1.8.4. *Note that the* (ϵ, K)*-condition is satisfied in this setting in view of* (1.8.3). *We call* Π *the projective limit of the* Π_ψ.

For a proof of the general case we need a lemma.

Lemma 1.8.5. *Let* E *and* E' *be Hausdorff topological spaces. Let* E' *be endowed with an* $\mathcal{F}'$*-idempotent probability* Π'*, where* $\mathcal{F}'$ *is the collection of closed subsets of* E'. *Let an injective mapping* $h : E \to E'$ *and a collection* $\hat{\mathcal{K}}$ *of compact subsets of* E *be such that the restrictions of* h *to the elements of* $\hat{\mathcal{K}}$ *are continuous and* $h(\hat{\mathcal{K}})$ *is a tightening collection for* Π'. *Then there exists a unique deviability* Π *on* E *such that* $\Pi' = \Pi \circ h^{-1}$. *It is specified by the equality* $\Pi(A) = \Pi'(h(A))$, $A \subset E$.

Proof. We define Π as in the statement of the lemma. In order to check that Π is τ-smooth relative to $\mathcal{F}$, let us consider a decreasing net $\{F_\phi, \phi \in \Phi\}$ of closed subsets of E. For $\epsilon > 0$, we choose a compact $K \in \hat{\mathcal{K}}$ such that $\Pi'(E' \setminus h(K)) < \epsilon$. Then by the fact that h is injective

$$\Pi(F_\phi) = \Pi'(h(F_\phi)) \le \Pi'\big(h(F_\phi) \cap h(K)\big) + \epsilon = \Pi'\big(h(F_\phi \cap K)\big) + \epsilon.$$

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Since h is continuous when restricted to K and Π' is τ-smooth relative to $\mathcal{F}'$, we have that

$$\inf_{\phi\in\Phi} \Pi'\big(h(F_\phi \cap K)\big) = \Pi'\Big(\bigcap_{\phi\in\Phi} h(F_\phi \cap K)\Big) \le \Pi'\Big(h\Big(\bigcap_{\phi\in\Phi} F_\phi\Big)\Big) = \Pi\Big(\bigcap_{\phi\in\Phi} F_\phi\Big).$$

Thus,

$$\inf_{\phi\in\Phi} \Pi(F_\phi) \le \Pi\Big(\bigcap_{\phi\in\Phi} F_\phi\Big) + \epsilon,$$

implying that Π is τ-smooth relative to $\mathcal{F}$. Finally, Π is tight since by h being injective $\Pi(K^c) = \Pi'(h(K)^c)$. □

Proof of Theorem 1.8.1. Let E' be the projective limit of the (E_ψ), let $K'_\varepsilon, \varepsilon > 0$, be the respective projective limits of the $(K_{\varepsilon,\psi})$, and let $\pi'_\psi : E' \to E_\psi$ be the canonical projections. By the part of the theorem already proved there exists a unique deviability Π' on E' such that $\Pi_\psi = \Pi' \circ {\pi'}_\psi^{-1}$. On the other hand, it is easy to see that there exists a map $h : E \to E'$, which is continuous when restricted to the sets K_ε, and is such that $\pi'_\psi \circ h = \pi_\psi$. By the (ε, K)-condition $\pi_\psi K_\epsilon \supset K_{\epsilon,\psi}$, which implies that $h(K_\varepsilon) \supset K'_\varepsilon$. Therefore, $\{h(K_\varepsilon)\}$ is a tightening collection for Π'. Since the family π_ψ separates points in E, h is injective. Thus, by Lemma 1.8.5 there exists a unique deviability Π on E such that $\Pi' = \Pi \circ h^{-1}$, which implies that $\Pi_\psi = \Pi \circ \pi_\psi^{-1}$. Also

$$\Pi(z) = \Pi'(h(z)) = \inf_{\psi\in\Psi} \Pi_\psi\big(\pi'_\psi \circ h(z)\big) = \inf_{\psi\in\Psi} \Pi_\psi(\pi_\psi z),\ z \in E.$$

□

We now consider an application to product spaces. Let $\{E_j, j \in J\}$ be a family of Hausdorff topological spaces with collections $\mathcal{F}_j$ of closed sets. Let Ψ be the set of finite subsets of elements of J. For $\psi \in \Psi$ let E_ψ denote the Cartesian product $\prod_{j\in\psi} E_j$ with product topology; E_ψ is endowed with the collection $\mathcal{F}_\psi$ of closed sets. Let the sets E_ψ be equipped with $\mathcal{F}_\psi$-idempotent probabilities Π_ψ. As above we denote $K_{\varepsilon,\psi} = \{z_\psi \in E_\psi : \Pi_\psi(z_\psi) \ge \varepsilon\}$. For $\psi \subset \phi$, let

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$\pi_{\psi\phi}$ denote the canonical projection $E_\phi \to E_\psi$. Let $E \subset \prod_{j\in J} E_j$ be equipped with a Hausdorff topology, which is finer than the relative product topology, and π^E_ψ denote the restriction to E of the canonical projection $\prod_{j\in J} E_j \to E_\psi$. The next result follows by Theorem 1.8.1.

Theorem 1.8.6. *Let the idempotent probabilities Π_ψ form a projective system, i.e., $\Pi_\psi = \Pi_\phi \circ \pi^{-1}_{\psi\phi}$ if $\psi \subset \phi$. If for every $\epsilon > 0$ there exists a compact subset K_ϵ of E such that $\Pi_\psi\big((\pi^E_\psi K_\epsilon)^c\big) \le \epsilon$ for all $\psi \in \Psi$, then there exists a tight $\mathcal{F}$-idempotent probability Π on E such that $\Pi_\psi = \Pi \circ \pi^{E^{-1}}_\psi, \psi \in \Psi$. It is uniquely specified by $\Pi(z) = \inf_{\psi\in\Psi} \Pi_\psi(\pi^E_\psi z)$. In particular, Π exists if E is equipped with product topology and the idempotent probabilities $\Pi_j, j \in J$, are tight.*

Proof. Only the last claim requires proof. We check that the Π_ψ are tight if the $\Pi_j, j \in J$, are tight. For $\varepsilon > 0$, let $\hat{K}_{\varepsilon,\psi} = \prod_{j\in\Psi} K_{\varepsilon,j}$, where $\Pi_j(K^c_{\varepsilon,j}) \le \varepsilon$. Then $\hat{K}_{\varepsilon,\psi}$ is compact and

$$\Pi_\psi\big(\hat{K}^c_{\varepsilon,\psi}\big) = \Pi_\psi\Big(\bigcup_{j\in\psi} \pi^{-1}_{j\psi}\big(K^c_{\varepsilon,j}\big)\Big) = \sup_{j\in\psi} \Pi_\psi\big(\pi^{-1}_{j\psi}\big(K^c_{\varepsilon,j}\big)\big)$$
$$= \sup_{j\in\psi} \Pi_j\big(K^c_{\varepsilon,j}\big) \le \varepsilon.$$

For the (ϵ, K)-condition, we can take $K_\epsilon = \cap_j \pi^{E^{-1}}_j K_{\epsilon,j}$. ☐

The following consequence of Lemma 1.8.5 complements Corollary 1.7.12. Its weaker version has been used in the proof of Theorem 1.7.25.

Corollary 1.8.7. *Let E and E' be Hausdorff topological spaces such that $E \subset E'$ and the topology of E is finer than the topology induced by the topology of E'. Let Π' be a deviability on E'. If the collection of compact subsets of E is a tightening collection for Π', then the set function Π on E defined by $\Pi(A) = \Pi'(A)$ is a deviability on E.*

We now give topological versions of the results of Section 1.5.

Definition 1.8.8. *Let E and E' be Hausdorff topological spaces. A function $k(z, A') : E \times \mathcal{P}(E') \to [0, 1]$ is called a deviability transition kernel from E into E' if the following conditions hold:*

1. *$k(z, A')$ is upper semi-continuous in z for every closed set $A' \subset E'$,*

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2. *$k(z, A')$ is a deviability in A' for every $z \in E$ and is uniformly tight on compact subsets of E in the sense that for every compact $K \subset E$ and $\epsilon > 0$ there exists a compact $K' \subset E'$ such that* $\sup_{z \in K} k(z, E' \setminus K') \le \epsilon$.

Theorem 1.5.8 yields the following result.

Theorem 1.8.9. *Let μ be a tight $\mathcal{F}$-idempotent measure on a Hausdorff topological space E. Let E' be another Hausdorff topological space and $E \times E'$ be equipped with product topology. Let $k(z, A) : E \times \mathcal{P}(E') \to [0, 1]$ be a deviability transition kernel from E into E'. Then the idempotent measure $\tilde{\mu}$ on $E \times E'$ defined by $\tilde{\mu}(z, z') = k(z, z')\mu(z)$ is tight and τ-smooth relative to the collection of closed subsets of $E \times E'$.*

Proof. By Theorem 1.5.8 $\tilde{\mu}$ is τ-smooth relative to the collection $\{F \times F'\}$, where F and F' are closed in E and E', respectively, and has the tightening collection $\{K \times K'\}$, where K and K' are compact in E and E', respectively. The τ-smoothness property implies that $\tilde{\mu}(z, z')$ is an upper semi-continuous function on $E \times E'$, and the tightness property then allows us to deduce that the function is actually upper compact. ☐

Corollary 1.8.10. *If Π and Π' are deviabilities on respective Hausdorff topological spaces E and E', then the product idempotent measure $\tilde{\Pi} = \Pi \times \Pi'$ is a deviability on $E \times E'$ equipped with product topology.*

Combining the latter with Theorem 1.8.6 yields the following existence result.

Corollary 1.8.11. *Let Π_j, $j \in J$, be a collection of deviabilities on respective Hausdorff spaces E_j. Then there exists an idempotent probability space (Ω, Π) and independent Luzin idempotent variables f_j, $j \in J$, on (Ω, Π) whose respective deviability distributions are the Π_j.*

The following condition will be used for checking that a function is a deviability transition kernel.

Lemma 1.8.12. *Let E and E' be Hausdorff topological spaces and let E be first countable. If a function $k(z, z') : E \times E' \to [0, 1]$ is*

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upper semi-continuous in (z, z'), *the sets* $\{z' \in E' : \sup_{z\in K} k(z,z') \geq a\}$, *where* $a \in (0,1]$, *are relatively compact subsets of* E' *for every compact* $K \subset E$, *and* $\sup_{z'\in E'} k(z,z') = 1$ *for every* $z \in E$, *then* $k(z, A') = \sup_{z'\in A'} k(z,z')$ *is a deviability transition kernel from* E *into* E'.

Proof. We first note that the hypotheses imply that the function $\sup_{z\in K} k(z,z')$ is upper compact for every compact $K \subset E$ so that condition 2 in the definition of a deviability transition kernel holds. We check condition 1. Let A' be a closed subset of E' and $z_n \to z$ as $n \to \infty$. The set $K = \cup_{n\in\mathbb{N}}\{z_n\} \cup \{z\}$ is a compact subset of E. Therefore, given an arbitrary $\epsilon > 0$, there exists compact $K' \subset E'$ such that $k(z_n, E' \setminus K') \leq \epsilon$, $n \in \mathbb{N}$. Hence,

$$\limsup_{n\to\infty} k(z_n, A') \leq \limsup_{n\to\infty} k(z_n, K' \cap A') + \epsilon \leq k(z, K' \cap A') + \epsilon \leq k(z, A') + \epsilon,$$

where the second inequality follows by upper semi-continuity of $k(z, z')$ and the fact that $K' \cap A'$ is compact. ☐

1.9 Topological spaces of idempotent probabilities

In this section we consider topologies on the space of idempotent probability measures on a topological space, our main concern being the weak topology. Let E be a topological space and let $\mathcal{IM}(E)$ denote the set of $\mathcal{F}$-idempotent probabilities on E, where $\mathcal{F}$ is the collection of closed subsets of E. As above we denote by $C_b^+(E)$ the set of all $\mathbb{R}_+$-valued bounded continuous functions on E.

Definition 1.9.1. *The weak topology on* $\mathcal{IM}(E)$ *is the coarsest topology for which the maps* $\Pi \to \bigvee_E h\, d\Pi$ *are continuous for all* $h \in C_b^+(E)$.

According to the definition, a base for the weak topology consists of sets $\{\Pi' \in \mathcal{IM}(E) : |\bigvee_E h_i\, d\Pi' - \bigvee_E h_i\, d\Pi| < \varepsilon,\ i = 1, \dots, k\}$, where $\Pi \in \mathcal{IM}(E), h_i \in C_b^+(E), \varepsilon > 0$. For most of the section we assume that E is a Tihonov topological space, in which case $\mathcal{IM}(E)$ is a Hausdorff topological space by Theorem 1.7.27. We denote convergence in the weak topology as $\stackrel{iw}{\to}$. Our purpose is to show that

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the weak convergence of idempotent probabilities has many of the properties of the weak convergence of probability measures.

Let us denote by $\overline{C}_b^+(E)$ the set of all upper semi-continuous bounded $\mathbb{R}_+$-valued functions on E, and by $\underline{C}_b^+(E)$ the set of all lower semi-continuous bounded $\mathbb{R}_+$-valued functions on E. For a function $h : E \to \mathbb{R}_+$, let $\overline{h}$ and $\underline{h}$ denote the respective upper semi-continuous and lower semi-continuous envelopes of h defined by

$$\overline{h} = \inf_{\substack{f \in \overline{C}_b^+(E): \\ f \geq h}} f \quad \text{and} \quad \underline{h} = \sup_{\substack{g \in \underline{C}_b^+(E): \\ g \leq h}} g.$$

We say that h is continuous relative to Π if $\bigvee_E \overline{h}\, d\Pi = \bigvee_E \underline{h}\, d\Pi$. We say that h is upper semi-continuous (respectively, lower semi-continuous) relative to Π if $\bigvee_E \overline{h}\, d\Pi = \bigvee_E h\, d\Pi$ (respectively, $\bigvee_E \underline{h}\, d\Pi = \bigvee_E h\, d\Pi$).

We adopt a similar terminology for sets. We call a set $H \subset E$ continuous relative to Π if $\Pi(\text{int}\, H) = \Pi(\text{cl}\, H)$. We call a set $H \subset E$ closed (respectively, open) relative to Π if $\Pi(H) = \Pi(\text{cl}\, H)$ (respectively, $\Pi(H) = \Pi(\text{int}\, H)$). If E is a Tihonov space, a set is continuous (closed or open, respectively) relative to Π if and only if its indicator function is continuous (upper semi-continuous or lower semi-continuous, respectively) relative to Π.

Theorem 1.9.2. *(Portmanteau theorem) Let E be a Tihonov topological space. Let $\Pi \in \mathcal{IM}(E)$ and $\Pi_\phi \in \mathcal{IM}(E), \phi \in \Phi$, be a net. The following conditions are equivalent.*

1. $$\Pi_\phi \overset{iw}{\to} \Pi$$

2. $$\bigvee_E h\, d\Pi_\phi \to \bigvee_E h\, d\Pi \textit{ for all } h \in C_b^+(E)$$

3. (i) $$\liminf_\phi \bigvee_E g\, d\Pi_\phi \geq \bigvee_E g\, d\Pi \quad \textit{for all } g \in \underline{C}_b^+(E)$$

   (ii) $$\limsup_\phi \bigvee_E f\, d\Pi_\phi \leq \bigvee_E f\, d\Pi \quad \textit{for all } f \in \overline{C}_b^+(E)$$

3′. *The inequalities of part 3 hold for all lower semi-continuous relative to Π, bounded functions $g : E \to \mathbb{R}_+$ and all upper semi-continuous relative to Π, bounded functions $f : E \to \mathbb{R}_+$, respectively*

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4. (*i*) $\liminf_\phi \Pi_\phi(G) \geq \Pi(G)$ *for all open* $G \subset E$

(*ii*) $\limsup_\phi \Pi_\phi(F) \leq \Pi(F)$ *for all closed* $F \subset E$

4′. *The inequalities of part 4 hold for all open relative to* Π *sets* G *and closed relative to* Π *sets* F, *respectively*

5. $\lim_\phi \Pi_\phi(H) = \Pi(H)$ *for all continuous relative to* Π *sets* $H \subset E$

6. $\lim_\phi \bigvee_E h\, d\Pi_\phi = \bigvee_E h\, d\Pi$ *for all continuous relative to* Π *bounded functions* $h : E \to \mathbb{R}_+$

7. $\lim_\phi \bigvee_E h\, d\Pi_\phi = \bigvee_E h\, d\Pi$ *for all bounded functions* $h : E \to \mathbb{R}_+$ *that are uniformly continuous with respect to a given uniformity on* E

Proof. Conditions 1 and 2 are equivalent by the definition of the weak topology. Clearly, $2 \Rightarrow 7$, $3 \Leftrightarrow 3'$, $3 \Rightarrow 2$, $3 \Rightarrow 4$, $3' \Rightarrow 6$, $4 \Leftrightarrow 4'$, $4' \Rightarrow 5$, and $6 \Rightarrow 2$.

We prove the implication $2 \Rightarrow 4$. To prove $2 \Rightarrow 4(i)$, we note that, since E is Tihonov and G is open, $\mathbf{1}(G) = \sup\, h$ over $h \in C_b^+(E)$ such that $h \leq \mathbf{1}(G)$. Therefore, by Theorem 1.4.4 $\Pi(G) = \sup_h \bigvee_E h\, d\Pi$, so that if $h_\epsilon \leq \mathbf{1}(G)$ is such that $\Pi(G) \leq \bigvee_E h_\varepsilon\, d\Pi + \varepsilon$, then

$$\liminf_\phi \Pi_\phi(G) \geq \lim_\phi \bigvee_E h_\varepsilon d\Pi_\phi = \bigvee_E h_\varepsilon d\Pi \geq \Pi(G) - \varepsilon.$$

The proof of $4(ii)$ is analogous if we note that $\mathbf{1}_F = \inf\, h$ over $h \in C_b^+(E)$ such that $h \geq \mathbf{1}(F)$ so that by Theorem 1.7.7 $\Pi(F) = \inf_h \bigvee_E h\, d\Pi_\phi$.

We prove that $4(i) \Rightarrow 3(i)$ and $4(ii) \Rightarrow 3(ii)$. For $g \in \underline{C}_b^+(E)$ such that $\|g\| = 1$ let

$$g_k(z) = \max_{i=0,\dots,k-1} \left[\frac{i}{k}\, \mathbf{1}\Big(g(z) > \frac{i}{k}\Big) \right],\ k \in \mathbb{N}.$$

Since $\bigvee_E g_k\, d\Pi_\phi = \max_{i=0,\dots,k-1}\big(i/k\, \Pi_\phi(g(z) > i/k)\big)$ and the sets $\{z : g(z) > x\}$ are open by the lower semi-continuity of g, $4(i)$ yields

$$\liminf_\phi \bigvee_E g_k\, d\Pi_\phi \geq \bigvee_E g_k\, d\Pi.$$

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As $g_k(z) < g(z) \le g_k(z) + 1/k$, by Theorem 1.4.4

$$\liminf_{\phi} \bigvee_E g d\Pi_\phi \ge \liminf_{\phi} \bigvee_E g_k d\Pi_\phi \ge \bigvee_E g_k d\Pi \ge \bigvee_E g d\Pi - \frac{1}{k},$$

which yields $3(i)$.

The proof of $4(ii) \Rightarrow 3(ii)$ is similar if we consider the functions $f_k(z) = \max_{i=0,\dots,k-1}(i+1)/k\ \mathbf{1}\big(f(z) \ge i/k\big)$.

Now we prove $5 \Rightarrow 4$. Let G be open and $\delta > 0$. Let h be a function from $C_b^+(E)$ such that $h \le \mathbf{1}(G)$ and $\bigvee_E h\, d\Pi \ge \Pi(G) - \delta$. Let $H_u = \{z \in E : h(z) \ge u\}$, $u \in [0,1]$. Then the function $\Pi(H_u)$ increases as $u \downarrow 0$ so it has at most countably many jumps. Also $\Pi(H_u) \ge \bigvee_E h\, d\Pi - u$, so $\Pi(H_u) \ge \Pi(G) - 2\delta$ for u small enough. Thus, there exists $\varepsilon > 0$ such that $\Pi(H_\varepsilon) \ge \Pi(G) - 2\delta$ and $\Pi(H_u)$ is continuous at ε. By τ-maxitivity of Π the latter is equivalent to H_ε being continuous relative to Π. Thus, we conclude that

$$\liminf_{\phi} \Pi_\phi(G) \ge \lim_{\phi} \Pi_\phi(H_\varepsilon) = \Pi(H_\varepsilon) \ge \Pi(G) - 2\delta.$$

The proof of $4(ii)$ is similar.

We prove that $7 \Rightarrow 4(ii)$. Let $\mathcal{V}$ be a uniformity on E and F be a closed subset of E. Let $\{\rho_\alpha\}$ be a collection of pseudo-metrics on E, uniformly continuous with respect to $\mathcal{V}$, which is closed under the formation of maximums and such that $\mathbf{1}(F) = \inf_{\varepsilon>0} \inf_\alpha (1 - \rho_\alpha(z,F)/\varepsilon)^+$, where $\rho_\alpha(z,F) = \inf_{z' \in F} \rho_\alpha(z,z')$. The functions $(1 - \rho_\alpha(z,F)/\varepsilon)^+$ are bounded and uniformly continuous with respect to $\mathcal{V}$ so that by Theorem 1.7.7

$$\limsup_{\phi} \Pi_\phi(F) \le \inf_{\varepsilon>0} \inf_\alpha \lim_\phi \bigvee_E (1 - \rho_\alpha(z,F)/\varepsilon)^+\, d\Pi_\phi$$

$$= \inf_{\varepsilon>0} \inf_\alpha \bigvee_E (1 - \rho_\alpha(z,F)/\varepsilon)^+\, d\Pi$$

$$= \bigvee_E \inf_{\varepsilon>0} \inf_\alpha (1 - \rho_\alpha(z,F)/\varepsilon)^+\, d\Pi = \Pi(F).$$

The implication $7 \Rightarrow 4(i)$ is proved in an analogous manner. □

Remark 1.9.3. *As the proof shows, in part 7 it is enough to require that the convergences hold for functions h that are Lipshitz continuous with respect to the pseudo-metrics specifying the uniformity.*

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Remark 1.9.4. *In particular, Theorem 1.9.2 implies that the weak topology on* $\mathcal{IM}(E)$ *is also generated by the subbase* $\{\Pi' \in \mathcal{IM}(E) : \Pi'(G) > \Pi(G) - \varepsilon\}, \{\Pi' \in \mathcal{IM}(E) : \Pi'(F) < \Pi(F) + \varepsilon\}$, *where the* G *are open,* F *are closed,* $\varepsilon > 0$, $\Pi \in \mathcal{IM}(E)$, *as well as by the subbase* $\{\Pi' \in \mathcal{IM}(E) : |\Pi'(H) - \Pi(H)| < \varepsilon\}$, *where the* H *are continuous relative to* Π, $\varepsilon > 0, \Pi \in \mathcal{IM}(E)$.

Remark 1.9.5. *The definition of the weak topology also applies to arbitrary finite* $\mathcal{F}$*-idempotent measures. Theorem 1.9.2 is retained.*

Remark 1.9.6. *For general Hausdorff topological spaces the convergences in part 4 of Theorem 1.9.2, which specify the narrow topology, see, e.g., O'Brien and Vervaat [97], imply convergence in the weak topology. On the other hand, if we defined the weak topology in analogy with Topsøe [125] by requiring that it be the weakest topology such that the evaluations* $\Pi \to \bigvee_E g\, d\Pi$ *are lower semi-continuous for all* $g \in \underline{C}_b^+(E)$ *and the evaluations* $\Pi \to \bigvee_E f\, d\Pi$ *are upper semi-continuous for all* $f \in \overline{C}_b^+(E)$, *then the weak topology would be equivalent to the narrow topology. Also for this topology the requirement of* E *being Tihonov in Theorem 1.9.17 below can be relaxed.*

Corollary 1.9.7. *Let* E *be a Tihonov topological space. Let* E_0 *be a subset of* E *equipped with relative topology. Let* $\Pi_\phi \in \mathcal{IM}(E)$ *and* $\Pi \in \mathcal{IM}(E)$ *be such that* $\Pi_\phi(E \setminus E_0) = \Pi(E \setminus E_0) = 0$ *and the restrictions of* Π_ϕ *and* Π *to* E_0, *which are denoted by* $\tilde{\Pi}_\phi$ *and* $\tilde{\Pi}$, *respectively, are* τ*-smooth relative to the collection of closed subsets of* E_0. *Then* $\Pi_\phi \stackrel{iw}{\to} \Pi$ *if and only if* $\tilde{\Pi}_\phi \stackrel{iw}{\to} \tilde{\Pi}$.

Remark 1.9.8. *The* τ*-smoothness property in the hypotheses holds if either* E_0 *is a closed subset of* E *or the* Π_ϕ *and* Π *are tight* $\mathcal{F}$*-idempotent probabilities on* E.

We next give sufficient conditions for continuity relative to Π and the other related notions. Let $E_0 \subset E$.

Definition 1.9.9. *We say that a set* $H \subset E$ *is* E_0*–closed if it contains all its accumulation points that are in* E_0, *i.e.,* $\mathrm{cl}\, H \cap E_0 \subset H$. *We say that a set* $H \subset E$ *is* E_0*–open if every point of* $H \cap E_0$ *is an interior point of* H, *i.e.,* $H \cap E_0 \subset \mathrm{int}\, H$.

Remark 1.9.10. *Note that both the interior and closure are taken in* E. *Also,* H *is* E_0*–open if and only if its complement in* E *is* E_0*–closed.*

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Definition 1.9.11. *A function $h : E \to \mathbb{R}_+$ is called E_0–upper (respectively, E_0–lower) semi-continuous if the sets $\{z \in E : f(z) \geq a\}$ (respectively, $\{z \in E : f(z) \leq a\}$), $a \in \mathbb{R}_+$, are E_0–closed. A function $h : E \to \mathbb{R}_+$ is said to be E_0–continuous if $h^{-1}(G)$ is E_0–open for each open $G \subset \mathbb{R}_+$.*

Remark 1.9.12. *An indicator function $\mathbf{1}(A)$, $A \subset E$, is E_0–upper (respectively, E_0–lower) semi-continuous if and only if A is E_0–closed (respectively, E_0–open).*

Remark 1.9.13. *If E is Hausdorff, then a function h is E_0-upper (respectively, E_0-lower) semi-continuous if and only if $\limsup_{\phi \in \Phi} h(z_\phi) \leq h(z)$ (respectively, $\liminf_{\phi \in \Phi} h(z_\phi) \geq h(z)$) for every net $z_\phi \to z \in E_0$. Similarly, $h : E \to \mathbb{R}_+$ is E_0–continuous if and only if $\lim_{\phi \in \Phi} h(z_\phi) = h(z)$ for every net $z_\phi \to z \in E_0$.*

Let us say that Π is supported by E_0 if $\Pi(E \setminus E_0) = 0$.

Lemma 1.9.14. *Let E be Hausdorff and Π be supported by E_0.*

1. *If a function $h : E \to \mathbb{R}_+$ is E_0–continuous (E_0–upper-semi-continuous or E_0–lower-semi-continuous, respectively), then it is continuous (upper semi-continuous or lower semi-continuous, respectively) relative to Π.*

2. *If a set $H \subset E$ is E_0–continuous (E_0–closed or E_0–open, respectively), then it is continuous (closed or open, respectively) relative to Π.*

Proof. Part 1 follows by the fact that on Hausdorff spaces

$$\overline{h}(z) = \lim_{U \in \mathcal{U}_z} \sup_{z' \in U} h(z') \quad \text{and} \quad \underline{h}(z) = \lim_{U \in \mathcal{U}_z} \inf_{z' \in U} h(z'),$$

where $\mathcal{U}_z$ is the collection of open neighbourhoods of z ordered by inclusion. Part 2 is a consequence of the definitions. □

Our next goal is to prove a Prohorov criterion of weak relative compactness. We denote by $\mathcal{IM}_t(E)$ the set of tight $\mathcal{F}$-idempotent probabilities on E.

Definition 1.9.15. *1. A subset $\mathcal{A}$ of $\mathcal{IM}_t(E)$ is called tight if* $\inf_{K \in \mathcal{K}} \sup_{\Pi \in \mathcal{A}} \Pi(K^c) = 0$.

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2. A net $\{\Pi_\phi, \phi \in \Phi\}$ in $\mathcal{IM}(E)$ is called tight if $\inf_{K \in \mathcal{K}} \limsup_{\phi \in \Phi} \Pi_\phi(K^c) = 0$.

Definition 1.9.16. *A net $\{\Pi_\phi, \phi \in \Phi\}$ in $\mathcal{IM}(E)$ is called relatively compact if every subnet of $\{\Pi_\phi, \phi \in \Phi\}$ contains a weakly convergent subsubnet.*

We also use the standard definition that a subset of $\mathcal{IM}(E)$ is relatively compact for the weak topology if its closure is compact.

Theorem 1.9.17. *1. Let E be a Tihonov topological space. If a subset $\mathcal{A}$ of $\mathcal{IM}_t(E)$ (respectively, a net $\{\Pi_\phi, \phi \in \Phi\}$ in $\mathcal{IM}(E)$) is tight, then $\mathcal{A}$ (respectively, $\{\Pi_\phi, \phi \in \Phi\}$) is relatively compact, the accumulation points being elements of $\mathcal{IM}_t(E)$.*

2. Let E be homeomorphic to a complete metric space. If a subset $\mathcal{A}$ of $\mathcal{IM}(E)$ is relatively compact, then $\mathcal{A}$ is tight.

3. Let E be locally compact and Hausdorff. If a subset $\mathcal{A}$ of $\mathcal{IM}(E)$ (respectively, a net $\{\Pi_\phi, \phi \in \Phi\}$ in $\mathcal{IM}(E)$) is relatively compact, then $\mathcal{A}$ (respectively, $\{\Pi_\phi, \phi \in \Phi\}$) is tight.

Proof. We prove part 1 by proving that every tight net $\{\Pi_\phi, \phi \in \Phi\}$ in $\mathcal{IM}(E)$ contains a subnet converging to an element of $\mathcal{IM}_t(E)$. Let $C^+_{b,1}(E) = \{f \in C^+_b(E) : \|f\| \le 1\}$. The mapping $\Pi \to (\bigvee_E f\, d\Pi, f \in C^+_{b,1}(E))$ defines a homeomorphism between space $\mathcal{IM}(E)$ and a subspace of space $[0,1]^{C^+_{b,1}(E)}$ with product topology. The latter space being compact by Tihonov's theorem and Hausdorff, there exists a subnet $\{\Pi_{\phi'}, \phi' \in \Phi'\}$ of $\{\Pi_\phi, \phi \in \Phi\}$ that converges to an element of $[0,1]^{C^+_{b,1}(E)}$. By the definition of topology on $[0,1]^{C^+_{b,1}(E)}$ and properties of idempotent integral this implies that $\bigvee_E f\, d\Pi_{\phi'}$ converges for every $f \in C^+_b(E)$. Denoting the limits by $V(f)$ we conclude in view of Theorem 1.4.4 that the functional $f \to V(f)$ has properties $(V0)$–$(V2)$. Tightness of $\{\Pi_\phi, \phi \in \Phi\}$ implies that the functional is tight in the sense of Theorem 1.7.25. Thus, the functional $V(f)$ satisfies all the conditions of Theorem 1.7.25; according to the theorem $V(f) = \bigvee_E f\, d\Pi$, $f \in C^+_b(E)$, for some tight $\mathcal{F}$-idempotent probability Π, which implies that $\Pi_{\phi'} \overset{iw}{\to} \Pi$. This completes the proof of part 1.

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For part 2, we may assume that E is a complete metric space. Also replacing $\mathcal{A}$ by its closure, we may assume that $\mathcal{A}$ is a compact subset of $\mathcal{IM}(E)$. We first show that for all $\varepsilon > 0$ and $\delta > 0$ there exist open δ-balls $A_1, \dots, A_k$ such that

$$\Pi\Big(E\setminus\bigcup_{i=1}^{k} A_i\Big) < \varepsilon,\, \Pi \in \mathcal{A}. \tag{1.9.1}$$

Since each $\Pi \in \mathcal{A}$ is tight by Theorem 1.7.8, we can choose compacts K_Π in E such that $\Pi(E\setminus K_\Pi) < \varepsilon/2$. Let $B_{\Pi,1}, \dots, B_{\Pi,l_\Pi}$ be open δ-balls that cover K_Π so that

$$\Pi\Big(E\setminus\bigcup_{i=1}^{l_\Pi} B_{\Pi,i}\Big) < \varepsilon/2,\, \Pi \in \mathcal{A}. \tag{1.9.2}$$

Let

$$G_\Pi = \Big\{\Pi' \in \mathcal{IM}(E):\ \Pi'\Big(E\setminus \bigcup_{i=1}^{l_\Pi} B_{\Pi,i}\Big) < \Pi\Big(E\setminus \bigcup_{i=1}^{l_\Pi} B_{\Pi,i}\Big) + \frac{\varepsilon}{2}\Big\},\ \Pi \in \mathcal{A}. \tag{1.9.3}$$

Since by Remark 1.9.4 $\{G_\Pi, \Pi \in \mathcal{A}\}$ is an open cover of the compact set $\mathcal{A}$, there exist $G_{\Pi 1}, \dots, G_{\Pi p}$ that also cover $\mathcal{A}$ so $\Pi \in \bigcup_{j=1}^{p} G_{\Pi j}$, $\Pi \in \mathcal{A}$. Then denoting $k = \sum_{j=1}^{p} l_{\Pi j}$ and $C_j = \bigcup_{i=1}^{l_{\Pi_j}} B_{\Pi_j,i}$, $j = 1, \dots, p$, and taking $A_1 = B_{\Pi_1,1}, \dots, A_{l_{\Pi_1}} = B_{\Pi_1,l_{\Pi_1}}, A_{l_{\Pi_1}+1} = B_{\Pi_2,1}, \dots, A_k = B_{\Pi_p,l_{\Pi_p}}$ we have by (1.9.3) and (1.9.2)

$$\Pi\Big(E\setminus \bigcup_{j=1}^{k} A_j\Big) \le \min_{j=1,\dots,p} \Pi(E\setminus C_j) \le \max_{j=1,\dots,p} [\Pi_j(E\setminus C_j) + \varepsilon/2] < \varepsilon,\ \Pi \in \mathcal{A},$$

which is the required property.

Therefore for arbitrary $\varepsilon > 0$ and $k = 1, 2, \dots,$ there exist open $1/k$-balls $A_{k1}, \dots, A_{kn_k}$ such that

$$\Pi\Big(E\setminus\bigcup_{i=1}^{n_k} A_{ki}\Big) < \varepsilon, \Pi \in \mathcal{A}. \tag{1.9.4}$$

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The set $A = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded and hence relatively compact by completeness of E. At the same time, by (1.9.4) and τ-maxitivity of Π

$$\Pi(E \backslash A) = \sup_{k \in \mathbb{N}} \Pi\Big(E \backslash \bigcup_{i=1}^{n_k} A_{ki}\Big) \leq \varepsilon, \Pi \in \mathcal{A},$$

i.e., the set $\mathcal{A}$ is tight.

Now let E be locally compact and Hausdorff. The case of a relatively compact set $\mathcal{A}$ is tackled similarly to part 2 in that one can show that there exist open sets A_i with compact closures such that (1.9.1) holds. Now, let a net $\{\Pi_\phi, \phi \in \Phi\}$ be relatively compact in $\mathcal{IM}(E)$. It is sufficient to prove that for every $\varepsilon > 0$ there exist open sets $A_1, \ldots, A_k$ with compact closures such that

$$\limsup_{\phi \in \Phi} \Pi_\phi\Big(E \backslash \bigcup_{i=1}^{k} A_i\Big) \leq \varepsilon. \tag{1.9.5}$$

We introduce a partial order on the collection $\mathcal{O}$ of finite unions O of open sets with compact closures so that $O \leq O'$ if $O \subset O'$. Let Ψ denote the set of pairs (ϕ, O). We turn Ψ into a directed set by defining that $(\phi, A) \leq (\phi', A')$ if $\phi \leq \phi'$ and $A \leq A'$. We denote $\Pi_\phi(E \setminus O) = x_\psi$. Let $\{x_{\psi'}, \psi' \in \Psi'\}$ be a subnet of $\{x_\psi, \psi \in \Psi\}$ such that $\limsup_{O \in \mathcal{O}} \limsup_{\phi \in \Phi} x_\psi = \lim_{\psi' \in \Psi'} x_{\psi'}$ (cf. Kelley [71]). The mapping γ from Ψ' to Ψ in the definition of a subnet induces a mapping from Ψ' to Φ by associating with elements of Ψ' the first components of the corresponding elements of Ψ. This defines a subnet $\{\Pi_{\psi'}\}$ of Π_ϕ. Since for every $O \in \mathcal{O}$ the second component in $\gamma(\psi')$ contains O for all ψ' large enough, we have that $\lim_{\psi' \in \Psi'} x_{\psi'} \leq \limsup_{\psi' \in \Psi'} \Pi_{\psi'}(E \setminus O)$. Let $\{\Pi_{\psi''}, \psi'' \in \Psi''\}$ be a subnet of $\{\Pi_{\psi'}, \psi' \in \Psi'\}$ that weakly converges to a deviability Π. Defining $x_{\psi''} = x_{\psi'}$, where ψ' is the image of ψ'' under the mapping from Ψ'' to Ψ' in the definition of a subnet, we conclude that $\{x_{\psi''}\}$ is a subnet of $\{x_{\psi'}\}$. Therefore, for arbitrary $O \in \mathcal{O}$ by Theorem 1.9.2

$$\Pi(E \backslash O) \geq \limsup_{\psi'' \in \Psi''} \Pi_{\psi''}(E \backslash O) \geq \lim_{\psi'' \in \Psi''} x_{\psi''} = \limsup_{O \in \mathcal{O}} \limsup_{\phi \in \Phi} x_\psi.$$

Since Π is τ-smooth relative to the collection of closed subsets of E and the union of the sets O equals E by local compactness of E, it follows that $\lim_{O \in \mathcal{O}} \Pi(E \setminus O) = \Pi(\emptyset) = 0$, so we conclude that $\limsup_{O \in \mathcal{O}} \limsup_{\phi \in \Phi} \Pi_\phi(E \backslash O) = 0$, as claimed. ☐

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We now assume that E is a metric space with metric ρ and prove that the weak topology on $\mathcal{IM}(E)$ is metrisable. The next lemma has been proved in the proof of Theorem 1.7.8.

Lemma 1.9.18. *If* $\Pi \in \mathcal{IM}(E)$, *then* Π *is separable in the sense that for every* $\epsilon > 0$ *and* $\delta > 0$ *there exists a finite collection of open* δ*-balls* $A_1, A_2, \dots, A_k$ *such that* $\Pi\big(E \setminus \cup_{i=1}^k A_i\big) < \epsilon$.

We now define an idempotent analogue of the Prohorov metric. We denote by $B_\epsilon(z)$ the closed ball of radius ϵ about z, by A^ϵ the closed ϵ–neighbourhood of a set A, and let $A^{-\epsilon} = E \setminus (E \setminus A)^\epsilon$.

Definition 1.9.19. *Given* $\Pi, \Pi' \in \mathcal{IM}(E)$, *we define*

$$p(\Pi, \Pi') = \inf\big\{\epsilon > 0 : \Pi(z) \le \Pi'\big(B_\epsilon(z)\big) + \epsilon, \\ \Pi'(z) \le \Pi\big(B_\epsilon(z)\big) + \epsilon \text{ for all } z \in E\big\}.$$

It is not difficult to check that p is a metric on $\mathcal{IM}(E)$. The next lemma follows by τ-maxitivity of an idempotent measure.

Lemma 1.9.20. *Let* $\Pi, \Pi' \in \mathcal{IM}(E)$. *Then*

$$p(\Pi, \Pi') = \inf\big\{\epsilon > 0 : \Pi(A) \le \Pi'\big(A^\epsilon\big) + \epsilon, \\ \Pi'(A) \le \Pi\big(A^\epsilon\big) + \epsilon \text{ for all } A \subset E\big\}.$$

Remark 1.9.21. *One obtains the same metric as* p *if, as in the standard definition of the Prohorov metric, one considers open rather than closed* ϵ*-neighbourhoods of* z *and* A, *respectively.*

Theorem 1.9.22. *The metric* p *is compatible with the weak topology on* $\mathcal{IM}(E)$.

Proof. We first prove that the topology induced by p is finer than the weak topology. By Remark 1.9.4 it is sufficient to prove that, given $\Pi \in \mathcal{IM}(E)$, a closed set F, an open set G, and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\{\Pi' \in \mathcal{IM}(E) : p(\Pi, \Pi') < \delta\} \\ \subset \{\Pi' \in \mathcal{IM}(E) : \Pi'(F) < \Pi(F) + \epsilon\}$$

and

$$\{\Pi' \in \mathcal{IM}(E) : p(\Pi, \Pi') < \delta\} \\ \subset \{\Pi' \in \mathcal{IM}(E) : \Pi'(G) > \Pi(G) - \epsilon\}.$$

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Since Π is τ-smooth relative to $\mathcal{F}$, there exists $\delta \in (0, \epsilon/2)$ such that $\Pi(F) \geq \Pi(F^\delta) - \epsilon/2$. Therefore, if $p(\Pi, \Pi') < \delta$, then $\Pi'(F) < \Pi(F^\delta) + \delta \leq \Pi(F) + \epsilon$ proving the first inclusion. For the second inclusion, using τ-maxitivity of Π, we choose $\delta \in (0, \epsilon/2)$ such that $\Pi(G) \leq \Pi(G^{-\delta}) + \epsilon/2$. Then, if $p(\Pi, \Pi') < \delta$, then $\Pi(G) \leq \Pi(G^{-\delta}) + \epsilon/2 < \Pi'(G) + \epsilon$.

Conversely, we show using again Remark 1.9.4 that given Π and $\epsilon > 0$ there exists a collection H_i, $i = 1, \dots, k$ of sets, which are continuous relative to Π, and $\delta > 0$ such that

$$\{\Pi' \in \mathcal{IM}(E) : |\Pi'(H_i) - \Pi(H_i)| < \delta, i = 1, \dots, k\} \subset \{\Pi' \in \mathcal{IM}(E) : p(\Pi, \Pi') < \epsilon\}.$$

Let $\delta < \epsilon/3$. By separability of Π there exist closed $\delta/2$-balls $B_1, \dots, B_{k-1}$ centred at $z_1, \dots, z_{k-1}$, respectively, such that $\Pi\big(E \setminus \cup_{i=1}^{k-1} B_i\big) < \delta$. By τ-maxitivity of Π for each $i = 1, 2, \dots, k-1$ there exists a closed ball H_i centred at z_i of radius not less than $\delta/2$ and not greater than δ, which is a continuous set relative to Π. We also take $H_k = \big(E \setminus \cup_{i=1}^{k-1} B_i\big)^{\delta'}$, where $\delta' > 0$ is chosen such that H_k is continuous relative to Π and $\Pi(H_k) \leq 2\delta$. Let $\Pi' \in \mathcal{IM}(E)$ be such that $|\Pi'(H_i) - \Pi(H_i)| < \delta$, $i = 1, \dots, k$. If $z \in H_i$ for some $i = 1, \dots, k-1$, then $\Pi(z) \leq \Pi(H_i) < \Pi'(H_i) + \delta \leq \Pi'(B_{2\delta}(z)) + \delta$. Similarly, $\Pi'(z) < \Pi(B_{2\delta}(z)) + \delta$. If $z \notin \cup_{i=1}^{k-1} H_i$, then $\Pi(z) < \delta$; also, since $E \setminus \cup_{i=1}^{k-1} H_i \subset H_k$, we have that $\Pi'(z) \leq \Pi'(H_k) < \Pi(H_k) + \delta \leq 3\delta$. We thus conclude that $p(\Pi, \Pi') < \epsilon$. □

We next show that the weak topology on $\mathcal{IM}(E)$ is also metrised by a Kantorovich-Wasserstein metric. For $f \in C_b^+(E)$ let

$$\|f\|_{BL} = \big(\sup_{z \in E} f(z)\big) \vee \Big(\sup_{z \neq z'} \frac{|f(z) - f(z')|}{\rho(z, z')}\Big).$$

Clearly, if $\|f\|_{BL} < \infty$, then f is bounded and Lipshitz-continuous.

Definition 1.9.23. *For* $\Pi, \Pi' \in \mathcal{IM}(E)$, *we let*

$$\rho_{BL}(\Pi, \Pi') = \sup_{\substack{f \in C_b^+(E): \\ \|f\|_{BL} \leq 1}} \Big| \bigvee_E f \, d\Pi - \bigvee_E f \, d\Pi' \Big|.$$

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It is not difficult to check that ρ_{BL} is a metric on $\mathcal{IM}(E)$.

Lemma 1.9.24. *Let* Π *and* Π' *be in* $\mathcal{IM}(E)$. *Then* $\rho_{BL}(\Pi, \Pi') \leq 2p(\Pi, \Pi')$.

Proof. We have for f such that $\|f\|_{BL} \leq 1$ and $\delta > 0$

$$\Big|\bigvee_E f\, d\Pi - \bigvee_E f\, d\Pi'\Big| \leq \sup_{z\in E}\Big|\bigvee_{B_\delta(z)} f\, d\Pi - \bigvee_{B_\delta(z)} f\, d\Pi'\Big|$$
$$\leq \sup_{z\in E}\Big(\Big|\bigvee_{B_\delta(z)} f\, d\Pi - f(z)\Pi'(z)\Big| \vee \Big|f(z)\Pi(z) - \bigvee_{B_\delta(z)} f\, d\Pi'\Big|\Big)$$
$$\leq \delta + \sup_{z\in E}\Big(\big|\Pi(z) - \Pi'\big(B_\delta(z)\big)\big| \vee \big|\Pi\big(B_\delta(z)\big) - \Pi'(z)\big|\Big),$$

yielding the required. □

As a consequence, we have the following result.

Theorem 1.9.25. *The metric* ρ_{BL} *metrises the weak topology on* $\mathcal{IM}(E)$.

Proof. By Theorem 1.9.2 and Remark 1.9.3 the convergence $\rho_{BL}(\Pi_\phi, \Pi) \to 0$ implies the convergence $\Pi_\phi \to \Pi$. Thus, the topology induced by the metric ρ_{BL} is finer than the weak topology. The converse follows by Theorem 1.9.22 and Lemma 1.9.24. □

Since on a metric space the notions of sequential compactness and compactness are identical, metrisability of $\mathcal{IM}(E)$ allows us to give criteria for sequential compactness. We first recall relevant definitions.

Definition 1.9.26. *A subset* $\mathcal{A}$ *of* $\mathcal{IM}(E)$ *is called relatively sequentially compact (for the weak topology) if every sequence* $\{\Pi_n, n \in \mathbb{N}\}$ *of elements of* $\mathcal{A}$ *contains a weakly convergent subsubsequence.*

Combining Theorem 1.9.17 and Theorem 1.9.22 yields the following result.

Theorem 1.9.27. *Let* E *be a metric space.*

1. *If a subset* $\mathcal{A}$ *of* $\mathcal{IM}(E)$ *is tight, then* $\mathcal{A}$ *is relatively sequentially compact, the accumulation points being elements of* $\mathcal{IM}_t(E)$.

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2. Let E be homeomorphic to a complete metric space. If a subset $\mathcal{A}$ of $\mathcal{IM}(E)$ is relatively sequentially compact, then $\mathcal{A}$ is tight.

We give, however, a proof of part 1 that does not use Theorem 1.9.17.

Proof. Let $\Pi_n \in \mathcal{IM}(E)$, $n \in \mathbb{N}$. Let us assume first that E is a compact metric space. The set $C^+_{b,1}(E)$ of $\mathbb{R}_+$-valued continuous functions on E that are bounded by 1, endowed with the topology of uniform convergence, is a separable metric space. Let $C^+_{b,1,d}(E)$ denote its countable dense subset. The set $[0,1]^{C^+_{b,1,d}(E)}$ with product topology is sequentially compact, so by a diagonal argument there exists a subsequence n_k such that the sequences $\{\bigvee_E f\, d\Pi_{n_k},\, k \in \mathbb{N}\}$ converge for all $f \in C^+_{b,1,d}(E)$. Since $C^+_{b,1,d}(E)$ is dense in $C^+_{b,1}(E)$, it follows by properties of idempotent integral (more specifically, by part $(JS5)$ of Theorem 1.4.4) that the sequences $\{\bigvee_E f\, d\Pi_{n_k}, k \in \mathbb{N}\}$ converge for all $f \in C^+_{b,1}(E)$, which implies in analogy with the proof of Theorem 1.9.17 that there exists an $\mathcal{F}$-idempotent probability Π on E such that $\Pi_{n_k} \stackrel{iw}{\to} \Pi$ as $k \to \infty$.

Let us now assume that E is a separable metric space. Then it is embedded into a compact metric space E'. We extend idempotent probabilities on E to idempotent probabilities on E' by letting $\Pi'(A) = \Pi\big(A' \cap E\big)$, $A' \subset E'$. Let $\{\Pi'_{n_k}\}$ be a subsequence of $\{\Pi'_n,\, n \in \mathbb{N}\}$ which weakly converges to a deviability Π' on E'. Tightness of $\{\Pi_n,\, n \in \mathbb{N}\}$ implies that the collection of compact subsets of E is a tightening collection for Π' so that the set function Π defined by $\Pi(A) = \Pi'(A), A \subset E$, is a tight $\mathcal{F}$-idempotent probability on E. Also $\Pi'(E' \setminus E) = 0$. We check that $\Pi_{n_k} \stackrel{iw}{\to} \Pi$. Let f be a uniformly continuous function from $C^+_b(E)$ and f' be an element of $C^+_b(E')$ that extends f. By Theorem 1.9.2 $\bigvee_{E'} f'\, d\Pi'_{n_k} \to \bigvee_{E'} f'\, d\Pi'$. Since the Π'_{n_k} and Π' are supported by E, we conclude that $\bigvee_E f\, d\Pi_{n_k} \to \bigvee_E f\, d\Pi$, which proves the required by Theorem 1.9.2.

Now, if E is an arbitrary metric space, then by the tightness condition there exists a σ-compact metric space $E' \subset E$ such that $\Pi_n(E \setminus E') = 0$ for all n. Since E' is separable in relative topology, applying the part just proved to the restrictions Π'_n of the Π_n to E', we deduce existence of a subsequence $\{\Pi'_{n_k}\}$ that weakly converges to a deviability Π' on E'. Let $\Pi(A) = \Pi'(A \cap E')$, $A \subset E$. Then

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Π is a tight $\mathcal{F}$-idempotent probability on E. Finally, if $f \in C_b^+(E)$, then its restriction f' to E' belongs to $C_b^+(E')$. Since $\Pi'_n(E \setminus E') = \Pi'(E \setminus E') = 0$, we have that $\bigvee_{E'} f' \, d\Pi'_{n_k} = \bigvee_E f \, d\Pi_{n_k}$ and $\bigvee_{E'} f' \, d\Pi' = \bigvee_E f \, d\Pi$. Thus, convergence $\bigvee_{E'} f' \, d\Pi'_{n_k} \to \bigvee_{E'} f' \, d\Pi'$ yields convergence $\bigvee_E f \, d\Pi_{n_k} \to \bigvee_E f \, d\Pi$. □

A modification of the argument used in the proof of Lemma 1.9.24 allows us to obtain the following result.

Theorem 1.9.28. *Let $\{\Pi_\phi,\ \phi \in \Phi\}$ be a net of $\mathcal{F}$-idempotent probabilities on a Tihonov space E that weakly converges to $\Pi \in \mathcal{IM}(E)$. Let $\mathcal{G}$ be a subset of $C_b^+(E)$ consisting of uniformly bounded and pointwise equicontinuous functions, i.e., $\sup_{f\in\mathcal{G}} \sup_{z\in E} f(z) < \infty$, and for every $\epsilon > 0$ and $z \in E$ there exists an open neighbourhood U_z of z such that $\sup_{f\in\mathcal{G}} \sup_{z'\in U_z} |f(z) - f(z')| \le \epsilon$. Then*

$$\lim_{\phi} \sup_{f\in\mathcal{G}} \Big|\bigvee_E f \, d\Pi_\phi - \bigvee_E f \, d\Pi\Big| = 0.$$

Proof. We fix $\epsilon > 0$. For each $z \in E$ let U_z be as in the statement of the theorem. We show that the U_z can be assumed to be continuous relative to Π. Let f_z be continuous functions with values in $[0,1]$ such that $f_z(z) = 1$ and $f_z(z') = 0$ on $E \setminus U_z$. The function $\Pi\big(f_z^{-1}((x,1])\big)$, $x \in [0,1)$, is monotonically decreasing, so it has continuity points. Since $f_z^{-1}\big((x,1]\big)$ is an open Π-continuous set if $\Pi\big(f_z^{-1}((x,1])\big)$ is continuous at x by τ-smoothness of Π, the claim has been proved.

Let $U_{z_1}, \ldots, U_{z_k}$ be such that $\Pi\big(E \setminus \cup_{i=1}^k U_{z_i}\big) < \epsilon$. Then, denoting $a = \sup_{f\in\mathcal{G}} \sup_{z\in E} f(z)$, we have

$$\begin{aligned}
\Big|\bigvee_E f \, d\Pi_\phi - \bigvee_E f \, d\Pi\Big| &\le \bigvee_{E\setminus\cup_{i=1}^k U_{z_i}} f \, d\Pi_\phi + \bigvee_{E\setminus\cup_{i=1}^k U_{z_i}} f \, d\Pi \\
&+ \max_{i=1,\ldots,k} \Big|\bigvee_{U_{z_i}} f \, d\Pi_\phi - \bigvee_{U_{z_i}} f \, d\Pi\Big| \\
&\le a\Pi_\phi\big(E \setminus \cup_{i=1}^k U_{z_i}\big) + a\epsilon + 2 \max_{i=1,\ldots,k} \sup_{z\in U_{z_i}} |f(z) - f(z_i)| \\
&\qquad + \max_{i=1,\ldots,k} f(z_i) |\Pi_\phi(U_{z_i}) - \Pi(U_{z_i})|.
\end{aligned}$$

Since by "the Portmanteau theorem" the latter maximum converges to 0 as $\phi \in \Phi$ and $\limsup_{\phi\in\Phi} \Pi_\phi\big(E \setminus \cup_{i=1}^k U_{z_i}\big) \le \Pi\big(E \setminus \cup_{i=1}^k U_{z_i}\big) < \epsilon$, the proof is complete. □

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If we replace space $C_b^+(E)$ in the definition of the weak topology by space $C_{\mathcal{K}}^+(E)$ of $\mathbb{R}_+$-valued continuous functions with compact support, then we arrive at the definition of the vague topology. However, to obtain nice properties, we have to consider the space of $\mathcal{K}$–idempotent measures.

Definition 1.9.29. *The vague topology on the set of $\mathcal{K}$–idempotent measures on a topological space E is the coarsest topology for which the maps $\Pi \to \bigvee_E h\, d\Pi$ are continuous for all $h \in C_{\mathcal{K}}^+(E)$.*

If E is locally compact and Hausdorff the vague topology has properties similar to the above properties of the weak topology. For instance, the space of $\mathcal{K}$–idempotent measures is a Hausdorff topological space and there is an easy analogue of Theorem 1.9.2. A distinguishing feature of the vague topology is that the space of $\mathcal{K}$–idempotent measures is compact.

Theorem 1.9.30. *Let E be a locally compact Hausdorff topological space. Then the space of $\mathcal{K}$–idempotent measures with the vague topology is compact.*

The proof is similar to the proof of part 1 of Theorem 1.9.17, the main distinction being the use of Theorem 1.7.21 in place of Theorem 1.7.25. We end the section by indicating a connection between the vague and weak topologies, on the one hand, and Mosco convergence, on the other hand. The proof is straightforward.

Theorem 1.9.31. *I. Let E be a locally compact Hausdorff topological space. Let Π be a $\mathcal{K}$-idempotent probability and $\{\Pi_\phi,\ \phi \in \Phi\}$ be a net of $\mathcal{K}$-idempotent probabilities on E. The following pairs of conditions are equivalent:*

- (M)
  1. *for every $z \in E$ and net $z_\phi \to z$, $\limsup_{\phi\in\Phi} \Pi_\phi(z_\phi) \le \Pi(z)$,*
  2. *for every $z \in E$ there exists a net $z_\phi \to z$ such that $\lim_{\phi\in\Phi} \Pi_\phi(z_\phi) = \Pi(z)$,*
- (V)
  1. *for every compact set $K \subset E$, $\limsup_{\phi\in\Phi} \Pi_\phi(K) \le \Pi(K)$,*
  2. *for every open set $G \subset E$, $\liminf_{\phi\in\Phi} \Pi_\phi(G) \ge \Pi(G)$.*

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II. Let E be a Hausdorff topological space. Let Π be a tight $\mathcal{F}$-idempotent probability and $\{\Pi_\phi,\ \phi \in \Phi\}$ be a tight net of $\mathcal{F}$-idempotent probabilities on E. Then the next pair of conditions is equivalent to both of the above:

(N) *1. for every closed set $F \subset E$, $\limsup_{\phi\in\Phi} \Pi_\phi(F) \le \Pi(F)$,*

2. for every open set $G \subset E$, $\liminf_{\phi\in\Phi} \Pi_\phi(G) \ge \Pi(G)$.

1.10 Derived weak convergence

The results of this section give conditions for weak convergence of idempotent probabilities that are derived from weakly convergent idempotent probabilities. We also introduce convergence in idempotent distribution as an alternative way of viewing weak convergence of idempotent probabilities.

Definition 1.10.1. *Let $\{X^\phi,\ \phi \in \Phi\}$ be a net of idempotent variables defined on respective idempotent probability spaces (Ω_ϕ, Π_ϕ) and assuming values in a topological space E and X be an idempotent variable defined on an idempotent probability space (Ω, Π) and assuming values in E. Let the idempotent distributions of the X^ϕ and X be $\mathcal{F}$-idempotent probabilities on E. We say that the net $\{X^\phi,\ \phi \in \Phi\}$ converges in idempotent distribution to X if $\Pi_\phi \circ {X^\phi}^{-1} \overset{iw}{\to} \Pi \circ X^{-1}$.*

We denote convergence in idempotent distribution by $\overset{id}{\to}$. Since convergence in idempotent distribution is the weak convergence of induced idempotent laws, the theory of Section 1.9 applies, e.g., there is a version of the Portmanteau theorem. Depending on the concrete situation it can be more convenient to formulate results on weak convergence of idempotent laws as convergence in idempotent distribution or vice versa.

Lemma 1.10.2. *Let E be a Tihonov topological space, and Π_ϕ and Π be $\mathcal{F}$-idempotent probabilities on E such that $\Pi_\phi \overset{iw}{\to} \Pi$. Let functions $h_\phi : E \to \mathbb{R}_+$ be uniformly bounded and a function $h : E \to \mathbb{R}_+$ be such that*

$$\lim_{\phi\in\Phi} h_\phi(z_\phi) = h(z)$$

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for Π*-almost every* $z \in E$ *and every net* $z_\phi \to z$ *as* $\phi \in \Phi$*. Then*

$$\lim_{\phi\in\Phi} \bigvee_E h_\phi(z)\, d\Pi_\phi(z) = \bigvee_E h(z)\, d\Pi(z).$$

Proof. Let

$$\overline{h}_\phi(z) = \inf_{U\in\mathcal{U}_z} \sup_{z'\in U} \sup_{\phi'\geq\phi} h_\phi(z'), \tag{1.10.1}$$

where $\mathcal{U}_z$ denotes the set of open neighbourhoods of z. Since the convergence condition in the hypotheses equivalently requires that for every $z \in E$ such that $\Pi(z) > 0$ and $\epsilon > 0$ there exist an open neighbourhood U of z and ϕ such that $|h_{\phi'}(z') - h(z)| < \epsilon$ for all $z' \in U$ and $\phi' \geq \phi$, we conclude that, given $\epsilon > 0$ and $z \in E$ such that $\Pi(z) > 0$, there exists ϕ such that $\overline{h}_{\phi'}(z) \leq h(z) + \epsilon$ for all $\phi' \geq \phi$. Therefore, introducing $\overline{h}(z) = \inf_{\phi\in\Phi} \overline{h}_\phi(z)$, we have that $\overline{h}$ is upper semi-continuous and $\overline{h}(z) \leq h(z)$ Π-a.e. Also, since the net $\{\overline{h}_\phi\}$ consists of upper semi-continuous bounded functions, is monotonically decreasing and converges to $\overline{h}$, by Theorem 1.7.7 $\lim_{\phi\in\Phi} \bigvee_E \overline{h}_\phi\, d\Pi = \bigvee_E \overline{h}\, d\Pi$ so that for arbitrary $\epsilon > 0$ there exists ϕ_0 such that $\bigvee_E \overline{h}_{\phi_0}\, d\Pi \leq \bigvee_E \overline{h}\, d\Pi + \epsilon$. Using the fact that $h_\phi \leq \overline{h}_{\phi_0}$, $\phi \geq \phi_0$, by (1.10.1) and Theorem 1.9.2 applied to $\overline{h}_{\phi_0}$ we obtain

$$\begin{aligned}\limsup_{\phi\in\Phi} \bigvee_E h_\phi(z)\, d\Pi_\phi(z) &\leq \limsup_{\phi\in\Phi} \bigvee_E \overline{h}_{\phi_0}(z)\, d\Pi_\phi(z) \\ &\leq \bigvee_E \overline{h}_{\phi_0}(z)\, d\Pi(z) \leq \bigvee_E \overline{h}(z)\, d\Pi(z) + \epsilon \leq \bigvee_E h(z)\, d\Pi(z) + \epsilon.\end{aligned}$$

The complementary inequality

$$\liminf_{\phi\in\Phi} \bigvee_E h_\phi(z)\, d\Pi_\phi(z) \geq \bigvee_E h(z)\, d\Pi(z) - \epsilon$$

is proved by a symmetric argument. Specifically, we define

$$\underline{h}_\phi(z) = \sup_{U\in\mathcal{U}_z} \inf_{z'\in U} \inf_{\phi'\geq\phi} h_\phi(z'),\ \underline{h}(z) = \sup_{\phi\in\Phi} \underline{h}_\phi(z)$$

and note that the $\underline{h}_\phi$ are lower semi-continuous, $\underline{h}_\phi \leq h_{\phi'}$ if $\phi \leq \phi'$, $h \leq \underline{h}$ Π-a.e., and $\lim_{\phi\in\Phi} \bigvee_E \underline{h}_\phi\, d\Pi = \bigvee_E \underline{h}\, d\Pi$. Therefore, for an

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arbitrary $\epsilon > 0$ and suitable ϕ_1

$$\liminf_{\phi\in\Phi} \bigvee_E h_\phi(z)\, d\Pi_\phi(z) \geq \liminf_{\phi\in\Phi} \bigvee_E \underline{h}_{\phi_1}(z)\, d\Pi_\phi(z)$$
$$\geq \bigvee_E \underline{h}_{\phi_1}(z)\, d\Pi(z) \geq \bigvee_E \underline{h}(z)\, d\Pi(z) - \epsilon \geq \bigvee_E h(z)\, d\Pi(z) - \epsilon.$$

□

As a consequence, we obtain the following version of the continuous mapping theorem on preservation of weak convergence of probability measures under mappings. We formulate it in terms of convergence in idempotent distribution.

Theorem 1.10.3. *Let E and E' be Tihonov topological spaces, and X^ϕ and X be Luzin idempotent variables with values in E such that $X^\phi \stackrel{id}{\to} X$. Let functions $f_\phi : E \to E'$, $\phi \in \Phi$, be Luzin measurable relative to the respective idempotent distributions of the X^ϕ and $f : E \to E'$ be Luzin measurable relative to the idempotent distribution of X. If for almost every $z \in E$ with respect to the idempotent distribution of X and every net $z_\phi \to z$ we have that $f_\phi(z_\phi) \to f(z)$, then $f_\phi \circ X^\phi \stackrel{id}{\to} f \circ X$.*

Proof. Let Π and Π_ϕ denote the respective idempotent distributions of X and X^ϕ on E. Since the idempotent distribution of $f_\phi \circ X^\phi$ is $\Pi_\phi \circ f_\phi^{-1}$ and the idempotent distribution of $f \circ X$ is $\Pi \circ f^{-1}$, for an $\mathbb{R}_+$-valued bounded continuous function h on E' by a change of variables and Lemma 1.10.2

$$\lim_{\phi\in\Phi} \bigvee_{E'} h(z')\, d\Pi_\phi \circ f_\phi^{-1}(z') = \lim_{\phi\in\Phi} \bigvee_E h \circ f_\phi(z)\, d\Pi_\phi(z)$$
$$= \bigvee_E h \circ f(z)\, d\Pi(z) = \bigvee_{E'} h(z')\, d\Pi \circ f^{-1}(z').$$

□

We thus have the following "continuous mapping theorem".

Corollary 1.10.4. *Let E and E' be Tihonov topological spaces, and X^ϕ and X be Luzin variables with values in E. If $X^\phi \stackrel{id}{\to} X$ as $\phi \in \Phi$ and $f : E \to E'$ is Luzin measurable with respect to the distribution of X^ϕ for every $\phi \in \Phi$ and continuous a.e. with respect to the distribution of X, then $f \circ X^\phi \stackrel{id}{\to} f \circ X$.*

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We denote by $\mathcal{L}_i(X)$ the idempotent distribution of an idempotent variable X.

Lemma 1.10.5. *Let E be a metric space with metric ρ, and let X_ψ^ϕ and Y^ϕ, where $\phi \in \Phi$, $\psi \in \Psi$, Φ and Ψ are directed sets, be nets of idempotent variables with values in E defined on respective idempotent probability spaces (Ω_ϕ, Π_ϕ), whose idempotent distributions are $\mathcal{F}$-idempotent probabilities on E.*

Let

$$\lim_{\psi \in \Psi} \limsup_{\phi \in \Phi} \Pi_\phi\bigl(\rho(X_\psi^\phi, Y^\phi) \geq \varepsilon\bigr) = 0, \varepsilon > 0,$$

and

$$\mathcal{L}_i\bigl(X_\psi^\phi\bigr) \stackrel{iw}{\to} \tilde{\Pi}_\psi \text{ as } \phi \in \Phi,$$

where $\tilde{\Pi}_\psi$, $\psi \in \Psi$, are $\mathcal{F}$-idempotent probabilities on E.

Then, for an $\mathcal{F}$-idempotent probability Π on E, we have that

$$\mathcal{L}_i(Y^\phi) \stackrel{iw}{\to} \Pi \text{ as } \phi \in \Phi$$

if and only if

$$\tilde{\Pi}_\psi \stackrel{iw}{\to} \Pi \text{ as } \psi \in \Psi.$$

Proof. We prove sufficiency of the condition so we assume that $\tilde{\Pi}_\psi \stackrel{iw}{\to} \Pi$. Let F be a closed subset of E. Since

$$\Pi_\phi(Y^\phi \in F) \leq \Pi_\phi(X_\psi^\phi \in F^\epsilon) + \Pi_\phi\bigl(\rho(X_\psi^\phi, Y^\phi) \geq \varepsilon\bigr),$$

by hypotheses

$$\limsup_{\phi \in \Phi} \Pi_\phi(Y^\phi \in F) \leq \limsup_{\psi \in \Psi} \tilde{\Pi}_\psi(F^\epsilon) + \limsup_{\psi \in \Psi} \limsup_{\phi \in \Phi} \Pi_\phi(\rho(X_\psi^\phi, Y^\phi) \geq \varepsilon) \leq \Pi(F^\epsilon),$$

and hence by the τ-smoothness property of Π

$$\limsup_{\phi \in \Phi} \Pi_\phi(Y^\phi \in F) \leq \Pi(F). \tag{1.10.2}$$

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Let G be an open subset of E. Then $\{X_\psi^\phi \in G^{-\varepsilon}\} \subset \{Y^\phi \in G\} \cup \{\rho(X_\psi^\phi, Y^\phi) \geq \varepsilon\}$, hence, since the $G^{-\varepsilon}$ are open as well,

$$\liminf_{\phi\in\Phi} \Pi_\phi(Y^\phi \in G) \geq \liminf_{\psi\in\Psi} \liminf_{\phi\in\Phi} \Pi_\phi(X_\psi^\phi \in G^{-\varepsilon}) - \limsup_{\psi\in\Psi} \limsup_{\phi\in\Phi} \Pi_\phi(\rho(X_\psi^\phi, Y^\phi) \geq \varepsilon) \geq \Pi(G^{-\varepsilon}).$$

Observing that $\cup_{\varepsilon>0} G^{-\varepsilon} = G$ so that by τ-maxitivity $\Pi(G^{-\varepsilon}) \to \Pi(G)$ as $\varepsilon \to 0$, we conclude that

$$\liminf_{\phi\in\Phi} \Pi_\phi(Y^\phi \in G) \geq \Pi(G),$$

which together with (1.10.2) ends the proof of the sufficiency part. The converse is proved in an analogous manner. □

The following special case is useful. Given a net $\{Z^\phi,\ \phi \in \Phi\}$ of idempotent variables defined on (Ω_ϕ, Π_ϕ) and assuming values in a metric space E with metric ρ, we write that $Z^\phi \stackrel{\Pi_\phi}{\to} z \in E$, if $\lim_{\phi\in\Phi} \Pi_\phi\big(\rho(Z^\phi, z) > \epsilon\big) = 0$ for every $\epsilon > 0$.

Lemma 1.10.6. *Let E be a metric space with metric ρ, and let X^ϕ and Y^ϕ, where $\phi \in \Phi$, be nets of idempotent variables on (Ω_ϕ, Π_ϕ) with values in E, whose idempotent distributions are $\mathcal{F}$-idempotent probabilities on E. If $\mathcal{L}_i(X^\phi) \stackrel{iw}{\to} \Pi$, where Π is an $\mathcal{F}$-idempotent probability on E, and $\rho(X^\phi, Y^\phi) \stackrel{\Pi_\phi}{\to} 0$ as $\phi \in \Phi$, then $\mathcal{L}_i(Y^\phi) \stackrel{iw}{\to} \Pi$.*

Lemma 1.10.7. *Let $\{X^\phi, \phi \in \Phi\}$ be a net of idempotent variables and X be an idempotent variable. Let all the variables be defined on (Ω, Π), assume values in a metric space E and have $\mathcal{F}$-idempotent probabilities on E as distributions. If $X^\phi \stackrel{\Pi}{\to} X$, then $X^\phi \stackrel{id}{\to} X$. If $X^\phi \stackrel{id}{\to} z \in E$, then $X^\phi \stackrel{\Pi}{\to} z$.*

Proof. The first property follows by convergence properties of idempotent integrals (Theorem 1.4.19). For the second one, let ρ denote the metric on E, Π_ϕ^X denote the idempotent distribution of X^ϕ, and $\mathbf{1}_z$ denote the unit mass at z. Then the convergence $X^\phi \stackrel{id}{\to} z$ implies that

$$\lim_{\phi\in\Phi} \bigvee_E 1 \wedge \rho(z', z)\, d\Pi_\phi^X(z') = \bigvee_E 1 \wedge \rho(z', z)\, d\,\mathbf{1}_z(z') = 0.$$

□

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We now consider joint convergence.

Lemma 1.10.8. *Let X^ϕ and Y^ϕ, $\phi \in \Phi$, be nets of idempotent variables on respective idempotent probability spaces (Ω_ϕ, Π_ϕ) with values in Tihonov spaces E and E', respectively. Let X and Y be idempotent variables on an idempotent probability space (Ω, Π) with values in E and E', respectively. Let the idempotent distributions of the X^ϕ, Y^ϕ, X, and Y be τ-smooth relative to the associated collections of closed sets. Let $E \times E'$ be equipped with product topology.*

1. *If $X^\phi \overset{id}{\to} X$, $Y^\phi \overset{id}{\to} Y$, X and Y are independent, and X^ϕ and Y^ϕ are independent, then $(X^\phi, Y^\phi) \overset{id}{\to} (X, Y)$.*

2. *Let E and E' be metric spaces. If $X^\phi \overset{id}{\to} X$ and $Y^\phi \overset{\Pi_\phi}{\to} z$, then $(X^\phi, Y^\phi) \overset{id}{\to} (X, z)$.*

Proof. We prove part 1. Let $h(z, z')$, $(z, z') \in E \times E'$, be an $\mathbb{R}_+$-valued bounded function that is uniformly continuous with respect to a uniformity on $E \times E'$. By Theorem 1.9.28

$$\lim_{\phi \in \Phi} \sup_{z \in E} \Big| \sup_{z' \in E'} h(z, z') \Pi_\phi \circ {Y^\phi}^{-1}(z') - \sup_{z' \in E'} h(z, z') \Pi \circ Y^{-1}(z') \Big| = 0. \tag{1.10.3}$$

Since $\sup_{z' \in E'} h(z, z') \Pi \circ Y^{-1}(z')$ is continuous in $z \in E$,

$$\lim_{\phi \in \Phi} \sup_{z \in E} \big(\sup_{z' \in E'} h(z, z') \Pi \circ Y^{-1}(z') \big) \Pi_\phi \circ {X^\phi}^{-1}(z) = \sup_{z \in E} \big(\sup_{z' \in E'} h(z, z') \Pi \circ Y^{-1}(z') \big) \Pi \circ X^{-1}(z). \tag{1.10.4}$$

Equations (1.10.3) and (1.10.4) imply the required.

For part 2 we observe that $(\rho \times \rho')((X^\phi, Y^\phi), (X^\phi, z)) \overset{\Pi_\phi}{\to} 0$, where $\rho \times \rho'$ is a product metric on $E \times E'$, so that the required follows by part 1 and Lemma 1.10.6. □

1.11 Laplace-Fenchel transform

This section introduces idempotent analogues of the characteristic function and standard probability distributions, and develops related techniques. Let (Ω, Π) be an idempotent probability space. We denote the idempotent distribution $\Pi \circ f^{-1}$ of an idempotent variable $f : \Omega \to \mathbb{R}^d$ by Π^f.

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Remark 1.11.1. *Clearly, given an idempotent probability* $\hat{\Pi}$ *on* $\mathbb{R}^d$*, we can always construct an idempotent variable whose idempotent distribution is* $\hat{\Pi}$*. This is the "canonical" idempotent variable* $f(x) = x$*. We refer to representations like this as "canonical settings".*

We recall that for d-dimensional vectors x and y we denote as $x \cdot y$ the inner product.

Definition 1.11.2. *Given* $f : \Omega \to \mathbb{R}^d$*, the Laplace-Fenchel transform of* f *is the* $\overline{\mathbb{R}}_+$*-valued function*

$$L_f(\lambda) = S e^{\lambda \cdot f(\omega)}, \ \lambda \in \mathbb{R}^d.$$

By Theorem 1.4.6 we can also write

$$L_f(\lambda) = \bigvee_{\mathbb{R}^d} e^{\lambda \cdot x} \, d\Pi^f(x).$$

Remark 1.11.3. *Note that* $\ln L_f(\lambda)$ *is the convex conjugate, or the Legendre-Fenchel transform, of* $-\ln \Pi^f(x)$ *in that*

$$\ln L_f(\lambda) = \sup_{x \in \mathbb{R}^d} (\lambda \cdot x + \ln \Pi^f(x)).$$

Lemma 1.11.4. *An* $\overline{\mathbb{R}}_+$*-valued function* $L(\lambda)$*,* $\lambda \in \mathbb{R}^d$*, is the Laplace-Fenchel transform of an* $\mathbb{R}^d$*-valued idempotent variable if and only if* $\ln L(\lambda)$ *is convex and lower semi-continuous, and* $L(0) = 1$*.*

Proof. Necessity of the conditions follows from the definition of the Laplace-Fenchel transform and properties of convex conjugates. Conversely, let $\ln L(\lambda)$ be convex and lower semi-continuous, and $L(0) = 1$. We define $\Pi(x)$ by

$$-\ln \Pi(x) = \sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot x - \ln L(\lambda)), \ \lambda \in \mathbb{R}^d.$$

Then Π is an idempotent probability and L is the Laplace-Fenchel transform of the canonical variable on $(\mathbb{R}^d, \Pi)$ by properties of convex conjugates, Rockafellar [117, §26]. ☐

According to the above proof, we can recover Π^f from L_f by the equality

$$\Pi^f(x) = \inf_{\lambda \in \mathbb{R}^d} e^{-\lambda \cdot x} L_f(\lambda) \tag{1.11.1}$$

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if we know, in addition, that $-\ln \Pi^f(x)$ is lower semi-continuous and convex. We refer to (1.11.1) as the inversion formula. It is useful, however, to have conditions for the inversion formula to hold that are expressed only in terms of the properties of L_f.

We recall some notions of convex analysis, Rockafellar [117]. Let a function $g(\lambda), \lambda \in \mathbb{R}^d$, assume values in $(-\infty, \infty]$. The domain of g as defined as dom $g = \{\lambda \in \mathbb{R}^d : g(\lambda) < \infty\}$ and the function g is said to be essentially smooth if the following conditions hold, Rockafellar [117, §26],

(*a*) int(dom g) is not empty,

(b) g is differentiable on int(dom g),

(c) $\lim_{k\to\infty} |\nabla g(\lambda_k)| = \infty$ whenever $\{\lambda_k\}$ is a sequence of elements of int(dom g) converging to a boundary point of int(dom g).

We also denote as ri A the relative interior of a set A.

Lemma 1.11.5. *Let $L_f(\lambda)$, $\lambda \in \mathbb{R}^d$, be essentially smooth. Then the inversion formula holds.*

Proof. Let $\Pi(x)$ denote the right-hand side of (1.11.1). Since $-\ln \Pi(x), x \in \mathbb{R}^d$, is the convex conjugate of $\ln L_f(\lambda)$ and the latter is convex and lower semi-continuous, it follows that $\ln L_f(\lambda)$ is the convex conjugate of $-\ln \Pi(x)$, Rockafellar [117, §26]. Since $\ln L_f(\lambda)$ is essentially smooth, we conclude that $-\ln \Pi(x)$ is essentially strictly convex, Rockafellar [117, Theorem 26.3], hence, strictly convex on ri(dom $-\ln \Pi$). Since also $-\ln \Pi(x)$ is the bipolar of $-\ln \Pi^f(x)$, the two functions coincide by Lemma A.1 in Appendix A. ☐

Remark 1.11.6. *As the proof shows, one can weaken the requirement of essential smoothness of L_f to the requirement that the convex conjugate of $\ln L_f(\lambda)$ be strictly convex on the relative interior of its domain.*

We have a simple corollary, which shows that the Laplace-Fenchel transform can help us to identify Luzin idempotent variables.

Lemma 1.11.7. *If $L_f(\lambda)$, $\lambda \in \mathbb{R}^d$, is essentially smooth and $0 \in$ int(dom L_f), then f is a Luzin idempotent variable.*

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Proof. By the inversion formula $\Pi^f(x)$ is upper semi-continuous being the infimum of continuous functions of x so that by Lemma 1.7.4 Π^f is a $\mathcal{K}$-idempotent probability. By "the Chebyshev inequality" for $a > 0$ and $\epsilon > 0$ such that the ϵ-ball about the origin in $\mathbb{R}^d$ belongs to int(dom L_f)

$$\Pi^f(|x| \geq a) = \sup_{\substack{\lambda \in \mathbb{R}^d: \\ |\lambda| = \epsilon}} \Pi^f(\lambda \cdot x \geq a\epsilon) \leq e^{-a\epsilon} \sup_{\substack{\lambda \in \mathbb{R}^d: \\ |\lambda| = \epsilon}} L_f(\lambda).$$

The latter supremum being finite by continuity of $L_f(\lambda)$ on int(dom L_f), the right-most side tends to 0 as $a \to \infty$. Thus, Π^f is tight. □

The following property provides us with a means of proving independence of idempotent variables on Ω. For idempotent variables $f_1 : \Omega \to \mathbb{R}^{d_1}$ and $f_2 : \Omega \to \mathbb{R}^{d_2}$, we denote by $L_{f_1,f_2}(\lambda_1, \lambda_2)$, $\lambda_1 \in \mathbb{R}^{d_1}, \lambda_2 \in \mathbb{R}^{d_2}$, the Laplace-Fenchel transform of $(f_1, f_2) : \Omega \to \mathbb{R}^{d_1+d_2}$.

Lemma 1.11.8. *1. If f_1 and f_2 are independent, then $L_{f_1,f_2}(\lambda_1, \lambda_2) = L_{f_1}(\lambda_1) L_{f_2}(\lambda_2)$.*

2. Let $L_{f_1}(\lambda)$ and $L_{f_2}(\lambda)$ be essentially smooth. If $L_{f_1,f_2}(\lambda_1, \lambda_2) = L_{f_1}(\lambda_1) L_{f_2}(\lambda_2)$, then f_1 and f_2 are independent.

Proof. The first part follows by definition. The second part follows by Lemma 1.11.5 for denoting by $\Pi^{f_1,f_2}(x_1, x_2)$ the joint idempotent distribution of f_1 and f_2, we have by the lemma since $L_{f_1,f_2}(\lambda_1, \lambda_2) = L_{f_1}(\lambda_1) L_{f_2}(\lambda_2)$ is essentially smooth as well

$$\begin{aligned} \Pi^{f_1,f_2}(x_1, x_2) &= \inf_{\substack{\lambda_1 \in \mathbb{R}^{d_1}, \\ \lambda_2 \in \mathbb{R}^{d_2}}} e^{-\lambda_1 \cdot x_1 - \lambda_2 \cdot x_2} L_{f_1,f_2}(\lambda_1, \lambda_2) \\ &= \inf_{\lambda_1 \in \mathbb{R}^{d_1}} e^{-\lambda_1 \cdot x_1} L_{f_1}(\lambda_1) \inf_{\lambda_2 \in \mathbb{R}^{d_2}} e^{-\lambda_2 \cdot x_2} L_{f_2}(\lambda_2) \\ &= \Pi^{f_1}(x_1) \Pi^{f_2}(x_2). \end{aligned}$$

□

Corollary 1.11.9. *Let $\mathcal{A}$ be a τ-algebra on Ω and $f : \Omega \to \mathbb{R}^d$. If $S\big(\exp(\lambda \cdot f) | \mathcal{A}\big)(\omega)$ is constant for Π-almost ω and is an essentially smooth function of $\lambda \in \mathbb{R}^d$, then f is independent of $\mathcal{A}$.*

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Proof. From the hypotheses and properties of conditional idempotent expectations, for $\lambda_1 \in \mathbb{R}^d, \lambda_2 \in \mathbb{R}$ and $A \in \mathcal{A}$,

$$S\big(\exp(\lambda_1 \cdot f) \exp(\lambda_2 \, \mathbf{1}(A))\big) = S\big(\exp(\lambda_1 \cdot f)|\mathcal{A}\big) S\big(\exp(\lambda_2 \, \mathbf{1}(A))\big)$$
$$= S\big(\exp(\lambda_1 \cdot f)|\mathcal{A}\big)\big(\exp(\lambda_2)\Pi(A) \vee \Pi(A^c)\big).$$

The required follows by Lemma 1.11.8. □

We now introduce idempotent analogues of standard probability distributions by requiring that their Laplace-Fenchel transforms be identical to the Laplace transforms of their probabilistic counterparts.

Definition 1.11.10. *We say that $f : \Omega \to \mathbb{R}^d$ is idempotent Gaussian with parameters (m, Σ), where $m \in \mathbb{R}^d$ and Σ is a $d \times d$ positive semi-definite symmetric matrix, if $L_f(\lambda) = \exp\big(\lambda \cdot m + \lambda \cdot \Sigma\lambda/2\big)$.*

Remark 1.11.11. *Below we occasionally refer to m as the idempotent mean and Σ as the idempotent covariance of an idempotent Gaussian variable f.*

The next lemma follows by Lemma 1.11.5.

Lemma 1.11.12. *An idempotent variable $f : \Omega \to \mathbb{R}^d$ is idempotent Gaussian with parameters (m, Σ) if and only if $\Pi^f(x) = \exp\big(-(x - m) \cdot \Sigma^{\oplus}(x - m)/2\big)$ if $x - m$ is in the range of Σ and $\Pi^f(x) = 0$ otherwise.*

The following is a consequence of the definition.

Lemma 1.11.13. *An idempotent variable $f : \Omega \to \mathbb{R}^d$ is idempotent Gaussian if and only if $\lambda \cdot f : \Omega \to \mathbb{R}$ is idempotent Gaussian for every $\lambda \in \mathbb{R}^d$.*

Definition 1.11.14. *We say that $f : \Omega \to \mathbb{R}_+$ is idempotent Poisson with a parameter $\mu > 0$ if $L_f(\lambda) = \exp\big(\mu(e^\lambda - 1)\big)$, $\lambda \in \mathbb{R}$.*

An application of Lemma 1.11.5 yields the idempotent distribution of the Poisson idempotent variable.

Lemma 1.11.15. *An idempotent variable $f : \Omega \to \mathbb{R}_+$ is idempotent Poisson with a parameter $\mu > 0$ if and only if $\Pi^f(x) = \exp\big(-x \ln(x/\mu) + x - \mu\big)$, $x \in \mathbb{R}_+$, where $0 \ln 0 = 0$.*

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Remark 1.11.16. *By Lemma 1.11.7 both Gaussian and Poisson idempotent variables are Luzin idempotent variables.*

We now apply the Laplace-Fenchel transform to limit theorems. The following result is "an idempotent law of large numbers".

Lemma 1.11.17. *Let $\{f_i,\, i \in \mathbb{N}\}$ be a sequence of independent identically distributed $\mathbb{R}^d$-valued idempotent variables on an idempotent probability space (Ω, Π) such that $S\exp(\lambda \cdot f_1) < \infty$ for λ from a neighbourhood of the origin. If $\Pi(|f_1| > \epsilon) < 1$ for every $\epsilon > 0$, then*

$$\frac{1}{n}\sum_{i=1}^{n} f_i \stackrel{\Pi}{\to} 0 \quad \text{as } n \to \infty.$$

Proof. It is sufficient to check that for arbitrary $\delta > 0$ and arbitrary $\lambda \in \mathbb{R}^d$ such that $S\exp(\lambda \cdot f_1) < \infty$

$$\lim_{n\to\infty} \Pi\Big(\frac{1}{n}\sum_{i=1}^{n} \lambda\cdot f_i > |\lambda|\delta\Big) = 0. \tag{1.11.2}$$

Since by "the Chebyshev inequality" for $\alpha \in (0,1]$

$$\Pi\Big(\frac{1}{n}\sum_{i=1}^{n} \lambda\cdot f_i > |\lambda|\delta\Big) \le \frac{S\exp\big(\alpha\sum_{i=1}^{n}\lambda\cdot f_i\big)}{\exp(\alpha n|\lambda|\delta)} = \Big(\frac{Se^{\alpha\lambda\cdot f_1}}{e^{\alpha|\lambda|\delta}}\Big)^n,$$

the required would follow if there exists $\alpha \in (0,1]$ such that

$$\frac{Se^{\alpha\lambda\cdot f_1}}{e^{\alpha|\lambda|\delta}} < 1. \tag{1.11.3}$$

We have for $\epsilon > 0$

$$\begin{aligned} &Se^{\alpha\lambda\cdot f_1} \\ &= \big(Se^{\alpha\lambda\cdot f_1}\,\mathbf{1}(|\lambda\cdot f_1| > \epsilon|\lambda|)\big) \vee \big(Se^{\alpha\lambda\cdot f_1}\,\mathbf{1}(|\lambda\cdot f_1| \le |\lambda|\epsilon)\big) \\ &\qquad\le \big(Se^{\alpha\lambda\cdot f_1}\,\mathbf{1}(|\lambda\cdot f_1| > |\lambda|\epsilon)\big) \vee e^{\alpha|\lambda|\epsilon}. \end{aligned} \tag{1.11.4}$$

Let $\epsilon < \delta$. Then the second term on the right-most side of (1.11.4) is less than $e^{\alpha|\lambda|\delta}$. For the first term we have in view of the hypotheses that

$$\lim_{\alpha\to 0} \frac{Se^{\alpha\lambda\cdot f_1}\,\mathbf{1}(|\lambda\cdot f_1| > \epsilon|\lambda|)}{e^{\alpha|\lambda|\delta}} = \Pi(|\lambda\cdot f_1| > |\lambda|\epsilon) < 1.$$

Thus, (1.11.3) holds for $\alpha > 0$ small enough, which concludes the proof. ☐

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Remark 1.11.18. *Thus, an analogue of the expectation of a random variable for an idempotent variable f in the law of large numbers is an element a such that $\Pi(|f-a|>\epsilon)<1$ for every $\epsilon>0$.*

The following lemma is an analogue of the method of characteristic functions in weak convergence theory.

Lemma 1.11.19. *Let $\{\Pi_\phi, \phi\in\Phi\}$ be a net of deviabilities on $\mathbb{R}^d$. If $\bigvee_{\mathbb{R}^d} e^{\lambda\cdot x} d\Pi_\phi(x) \to L(\lambda)$ as $\phi\in\Phi$ for all $\lambda\in\mathbb{R}^d$, where $L(\lambda)$ is essentially smooth, lower semi-continuous and such that $0\in$ int(dom L), then $\Pi_\phi \stackrel{iw}{\to} \Pi$, where deviability Π is given by the inversion formula* (1.11.1).

Proof. Let us denote $L_\phi(\lambda) = \bigvee_{\mathbb{R}^d} e^{\lambda\cdot x}\, d\Pi_\phi(x)$. We show that the net $\{\Pi_\phi,\, \phi\in\Phi\}$ is tight. Indeed, since $L_\phi(\lambda)\to L(\lambda)$ and $L(\lambda)$ is finite in a neighbourhood of the origin by the fact that $0\in$ int (dom L), there exists $r>0$ such that $\limsup_\phi \bigvee_{\mathbb{R}^d} e^{r|x|}\, d\Pi_\phi(x) < \infty$, and then "the Chebyshev inequality" $\Pi_\phi(\{x : |x| > A\}) \le e^{-rA}\bigvee_{\mathbb{R}^d} e^{r|x|}\, d\Pi_\phi(x)$ yields the claim.

Therefore, by Theorem 1.9.17 $\{\Pi_\phi, \phi\in\Phi\}$ has accumulation points in $\mathcal{IM}_t(\mathbb{R}^d)$. Let $\{\Pi_{\phi'}, \phi'\in\Phi'\}$ be a subnet of $\{\Pi_\phi, \phi\in\Phi\}$ that weakly converges to $\tilde\Pi\in\mathcal{IM}_t(\mathbb{R}^d)$. Then the convergence $L_{\phi'}(\lambda)\to L(\lambda)$ implies that if $\lambda\in$ int(dom L), then for suitable $\varepsilon>0$ $\limsup_{\phi'}\bigvee_{\mathbb{R}^d} e^{(1+\varepsilon)\lambda\cdot x}\, d\Pi_{\phi'}(x) < \infty$. Hence, the function $(e^{\lambda\cdot x}, x\in\mathbb{R}^d)$ is uniformly maximable with respect to $\{\Pi_{\phi'}, \phi'\in\Phi'\}$, so $\lim_{\phi'}\bigvee_{\mathbb{R}^d} e^{\lambda\cdot x}\, d\Pi_{\phi'}(x) = \bigvee_{\mathbb{R}^d} e^{\lambda\cdot x}\, d\tilde\Pi(x)$, implying that

$$\bigvee_{\mathbb{R}^d} e^{\lambda\cdot x}\, d\tilde\Pi(x) = L(\lambda) \tag{1.11.5}$$

for all $\lambda\in$ int(dom L). We prove that (1.11.5) actually holds for all $\lambda\in\mathbb{R}^d$. Let $\tilde L(\lambda)$ denote the left-hand side of (1.11.5). Since $L(\lambda)$ is essentially smooth, it follows that $|\nabla\tilde L(\lambda_n)|\to\infty$ for every sequence λ_n in int(dom L) that converges to a boundary point of dom L. Since $\tilde L(\lambda)$ is convex, it follows that $\tilde L(\lambda)=\infty$ for all $\lambda\notin$ cl(dom L), which implies that $\tilde L(\lambda) = L(\lambda)$ for $\lambda\notin$ cl(dom L). Finally, if λ is a boundary point of dom L, then lower semi-continuity and convexity of $L(\lambda)$ imply that $L(\lambda) = \lim_n L(\lambda_n)$, where λ_n is a sequence of points from int(dom L) converging to λ. For the same reason this holds for $\tilde L(\lambda)$, so we conclude that $\tilde L(\lambda) = L(\lambda)$ for all $\lambda\in\mathbb{R}^d$. Then Lemma 1.11.5 implies that $\tilde\Pi = \Pi$, which ends the proof. ☐

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Chapter 2

Maxingales

In this chapter we develop elements of idempotent stochastic calculus. We are mostly interested in studying idempotent analogues of martingales and martingale problems (which we call maxingales and maxingale problems, respectively).

2.1 Idempotent stopping times

In this section we define stopping times with respect to τ-algebras and study their properties. The concepts, results and proofs are analogous to those in the general theory of stochastic processes, see Dellacherie [34] or Meyer [88]. Therefore, we omit proofs that are analogous to proofs in these books.

Let Ω be a set.

Definition 2.1.1. *An indexed collection* $\mathbf{A} = \{\mathcal{A}_t, t \in \mathbb{R}_+\}$ *of* τ-*algebras on* Ω *is called a flow of* τ-*algebras if* $\mathcal{A}_s \subset \mathcal{A}_t$ *for* $s \le t$. *We also refer to a flow of* τ-*algebras as a* τ-*flow. We say that the* τ-*flow* $\mathbf{A}$ *is right-continuous if* $\mathcal{A}_s = \bigcap_{t>s} \mathcal{A}_t$ *for all* $s \in \mathbb{R}_+$.

Remark 2.1.2. *Recall that* $\mathcal{A}_s \subset \mathcal{A}_t$ *if and only if the atoms of* $\mathcal{A}_s$ *are unions of the atoms of* $\mathcal{A}_t$.

Remark 2.1.3. *Given a* τ-*flow* $\mathbf{A}$, *there is a natural right-continuous* τ-*flow associated with* $\mathbf{A}$ *that is defined by* $\mathbf{A}_+ = (\mathcal{A}_{t_+}, t \in \mathbb{R}_+)$, *where* $\mathcal{A}_{t_+} = \cap_{\epsilon>0}\mathcal{A}_{t+\epsilon}$.

We assume as given a flow of τ-algebras $\mathbf{A}$ and a τ-algebra $\mathcal{A}_\infty$ such that $\mathcal{A}_s \subset \mathcal{A}_\infty$ for $s \in \mathbb{R}_+$.

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Definition 2.1.4. *An $\overline{\mathbb{R}}_+$-valued function σ on Ω is said to be an idempotent stopping time relative to* $\mathbf{A}$*, or, for short, an* $\mathbf{A}$*-stopping time, if* $\{\sigma \le s\} \in \mathcal{A}_s$ *for all* $s \in \mathbb{R}_+$.

Lemma 2.1.5.

1. *A function $\sigma : \Omega \to \overline{\mathbb{R}}_+$ is an* $\mathbf{A}$*-stopping time if and only if* $\{\sigma = s\} \in \mathcal{A}_s$ *for all* $s \in \mathbb{R}_+$ *(in particular, constants are stopping times); if* $\mathbf{A}$ *is right-continuous, an equivalent condition is that* $\{\sigma < s\} \in \mathcal{A}_s$ *for all* $s \in \mathbb{R}_+$.

2. *If σ and τ are* $\mathbf{A}$*-stopping times, then* $\sigma \vee \tau$, $\sigma \wedge \tau$ *and* $\sigma + \tau$ *are* $\mathbf{A}$*-stopping times.*

3. *If* $\sigma_\psi, \psi \in \Psi$, *are* $\mathbf{A}$*-stopping times, then* $\sup_{\psi\in\Psi} \sigma_\psi$ *is an* $\mathbf{A}$*-stopping time; if* $\mathbf{A}$ *is right-continuous, then* $\inf_{\psi\in\Psi} \sigma_\psi$, $\liminf_{\psi\in\Psi} \sigma_\psi$ *and* $\limsup_{\psi\in\Psi} \sigma_\psi$ *(in the latter two cases* Ψ *is a directed set) are* $\mathbf{A}$*-stopping times.*

We now introduce τ-algebras associated with stopping times. Let us first prove two simple facts. Recall that $[\omega]_{\mathcal{A}}$ denotes the atom of a τ-algebra $\mathcal{A}$ about ω.

Lemma 2.1.6. *Let σ be an* $\mathbf{A}$*-stopping time. If* $\omega'' \in [\omega']_{\mathcal{A}_{\sigma(\omega')}}$*, then* $\sigma(\omega'') = \sigma(\omega')$ *and* $[\omega']_{\mathcal{A}_{\sigma(\omega')}} = [\omega'']_{\mathcal{A}_{\sigma(\omega'')}}$.

Proof. Since $\omega' \in \{\omega : \sigma(\omega) \le \sigma(\omega')\} \in \mathcal{A}_{\sigma(\omega')}$, $[\omega']_{\mathcal{A}_{\sigma(\omega')}}$ is an atom of $\mathcal{A}_{\sigma(\omega')}$, and $\omega'' \in [\omega']_{\mathcal{A}_{\sigma(\omega')}}$, it follows that $\omega'' \in \{\omega : \sigma(\omega) \le \sigma(\omega')\}$ so that

$$\sigma(\omega'') \le \sigma(\omega'). \tag{2.1.1}$$

Therefore, $[\omega'']_{\mathcal{A}_{\sigma(\omega'')}} \supset [\omega'']_{\mathcal{A}_{\sigma(\omega')}} = [\omega']_{\mathcal{A}_{\sigma(\omega')}}$. Thus, $\omega' \in [\omega'']_{\mathcal{A}_{\sigma(\omega'')}}$. The argument of the proof of (2.1.1) with roles of ω' and ω'' switched then shows that we actually have equality in (2.1.1). Equality of the atoms in the statement is now self-evident. □

The following lemma is an easy consequence.

Lemma 2.1.7. *Let σ be an* $\mathbf{A}$*-stopping time. Then the collection* $\{[\omega]_{\mathcal{A}_{\sigma(\omega)}}, \omega \in \Omega\}$ *is a partition of Ω in that two arbitrary sets from the collection are either disjoint or coincide and the union of the sets from the collection equals Ω.*

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Definition 2.1.8. *Let σ be an $\mathbf{A}$-stopping time. We denote by $\mathcal{A}_\sigma$ the τ-algebra that has atoms $[\omega]_{\mathcal{A}_{\sigma(\omega)}}$.*

The next lemma shows that our definition is consistent with the corresponding definition in the general theory of stochastic processes.

Lemma 2.1.9. *The τ-algebra $\mathcal{A}_\sigma$ is the collection of subsets A of Ω such that $A \in \mathcal{A}_\infty$ and $A \cap \{\sigma \le s\} \in \mathcal{A}_s$ for all $s \in \mathbb{R}_+$.*

Proof. Clearly, $\mathcal{A}_\sigma \subset \mathcal{A}_\infty$. Let us show that $[\omega]_{\mathcal{A}_{\sigma(\omega)}} \cap \{\sigma \le s\} \in \mathcal{A}_s$. Let $\omega' \in [\omega]_{\mathcal{A}_{\sigma(\omega)}} \cap \{\sigma \le s\}$. Then by Lemma 2.1.6 $[\omega']_{\mathcal{A}_s} \subset [\omega']_{\mathcal{A}_{\sigma(\omega')}} = [\omega]_{\mathcal{A}_{\sigma(\omega)}}$. Also $[\omega']_{\mathcal{A}_s} \subset \{\sigma \le s\}$. Therefore, by Corollary 1.1.16 $[\omega]_{\mathcal{A}_{\sigma(\omega)}} \cap \{\sigma \le s\} \in \mathcal{A}_s$.

Conversely, let $A \in \mathcal{A}_\infty$ be such that $A \cap \{\sigma \le s\} \in \mathcal{A}_s$ for all $s \in \mathbb{R}_+$ and let $\omega \in A$. We prove that $A \in \mathcal{A}_\sigma$ by proving that $[\omega]_{\mathcal{A}_\sigma(\omega)} \subset A$. If $\sigma(\omega) < \infty$, then the required follows since $\omega \in A \cap \{\sigma \le \sigma(\omega)\} \in \mathcal{A}_{\sigma(\omega)}$ so that $[\omega]_{\mathcal{A}_\sigma(\omega)} \subset A \cap \{\sigma \le \sigma(\omega)\}$. If $\sigma(\omega) = \infty$, then $[\omega]_{\mathcal{A}_{\sigma(\omega)}}$ is an atom of $\mathcal{A}_\infty$ and the required follows since $A \in \mathcal{A}_\infty$. □

Remark 2.1.10. *The τ-algebra $\mathcal{A}_\sigma$ can also be defined as the collection of subsets A of Ω such that $A \cap \{\sigma = s\} \in \mathcal{A}_s$ for all $s \in \overline{\mathbb{R}}_+$.*

Lemma 2.1.11.

1. *Let σ be an $\mathbf{A}$-stopping time. Then σ is $\mathcal{A}_\sigma$-measurable.*
2. *Let σ be an $\mathbf{A}$-stopping time and $\tau \ge \sigma$ be $\mathcal{A}_\sigma$-measurable. Then τ is an $\mathbf{A}$-stopping time.*
3. *Let σ and τ be $\mathbf{A}$-stopping times such that $\tau \ge \sigma$. Then $\tau - \sigma$ is a stopping time relative to the τ-flow $\{\mathcal{A}_{\sigma+t},\, t \in \mathbb{R}_+\}$.*
4. *Let σ and τ be $\mathbf{A}$-stopping times such that $\sigma \le \tau$. Then $\mathcal{A}_\sigma \subset \mathcal{A}_\tau$.*
5. *Let σ and τ be $\mathbf{A}$-stopping times. Then the sets $\{\sigma < \tau\}$, $\{\sigma = \tau\}$ and $\{\sigma > \tau\}$ belong both to $\mathcal{A}_\sigma$ and $\mathcal{A}_\tau$. Also $\mathcal{A}_\sigma \bigcap \{\sigma = \tau\} = \mathcal{A}_\tau \bigcap \{\sigma = \tau\}$.*

Let, given $t > 0$, $\mathcal{A}_{t-}$ denote the τ-algebra generated by the τ-algebras $\mathcal{A}_s$ for $s < t$. It is obvious that the atoms of $\mathcal{A}_{t-}$ are of the form $[\omega]_{\mathcal{A}_{t-}} = \cap_{s<t}[\omega]_{\mathcal{A}_s}$. We also let $\mathcal{A}_{0-} = \mathcal{A}_0$ and $\mathcal{A}_{\infty-} = \tau\big(\mathcal{A}_s,\, s \in \mathbb{R}_+\big)$.

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Definition 2.1.12. *Let σ be an* $\mathbf{A}$*-stopping time. We denote by* $\mathcal{A}_{\sigma-}$ *the* τ*-algebra generated by the sets* $[\omega]_{\mathcal{A}_{\sigma(\omega)-}}$.

We now prove that similarly to $\mathcal{A}_\sigma$ the definition of $\mathcal{A}_{\sigma-}$ is compatible with the corresponding definition in stochastic calculus.

Lemma 2.1.13. *The* τ*-algebra* $\mathcal{A}_{\sigma-}$ *coincides with the* τ*-algebra* $\hat{\mathcal{A}}_{\sigma-}$ *that is generated by the elements of* $\mathcal{A}_0$ *and the sets of the form* $A \cap \{t < \sigma\}$, $A \in \mathcal{A}_t$, $t \in \mathbb{R}_+$.

Proof. The definition of $\mathcal{A}_{\sigma-}$ implies that $\mathcal{A}_0 \subset \mathcal{A}_{\sigma-}$. Let $A \in \mathcal{A}_t$ and $\omega \in A \cap \{t < \sigma\}$. In view of Corollary 1.1.16 we prove that $A \cap \{t < \sigma\} \in \mathcal{A}_{\sigma-}$ by proving that $[\omega]_{\sigma(\omega)-} \subset A \cap \{t < \sigma\}$. Since $\omega \in A$ and $A \in \mathcal{A}_t$, we have that $[\omega]_{\mathcal{A}_t} \subset A$. Since $\sigma(\omega) > t$, we have that $[\omega]_{\mathcal{A}_{\sigma(\omega)-}} \subset [\omega]_{\mathcal{A}_t}$. Therefore, $[\omega]_{\sigma(\omega)-} \subset A$. Since $\omega \in \{\sigma > t\} \in \mathcal{A}_t$, we have that $[\omega]_{\mathcal{A}_{\sigma(\omega)-}} \subset [\omega]_{\mathcal{A}_t} \subset \{\sigma > t\}$. The claim is proved. Thus, $\hat{\mathcal{A}}_{\sigma-} \subset \mathcal{A}_{\sigma-}$. The reverse inclusion follows since $[\omega]_{\mathcal{A}_{\sigma(\omega)-}} = \cap_{t<\sigma(\omega)}[\omega]_{\mathcal{A}_t} = \cap_{t<\sigma(\omega)}\big([\omega]_{\mathcal{A}_t} \cap \{t < \sigma\}\big) \in \hat{\mathcal{A}}_{\sigma-}$ if $\sigma(\omega) > 0$. ☐

In view of Lemmas 2.1.9 and 2.1.13, the τ-algebras $\mathcal{A}_\sigma$ and $\mathcal{A}_{\sigma-}$ share the properties of their stochastic counterparts, the same proofs applying.

Lemma 2.1.14. *Let σ be an* $\mathbf{A}$*-stopping time. Then* $\mathcal{A}_{\sigma-} \subset \mathcal{A}_\sigma$ *and* σ *is* $\mathcal{A}_{\sigma-}$*-measurable.*

Lemma 2.1.15. *Let σ and τ be* $\mathbf{A}$*-stopping times and* $A \in \mathcal{A}_\sigma$*. Then* $A \cap \{\sigma \le \tau\} \in \mathcal{A}_\tau$ *and* $A \cap \{\sigma < \tau\} \in \mathcal{A}_{\tau-}$*. If, in addition,* $\sigma \le \tau$*, then* $\mathcal{A}_{\sigma-} \subset \mathcal{A}_{\tau-}$*. If, in addition,* $\sigma < \tau$ *on the set* $\{0 < \tau < \infty\}$*, then* $\mathcal{A}_\sigma \subset \mathcal{A}_{\tau-}$.

The next lemma summarises limiting properties of nets of τ-algebras associated with idempotent stopping times.

Lemma 2.1.16.

1. *Let* $\{\sigma_\phi, \phi \in \Phi\}$ *be an increasing net of* $\mathbf{A}$*-stopping times and* $\sigma = \lim_{\phi\in\Phi}\sigma_\phi$*. Then* $\mathcal{A}_{\sigma-} = \tau\big(\mathcal{A}_{\sigma_\phi-}, \phi \in \Phi\big)$*. If, in addition,* $\sigma_\phi < \sigma$ *on the set* $\{0 < \sigma < \infty\}$ *for* $\phi \in \Phi$*, then* $\mathcal{A}_{\sigma-} = \tau\big(\mathcal{A}_{\sigma_\phi}, \phi \in \Phi\big)$.

2. *Let the* τ*-flow* $\mathbf{A}$ *be right-continuous. Let* $\{\sigma_\psi, \psi \in \Psi\}$ *be a decreasing net of* $\mathbf{A}$*-stopping times and* $\sigma = \lim_{\psi\in\Psi}\sigma_\psi$*. Then* $\mathcal{A}_\sigma = \bigcap_{\psi\in\Psi}\mathcal{A}_{\sigma_\psi}$*. If, in addition,* $\sigma < \sigma_\psi$ *on the sets* $\{0 < \sigma_\psi < \infty\}$ *for* $\psi \in \Psi$*, then* $\mathcal{A}_\sigma = \bigcap_{\psi\in\Psi}\mathcal{A}_{\sigma_\psi-}$.

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The next concept is also borrowed from the general theory of stochastic processes, see Dellacherie [34].

Definition 2.1.17. *For $A \subset \mathbb{R}_+ \times \Omega$, we define the début of A by*

$$D_A(\omega) = \inf\{t \in \mathbb{R}_+ : (t,\omega) \in A\},$$

where $D_A(\omega) = \infty$ if the latter set is empty.

The associated property is also similar though proved much more easily.

Lemma 2.1.18. *Let the sets $\{\omega : (t,\omega) \in A\}$ belong to $\mathcal{A}_t$ for $t \in \mathbb{R}_+$ and $\mathbf{A}$ be right-continuous. Then $D_A(\omega)$ is an $\mathbf{A}$-stopping time.*

Proof. By right continuity of $\mathbf{A}$ it is enough to prove that $\{\omega : D_A(\omega) < t\} \in \mathcal{A}_t$. We have

$$\{\omega : D_A(\omega) < t\} = \bigcup_{s<t} \{\omega : (s,\omega) \in A\} \in \mathcal{A}_t.$$

□

2.2 Idempotent processes

We define and study idempotent analogues of stochastic processes. Let (Ω, Π) be an idempotent probability space.

Definition 2.2.1. *An $\mathbb{R}^d$-valued idempotent stochastic process (or idempotent process, for short) on (Ω, Π) is a collection $(X_t(\omega), \omega \in \Omega), t \in \mathbb{R}_+$, of $\mathbb{R}^d$-valued idempotent variables on Ω. It is a proper idempotent process if the X_t are proper idempotent variables.*

Unless specified otherwise, we consider only $\mathbb{R}^d$-valued idempotent processes. As above we denote by Π^X the idempotent probability $\Pi \circ X^{-1}$ on $(\mathbb{R}^d)^{\mathbb{R}_+}$ that is the idempotent distribution of an idempotent process $X = (X_t, t \in \mathbb{R}_+)$ on (Ω, Π). According to the general definition of the image of an idempotent measure we equivalently have that $\Pi^X(\mathbf{x}) = \Pi\big(\omega \in \Omega : X_t(\omega) = \mathbf{x}_t, t \in \mathbb{R}_+\big)$ for $\mathbf{x} = (\mathbf{x}_t, t \in \mathbb{R}_+) \in (\mathbb{R}^d)^{\mathbb{R}_+}$ and $\Pi^X(A) = \sup_{\mathbf{x} \in A} \Pi^X(\mathbf{x})$ for $A \subset (\mathbb{R}^d)^{\mathbb{R}_+}$. If $(\mathbb{R}^d)^{\mathbb{R}_+}$ is endowed with idempotent probability $\hat{\Pi}$, then the idempotent process $(X_t, t \in \mathbb{R}_+)$ defined by $X_t(\mathbf{x}) = \mathbf{x}_t$ has

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idempotent distribution $\hat{\Pi}$ as well. We then call X the canonical idempotent process with distribution $\hat{\Pi}$.

We now define finite-dimensional idempotent distributions. Let $\mathcal{Q}(\mathbb{R}_+)$ denote the collection of finite subsets of $\mathbb{R}_+$. For $\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{R}_+}$ and $\alpha \in \mathcal{Q}(\mathbb{R}_+)$ we denote $\pi_\alpha \mathbf{x} = (\mathbf{x}_t, t \in \alpha) \in (\mathbb{R}^d)^\alpha$. We refer to idempotent probabilities Π_α^X on $(\mathbb{R}^d)^\alpha$ defined by $\Pi_\alpha^X(A) = \Pi(\omega : \pi_\alpha X(\omega) \in A)$ as finite-dimensional idempotent distributions of the idempotent process $X = (X_t, t \in \mathbb{R}_+)$. Note that Π_α^X, $\alpha \in \mathcal{Q}(\mathbb{R}_+)$, is a projective system. We have the following simple relationship between Π^X and Π_α^X. We assume throughout the product topology on $(\mathbb{R}^d)^{\mathbb{R}_+}$.

Theorem 2.2.2. *Let X be an idempotent process with idempotent distribution Π^X. For $\alpha \in \mathcal{Q}(\mathbb{R}_+)$ and $x \in (\mathbb{R}^d)^\alpha$,*

$$\Pi_\alpha^X(x) = \Pi^X \circ \pi_\alpha^{-1}(x) = \sup_{\mathbf{x}:\, \pi_\alpha \mathbf{x} = x} \Pi^X(\mathbf{x}).$$

If Π^X is an $\mathcal{F}$-idempotent probability on $(\mathbb{R}^d)^{\mathbb{R}_+}$, then

$$\Pi^X(\mathbf{x}) = \inf_{\alpha \in \mathcal{Q}(\mathbb{R}_+)} \Pi_\alpha^X(\pi_\alpha \mathbf{x}),\ \mathbf{x} \in (\mathbb{R}^d)^{\mathbb{R}_+}.$$

Proof. The first equality follows by definition. The second is a consequence of the representation $\{\mathbf{x}\} = \bigcap_{\alpha \in \mathcal{Q}(\mathbb{R}_+)} \pi_\alpha^{-1} \circ \pi_\alpha \mathbf{x}$ and τ-smoothness of Π^X relative to the collection of closed subsets of $(\mathbb{R}^d)^{\mathbb{R}_+}$. □

Remark 2.2.3. *If Π is supported by space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ of right-continuous with left limits $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$, then the infimum in the last equality of the statement of Theorem 2.2.2 can be taken over finite subsets of a set dense in $\mathbb{R}_+$.*

Thus, if Π^X is a deviability, it is uniquely specified by the finite-dimensional projections. The next adaptation of Theorem 1.8.6 shows that, conversely, a projective system of deviabilities on finite-dimensional spaces is uniquely extended to a deviability on $(\mathbb{R}^d)^{\mathbb{R}_+}$.

Theorem 2.2.4. *Let Π_α, $\alpha \in \mathcal{Q}(\mathbb{R}_+)$, be a projective system of deviabilities defined on respective spaces $(\mathbb{R}^d)^\alpha$. Then there exists a unique deviability Π' on $(\mathbb{R}^d)^{\mathbb{R}_+}$ such that $\Pi_\alpha = \Pi' \circ \pi_\alpha^{-1}$. It is specified by the equality $\Pi'(\mathbf{x}) = \inf_{\alpha \in \mathcal{Q}(\mathbb{R}_+)} \Pi_\alpha(\pi_\alpha \mathbf{x})$, $\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{R}_+}$.*

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We single out the class of idempotent processes whose idempotent distributions are deviabilities.

Definition 2.2.5. *We say that an idempotent process X is Luzin if the finite-dimensional projections $\pi_\alpha X$, $\alpha \in \mathcal{Q}(\mathbb{R}_+)$, are Luzin idempotent variables on (Ω, Π). Let, in addition, Ω be a Hausdorff topological space and Π be a deviability on Ω. An idempotent process X is called a strictly Luzin idempotent process on (Ω, Π) if the $X_t(\omega)$, $t \in \mathbb{R}_+$, are strictly Luzin idempotent variables on (Ω, Π).*

Thus, strictly Luzin idempotent processes are Luzin idempotent processes, and Luzin idempotent processes are proper idempotent processes. Theorem 2.2.4 and the definition of the product topology yield the following.

Corollary 2.2.6.
1. *An $\mathbb{R}^d$-valued idempotent process X is Luzin if and only if Π^X is a deviability on $(\mathbb{R}^d)^{\mathbb{R}_+}$.*
2. *Let Ω be a Hausdorff topological space and Π be a deviability. An $\mathbb{R}^d$-valued idempotent process X is a strictly Luzin idempotent process if and only if the mapping $\omega \to (X_t(\omega),\, t \in \mathbb{R}_+)$ is a strictly Luzin idempotent variable in $(\mathbb{R}^d)^{\mathbb{R}_+}$.*

Remark 2.2.7. *Note that if X is a Luzin idempotent process with distribution Π^X, then the canonical idempotent process on $\big((\mathbb{R}^d)^{\mathbb{R}_+}, \Pi^X\big)$ is a strictly Luzin idempotent process. Therefore, if we are only concerned with the idempotent distribution of a Luzin idempotent process, we can assume that it is a strictly Luzin idempotent process.*

We now consider properties of trajectories.

Definition 2.2.8. *An idempotent process X on Ω is said to be continuous in idempotent probability if for every $s \in \mathbb{R}_+$ and $\varepsilon > 0$*

$$\lim_{t \to s} \Pi(|X_t - X_s| > \varepsilon) = 0.$$

An idempotent process X on Ω is said to be continuous Π-a.e. if for every $s \in \mathbb{R}_+$ we have $\Pi\big(X_t$ does not converge to X_s as $t \to s\big) = 0$. An idempotent process X on Ω is said to have continuous paths Π-a.e. if the set of discontinuous trajectories of X has idempotent probability 0.

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We define left- and right-continuity notions in an analogous manner.

The following result shows that we do not need to distinguish between continuity (right- or left-continuity, respectively) Π-a.e. and path continuity (right- or left-continuity, respectively) Π-a.e.

Lemma 2.2.9. *An idempotent process X on Ω has continuous (right- or left-continuous, respectively) paths* Π*-a.e. if and only if it is continuous (right- or left-continuous, respectively)* Π*-a.e. on* $\mathbb{R}_+$.

Proof. We give a proof for continuity. Let X be continuous Π-a.e. at every $s \in \mathbb{R}_+$. Then

$$\Pi\big(\omega : (X_t(\omega), t \in \mathbb{R}_+) \text{ is not continuous}\big)$$
$$= \Pi\Big(\bigcup_{s \in \mathbb{R}_+} \{\omega : (X_t(\omega), t \in \mathbb{R}_+) \text{ is not continuous at } s\}\Big)$$
$$= \sup_{s \in \mathbb{R}_+} \Pi\big(\omega : (X_t(\omega), t \in \mathbb{R}_+) \text{ is not continuous at } s\big) = 0,$$

so X has continuous paths Π-a.e. The converse follows by definition. □

Remark 2.2.10. *Since convergence in idempotent probability is stronger than convergence* Π*-a.e., a process can be continuous* Π*-a.e. but not continuous in idempotent probability. For instance, let* $\Omega = [0,1]$, $\Pi(\omega) = 1$ *for all* ω, $X_t(\omega) = 0$, $\omega \geq 2t$, $X_t(\omega) = \omega/t$, $0 \leq \omega \leq t$, *and* $X_t(\omega) = 2 - \omega/t$, $t \leq \omega \leq 2t$ *if* $t > 0$, *and* $X_0(\omega) = 0$. *Then* X_t *is continuous at* $t = 0$ *for every* $\omega \in \Omega$ *but it is not continuous in idempotent probability.*

In the sequel we refer to idempotent processes with continuous (right-continuous, respectively) paths as continuous (right-continuous, respectively) idempotent processes. The following simple fact is useful.

Lemma 2.2.11. *If* X_0 *is a proper idempotent variable and* X *is continuous in idempotent probability, then* X *is a proper idempotent process.*

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We now introduce the class of Luzin-continuous processes, which is smaller than the class of processes continuous in idempotent probability. Let $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}_+$ equipped with the metric

$$d_C(\mathbf{x}, \mathbf{y}) = \sup_{t \in \mathbb{R}_+} \frac{\sup_{s \le t} |\mathbf{x}_s - \mathbf{y}_s| \wedge 1}{1 + t},$$

where $\mathbf{x} = (\mathbf{x}_t,\, t \in \mathbb{R}_+)$ and $\mathbf{y} = (\mathbf{y}_t,\, t \in \mathbb{R}_+)$. We recall that with this metric $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ is a complete separable metric space.

Definition 2.2.12. *We say that an idempotent process X with continuous trajectories is Luzin-continuous if the restriction of Π^X to $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ is a deviability on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$. Let, in addition, Ω be a Hausdorff topological space and Π be a deviability on Ω. An idempotent process X is called a strictly Luzin-continuous idempotent process on (Ω, Π) if the mapping $\omega \to (X_t(\omega),\, t \in \mathbb{R}_+)$ from Ω to $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ is a strictly Luzin idempotent variable.*

Theorem 2.2.13. *A Luzin (respectively, strictly Luzin) idempotent process X is Luzin-continuous (respectively, strictly Luzin-continuous) if and only if for arbitrary $T > 0$ and $\eta > 0$*

$$\lim_{\delta \to 0} \sup_{\substack{s,t \in [0,T]: \\ |s-t| \le \delta}} \Pi\big(|X_t - X_s| > \eta\big) = 0.$$

Proof. According to Corollary 1.8.7 a Luzin idempotent process X is Luzin-continuous if and only if $\inf_{K \in \mathcal{K}} \Pi(X \notin K) = 0$, where $\mathcal{K}$ is the class of compact subsets of $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$. By a standard argument based on Arzelá-Ascoli's theorem this is equivalent to the convergences

$$\lim_{A \to \infty} \Pi\big(|X_0| > A\big) = 0, \quad \lim_{\delta \to 0} \Pi\big(\sup_{\substack{s,t \in [0,T]: \\ |s-t| \le \delta}} |X_t - X_s| > \eta\big) = 0,$$

where $T > 0$ and $\eta > 0$ are arbitrary. The first condition is a consequence of X_0 being Luzin. The second condition is equivalent to the condition in the statement by τ-maxitivity of Π.

The proof for strictly Luzin idempotent processes is similar. ☐

We now consider measurability issues in the spirit of the general theory of stochastic processes. We assume the discrete τ-algebra on

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$\mathbb{R}^d$ unless otherwise specified. Let $\mathbf{A} = (\mathcal{A}_t, t \in \mathbb{R}_+)$ be a flow of τ-algebras on Ω.

Definition 2.2.14. *We say that an idempotent process X is $\mathbf{A}$-adapted if X_t is $\mathcal{A}_t$-measurable for every $t \in \mathbb{R}_+$.*

Let $\overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t$ be the product of the Lebesgue σ-algebra on $[0,t]$ and the τ-algebra $\mathcal{A}_t$ in the sense of Definition 1.5.9. We refer to elements of $\overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t$ as progressively measurable sets.

Definition 2.2.15. *An idempotent process X is said to be progressively measurable (or $\mathbf{A}$-progressively measurable) if the mappings $(s,\omega) \to X(s,\omega)$ from $[0,t] \times \Omega$ to $\mathbb{R}^d$ are $\overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t/\mathcal{B}(\mathbb{R}^d)$-measurable for all $t \in \mathbb{R}_+$.*

Lemma 2.2.16. *An idempotent process $X = (X_t(\omega),\, t \in \mathbb{R}_+, \omega \in \Omega)$ is $\mathbf{A}$-progressively measurable if and only if it is $\mathbf{A}$-adapted and the functions $(X_t(\omega),\, t \in \mathbb{R}_+)$ are $\overline{\mathcal{B}}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R}^d)$-measurable in t for all $\omega \in \Omega$.*

Proof. Let X be $\mathbf{A}$-progressively measurable. Then, given $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, we have $\{(s,\omega) : X_s(\omega) = x, s \in [0,t]\} \in \overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t$ so that $\{t\} \times \{\omega : X_t(\omega) = x\} \in \overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t$, which implies that $\{\omega : X_t(\omega) = x\} \in \mathcal{A}_t$. Thus, X is $\mathbf{A}$-adapted. Next, given $\hat{\omega}$ and a Borel subset Γ of $\mathbb{R}^d$, we have that $\{(s,\omega) : X_s(\omega) \in \Gamma\} \cap \big([0,t] \times [\hat{\omega}]_{\mathcal{A}_t}\big) \in \overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t$. Since $X_s(\omega) = X_s(\hat{\omega})$ for $s \in [0,t]$ if $\omega \in [\hat{\omega}]_{\mathcal{A}_t}$ by $\mathbf{A}$-adaptedness of X and Corollary 1.2.5, we conclude that $\{s \in [0,t] : X_s(\hat{\omega}) \in \Gamma\} \times [\hat{\omega}]_{\mathcal{A}_t} \in \overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t$ so that $\{s \in [0,t] : X_s(\hat{\omega}) \in \Gamma\} \in \overline{\mathcal{B}}\big([0,t]\big)$. Necessity is proved.

Conversely, since

$$\{(s,\omega) : X_s(\omega) \in \Gamma\} \bigcap \big([0,t] \times \Omega\big) = \bigcup_{\omega \in \Omega} \Big(\big(\{s : X_s(\omega) \in \Gamma\} \bigcap [0,t]\big) \times \{\omega\} \Big),$$

$\{s : X_s(\omega) \in \Gamma\} \cap [0,t] \in \overline{\mathcal{B}}\big([0,t]\big)$ for $\omega \in \Omega$, and $X_s(\omega) = X_s(\omega')$ if $s \in [0,t]$ and $\omega \overset{\mathcal{A}_t}{\sim} \omega'$, the definition of $\overline{\mathcal{B}}\big([0,t]\big) \otimes \mathcal{A}_t$ implies sufficiency. $\square$

Lemma 2.2.17. *Let X be an $\mathbf{A}$-progressively measurable idempotent process. Let the integrals $\int_0^t X_s(\omega)\, ds$, $t \in \mathbb{R}_+, \omega \in \Omega$, be well defined. Then the idempotent process $\big(\int_0^t X_s(\omega)\, ds,\, t \in \mathbb{R}_+\big)$ is $\mathbf{A}$-adapted.*

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Proof. Since X is $\mathbf{A}$-adapted, $\int_0^t X_s(\omega)\,ds$ is constant on the atoms of $\mathcal{A}_t$, so it is $\mathcal{A}_t$-measurable by Corollary 1.2.5. □

The next result is a version of Lemma 2.1.18.

Lemma 2.2.18. *Let X be $\mathbf{A}$-adapted and $D \subset \mathbb{R}^d$. Let $\tau_D = \inf\{t \in \mathbb{R}_+ : X_t \in D\}$. Let either one of the conditions hold:*

1. *X is right-continuous and D is closed,*
2. *$\mathbf{A}$ is right-continuous.*

Then τ_D is an $\mathbf{A}$-stopping time.

Proof. Under the first condition, we have by right-continuity of X and closedness of D

$$\{\omega : \tau_D(\omega) \le t\} = \bigcup_{s \le t} \{\omega : X_s(\omega) \in D\} \in \mathcal{A}_t.$$

The second part follows by Lemma 2.1.18. We can also adapt the proof of the lemma by writing

$$\{\omega : \tau_D(\omega) < t\} = \bigcup_{s < t} \{\omega : X_s(\omega) \in D\} \in \mathcal{A}_t.$$

□

Lemma 2.2.19. *Let X be $\mathbf{A}$-adapted and σ be a finite $\mathbf{A}$-stopping time. Then X_σ is $\mathcal{A}_\sigma$-measurable.*

Proof. We have for $x \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$

$$\begin{aligned}\{\omega : X_{\sigma(\omega)}(\omega) = x\} \bigcap \{\omega : \sigma(\omega) = s\} \\ = \{\omega : X_s(\omega) = x\} \bigcap \{\omega : \sigma(\omega) = s\} \in \mathcal{A}_s,\end{aligned}$$

and the required follows by Remark 2.1.10. □

Let $\tau(\xi_j,\, j \in J)$ denote the τ-algebra generated by $\mathbb{R}^d$-valued idempotent variables ξ_j, $j \in J$.

Definition 2.2.20. *We call the τ-flow $\mathbf{A}^X = (\mathcal{A}_s^X,\, s \in \mathbb{R}_+)$ defined by $A_s^X = \tau(X_t,\, t \in [0,s])$ the natural τ-flow associated with X. We also let $\mathcal{A}_\infty^X = \tau(X_t,\, t \in \mathbb{R}_+)$.*

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Lemma 2.2.21. *1. A function $\sigma : \Omega \to \mathbb{R}_+$ is an $\mathbf{A}^X$-stopping time if and only if $\sigma(\omega) = \sigma(\omega')$ for all ω and ω' such that $X_s(\omega) = X_s(\omega')$, $s \le \sigma(\omega)$.*

2. Let σ be an $\mathbf{A}^X$-stopping time. Then $\mathcal{A}_\sigma^X = \tau(X_{s\wedge\sigma},\, s \in \mathbb{R}_+)$.

Proof. We prove part 1. To check necessity, we note that

$$[\omega]_{\mathcal{A}^X_{\sigma(\omega)}} = \{\omega' \in \Omega : X_s(\omega') = X_s(\omega), s \le \sigma(\omega)\} \tag{2.2.1}$$

so that the equality $X_s(\omega) = X_s(\omega')$, $s \le \sigma(\omega)$, is equivalent to the inclusion $\omega' \in [\omega]_{\mathcal{A}^X_{\sigma(\omega)}}$ and the required follows by Lemma 2.1.6. By Corollary 1.1.16 sufficiency will follow if we prove that $[\omega]_{\mathcal{A}_t^X} \in \{\sigma \le t\}$ whenever $\omega \in \{\sigma \le t\}$. Let $\omega' \in [\omega]_{\mathcal{A}_t^X}$. Since $\omega \in \{\sigma \le t\}$, we have that $[\omega]_{\mathcal{A}_t^X} \subset [\omega]_{\mathcal{A}^X_{\sigma(\omega)}}$, so by (2.2.1) $X_s(\omega') = X_s(\omega)$, $s \le \sigma(\omega)$, and by hypotheses $\sigma(\omega') = \sigma(\omega)$, implying that $\sigma(\omega') \le t$.

We prove part 2. The inclusion $\mathcal{A}_\sigma^X \supset \tau(X_{s\wedge\sigma},\, s \in \mathbb{R}_+)$ follows since by Lemma 2.2.19 $X_{s\wedge\sigma}$ is $\mathcal{A}^X_{s\wedge\sigma}$-measurable, hence, it is $\mathcal{A}^X_\sigma$-measurable. To prove the reverse inclusion, we use the fact that the atoms of $\mathcal{A}_\sigma^X$ are the sets $[\omega]_{\mathcal{A}_\sigma^X(\omega)}$, $\omega \in \Omega$. Since by Lemma 2.1.6 $\sigma(\omega') = \sigma(\omega)$ for all $\omega' \in [\omega]_{\mathcal{A}^X_{\sigma(\omega)}}$, we have by (2.2.1) that $[\omega]_{\mathcal{A}^X_{\sigma(\omega)}} = \{\omega' \in \Omega : X_{s\wedge\sigma(\omega')}(\omega') = X_{s\wedge\sigma(\omega)}(\omega), s \in \mathbb{R}_+\}$, which set belongs to $\tau(X_{s\wedge\sigma},\, s \in \mathbb{R}_+)$. □

We introduce two more classes of idempotent processes, which are considered in the sequel. Let $p_t\mathbf{x} = (\mathbf{x}_{s\wedge t},\, s \in \mathbb{R}_+)$ and $\theta_t\mathbf{x} = (\mathbf{x}_{t+s} - \mathbf{x}_t,\, s \in \mathbb{R}_+)$ for $\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{R}_+}$ and $t \in \mathbb{R}_+$.

Definition 2.2.22. *We say that an idempotent process X has independent increments if for every $0 \le t_1 < t_2 < \ldots < t_k$ the increments $X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent idempotent variables. Given a τ-flow $\mathbf{A} = \{\mathcal{A}_t,\, t \in \mathbb{R}_+)$, we say that an $\mathbf{A}$-adapted idempotent process X has $\mathbf{A}$-independent increments if the idempotent process $\theta_t X = (X_{s+t} - X_t,\, s \in \mathbb{R}_+)$ is independent of $\mathcal{A}_t$ for all $t \in \mathbb{R}_+$.*

Remark 2.2.23. *Clearly, an idempotent process with independent increments has $\mathbf{A}^X$-independent increments.*

The next result follows by the definition of a Luzin idempotent process and deviability.

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Lemma 2.2.24. *1. A Luzin idempotent process X on (Ω, Π) has independent increments if and only if for $\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{R}_+}$, $\mathbf{x}' \in (\mathbb{R}^d)^{\mathbb{R}_+}$ such that $\Pi(p_t X = \mathbf{x}') > 0$, and $t \in \mathbb{R}_+$*

$$\Pi(\theta_t X = \mathbf{x} | p_t X = \mathbf{x}') = \Pi(\theta_t X = \mathbf{x}).$$

2. An $\mathbf{A}$-adapted Luzin idempotent process X on (Ω, Π) has $\mathbf{A}$-independent increments if and only if for $\mathbf{x} \in (\mathbb{R}^d)^{\mathbb{R}_+}$ and $t \in \mathbb{R}_+$ Π-a.e.

$$\Pi(\theta_t X = \mathbf{x} | \mathcal{A}_t) = \Pi(\theta_t X = \mathbf{x}).$$

Definition 2.2.25. *An idempotent process $X = (X_t,\ t \in \mathbb{R}_+)$ is said to be idempotent Gaussian if its finite-dimensional distributions are idempotent Gaussian.*

The next two theorems consider convergence in idempotent distribution for Luzin-continuous idempotent processes. We state the results in the form of weak convergence of associated deviabilities. The following tightness theorem is an obvious consequence of Arzelà-Ascoli's theorem.

Theorem 2.2.26. *A net $\{\Pi_\phi, \phi \in \Phi\}$ of deviabilities on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ is tight if and only if*

1.° $\lim\limits_{A \to \infty} \limsup\limits_{\phi} \Pi_\phi(\{\mathbf{x} : |\mathbf{x}_0| > A\}) = 0,$

2.° $\lim\limits_{\delta \to 0} \limsup\limits_{\phi} \Pi_\phi(\{\mathbf{x} : \sup\limits_{\substack{s,t \in [0,T]: \\ |s-t| < \delta}} |\mathbf{x}_t - \mathbf{x}_s| > \eta\}) = 0,\ \eta > 0, T > 0.$

Now we establish connection between convergence of deviabilities on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ and their finite dimensional projections. Given $0 \le t_1 < t_2 < \ldots < t_k$, we denote $\pi_{t_1 \ldots t_k} \mathbf{x} = \pi_{\{t_1 \ldots t_k\}} \mathbf{x} = (\mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_k})$, $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$.

Theorem 2.2.27. *Let $\{\Pi_\phi, \phi \in \Phi\}$ be a tight net of deviabilities on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$. If $\Pi_\phi \circ \pi^{-1}_{t_1,\ldots,t_k} \stackrel{iw}{\to} \Pi_{t_1 \ldots t_k}$ for $t_1, \ldots, t_k$ from a dense subset U of $\mathbb{R}_+$, where $\Pi_{t_1 \ldots t_k}$ are deviabilities on $(\mathbb{R}^d)^k$, then $\Pi_\phi \stackrel{iw}{\to} \Pi$, where Π is the deviability on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ defined by $\Pi(\mathbf{x}) = \inf_{t_1,\ldots,t_k \in U} \Pi_{t_1 \ldots t_k}(\mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_k})$.*

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Proof. Since the net $\{\Pi_\phi, \phi \in \Phi\}$ is tight, it has an accumulation point Π'. By "the continuous mapping theorem" (Corollary 1.10.4) $\Pi_\phi \circ \pi^{-1}_{t_1,\dots,t_k} \overset{iw}{\to} \Pi' \circ \pi^{-1}_{t_1,\dots,t_k}$ so that $\Pi' \circ \pi^{-1}_{t_1,\dots,t_k} = \Pi_{t_1\dots t_k}$. By Theorem 2.2.2 the proof is complete. □

2.3 Exponential maxingales

This section develops an idempotent analogue of martingale theory. Let (Ω, Π) be an idempotent probability space and $\mathbf{A} = \{\mathcal{A}_t, t \in \mathbb{R}_+\}$ be a flow of τ-algebras on Ω.

Definition 2.3.1. *An $\mathbb{R}_+$-valued idempotent process $M = (M_t(\omega),\ t \in \mathbb{R}_+, \omega \in \Omega)$ is an $\mathbf{A}$-exponential maxingale if the functions $\omega \to M_t(\omega)$ are $\mathcal{A}_t$-measurable and maximable for all $t \in \mathbb{R}_+$, and for all $s \le t$ Π-a.e.*

$$S(M_t|\mathcal{A}_s) = M_s. \tag{2.3.1}$$

If, in addition, the collection $\{M_t, t \in \mathbb{R}_+\}$ is uniformly maximable, then M is called a uniformly maximable $\mathbf{A}$-exponential maxingale.

Remark 2.3.2. *We occasionally refer to $\mathbf{A}$-exponential maxingales as exponential maxingales if the τ-flow $\mathbf{A}$ is understood.*

Exponential submaxingales and supermaxingales are defined similarly with the equality sign in (2.3.1) replaced by $\ge$ for exponential submaxingales and by $\le$ for exponential supermaxingales.

The following simple fact shows that uniform maximability is quite a strong property.

Lemma 2.3.3. *Let $X = (X_t, t \in \mathbb{R}_+)$, where $X_t : \Omega \to \mathbb{R}_+$, be $\mathbf{A}$-adapted. The collection $\{X_t, t \in \mathbb{R}_+\}$ is uniformly maximable if and only if the collection $\{X_g\}$, where g ranges in the set of all $\mathbb{R}_+$-valued functions on Ω, is uniformly maximable.*

Proof. Since $X_g \le \sup_t X_t$, the property follows by Theorem 1.4.13. □

The next property is a consequence of the definitions.

Lemma 2.3.4. *1. Let M be an exponential submaxingale and $p \ge 1$. If the M_t^p are maximable, then $(M_t^p, t \in \mathbb{R}_+)$ is an exponential submaxingale.*

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2. *Let M be an exponential supermaxingale and $0 \le p \le 1$. Then $(M_t^p, t \in \mathbb{R}_+)$ is an exponential supermaxingale.*

3. *Let M and N be exponential supermaxingales (exponential submaxingales or exponential maxingales, respectively). Then $M \vee N$ is an exponential supermaxingale (exponential submaxingale or exponential maxingale, respectively) and $M \wedge N$ is an exponential supermaxingale.*

The following characterisations of exponential maxingales are implied by Lemma 1.6.24 and Lemma 1.6.37.

Lemma 2.3.5. *Let $M = (M_t(\omega),\ t \in \mathbb{R}_+, \omega \in \Omega)$ be an $\mathbb{R}_+$-valued adapted idempotent process such that the $(M_t(\omega),\ \omega \in \Omega)$ are maximable.*

1. *The idempotent process M is an $\mathbf{A}$-exponential supermaxingale (respectively, an $\mathbf{A}$-exponential submaxingale or an $\mathbf{A}$-exponential maxingale) if and only if, for $s \le t$,*

$$S(M_t h) \le S(M_s h) \ \textit{(respectively, } S(M_t h) \ge S(M_s h) \ \textit{or } S(M_t h) = S(M_s h)\textit{)}$$

   for all $\mathcal{A}_s$-measurable functions $h : \Omega \to \mathbb{R}_+$.

2. *Let Π be τ-smooth relative to a collection $\mathcal{E}$ and have a tightening collection $\mathcal{T}$. Let the M_s, $s \in \mathbb{R}_+$, be Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable. Let $\mathcal{H}_s$ for $s \in \mathbb{R}_+$ be collections of Luzin $(\mathcal{E}, \mathcal{T})/\mathcal{U}$-measurable $\mathbb{R}_+$-valued functions on Ω, which contain the zero function, are closed under multiplication by nonnegative scalars and the formation of maximums and minimums, and are such that if $h \in \mathcal{H}_s$, then $(h-1) \vee 0 \in \mathcal{H}_s$ and $(1-h) \vee 0 \in \mathcal{H}_s$. Let also the τ-algebras generated by the elements of the $\mathcal{H}_s$ coincide with the $\mathcal{A}_s$. Then the assertion of part 1 holds if h only ranges in the collection $\mathcal{H}_s$.*

The following lemma shows that "likelihood ratios" are exponential maxingales. Let Π' be an idempotent probability on Ω, and Π_t and Π'_t denote the respective restrictions of Π and Π' to $\mathcal{A}_t$.

Lemma 2.3.6. *Let Π'_t be absolutely continuous with respect to Π_t for all $t \in \mathbb{R}_+$. Then the idempotent process $(d\Pi'_t/d\Pi_t,\ t \in \mathbb{R}_+)$ is an $\mathbf{A}$-exponential maxingale under Π.*

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Proof. Let $M_t = d\Pi'_t/d\Pi_t$, which is well-defined, $\mathcal{A}_t$-measurable and Π-maximable by Theorem 1.6.34. Let $s \le t$ and $A \in \mathcal{A}_s$. Then the maxingale property of M follows by the equalities

$$S_\Pi(M_t\,\mathbf{1}(A)) = S_{\Pi_t}(M_t\,\mathbf{1}(A)) = S_{\Pi'_t}(\,\mathbf{1}(A)) = S_{\Pi'_s}(\,\mathbf{1}(A))$$
$$= S_{\Pi_s}(M_s\,\mathbf{1}(A)) = S_\Pi(M_s\,\mathbf{1}(A)).$$

□

Our next goal is to prove an analogue of Doob's stopping theorem. We precede it with a lemma.

Lemma 2.3.7. *Let $f : \Omega \to \mathbb{R}_+$. Let σ and τ be finite* $\mathbf{A}$*-stopping times. Then $S(f|\mathcal{A}_\sigma) = S(f|\mathcal{A}_\tau)$ Π-a.e. on the set $\{\sigma = \tau\}$.*

Proof. It suffices to prove that $\Pi(\omega'|\mathcal{A}_\sigma) = \Pi(\omega'|\mathcal{A}_\tau)$ Π-a.e. on $\{\sigma = \tau\}$ for every $\omega' \in \Omega$. Let $\omega \in \Omega$ be such that $\Pi(\omega) > 0$ and $\sigma(\omega) = \tau(\omega)$. Then in view of Definition 2.1.8

$$\Pi(\omega'|\mathcal{A}_\sigma)(\omega) = \frac{\Pi(\omega')}{\Pi\big([\omega]_{\mathcal{A}_\sigma}\big)}\,\mathbf{1}\big(\omega' \in [\omega]_{\mathcal{A}_\sigma}\big)$$
$$= \frac{\Pi(\omega')}{\Pi\big([\omega]_{\mathcal{A}_{\sigma(\omega)}}\big)}\,\mathbf{1}\big(\omega' \in [\omega]_{\mathcal{A}_{\sigma(\omega)}}\big) = \frac{\Pi(\omega')}{\Pi\big([\omega]_{\mathcal{A}_{\tau(\omega)}}\big)}\,\mathbf{1}\big(\omega' \in [\omega]_{\mathcal{A}_{\tau(\omega)}}\big)$$
$$= \frac{\Pi(\omega')}{\Pi\big([\omega]_{\mathcal{A}_\tau}\big)}\,\mathbf{1}\big(\omega' \in [\omega]_{\mathcal{A}_\tau}\big) = \Pi(\omega'|\mathcal{A}_\tau)(\omega).$$

□

Let us say that an $\mathbb{R}_+$-valued idempotent process M on $(\Omega, \mathbf{A}, \Pi)$ is stopping-time-right-continuous in idempotent probability if $M_{\tau+\delta} - M_\tau \xrightarrow{\Pi} 0$ as $\delta \to 0$ for every finite $\mathbf{A}$-stopping time τ.

Theorem 2.3.8. *("the Doob stopping theorem")*

1. *Let $M = (M_t(\omega),\, t \in \mathbb{R}_+, \omega \in \Omega)$ be a right-continuous* $\mathbf{A}$*-exponential supermaxingale. If σ and τ are finite* $\mathbf{A}$*-stopping times, then (Π–a.e.)*

$$S(M_\tau|\mathcal{A}_\sigma) \le M_{\tau\wedge\sigma}.$$

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2. *Let* $M = (M_t(\omega), t \in \mathbb{R}_+, \omega \in \Omega)$ *be a right-continuous* $\mathbf{A}$-*exponential submaxingale and* σ *and* τ *be finite* $\mathbf{A}$-*stopping times. Let at least one of the following conditions hold:*

   (a) τ *assumes a finite number of values,*

   (b) M *is stopping-time-right-continuous in idempotent probability and* τ *is bounded,*

   (c) M *is stopping-time-right-continuous in idempotent probability and uniformly maximable, and* τ *is a proper idempotent variable.*

   Then (Π–*a.e.*)

$$S(M_\tau|\mathcal{A}_\sigma) \geq M_{\tau\wedge\sigma}.$$

3. *Let* $M = (M_t(\omega), t \in \mathbb{R}_+, \omega \in \Omega)$ *be a right-continuous* $\mathbf{A}$-*exponential maxingale and* σ *and* τ *be finite* $\mathbf{A}$-*stopping times. If either* τ *is bounded or* M *is uniformly maximable and* τ *is a proper idempotent variable, then* (Π–*a.e.*)

$$S(M_\tau|\mathcal{A}_\sigma) = M_{\tau\wedge\sigma}.$$

Proof. We routinely omit indications that certain relations hold Π-a.e., where the latter is understood. We begin with the case of exponential supermaxingales and assume, at first, that τ and σ take on a finite number of values:

$$\tau = \max_{i=1,\dots,k} a_i \, \mathbf{1}(A_i), \quad \sigma = \max_{j=1,\dots,l} b_j \, \mathbf{1}(B_j), \tag{2.3.2}$$

where $0 \leq a_1 < a_2 < \ldots < a_k$, $0 \leq b_1 < b_2 < \ldots < b_l$, $A_i \cap A_{i'} = B_j \cap B_{j'} = \emptyset$ unless either $i = i'$ or $j = j'$, $A_i \in \mathcal{A}_{a_i}, B_j \in \mathcal{A}_{b_j}$, $\cup_{i=1}^k A_i = \cup_{j=1}^l B_j = \Omega$. We assume furthermore that $\tau \geq \sigma$ so that $a_i \geq b_j$ if $A_i \cap B_j \neq \emptyset$.

By Lemma 1.6.21, Lemma 2.3.7 and (2.3.2), we have, introducing $i(j) = \min\{i = 1, \dots, k : a_i \geq b_j\}$ and defining $\max_\emptyset = 0$,

$$\begin{aligned}
S(M_\tau|\mathcal{A}_\sigma) &= \max_{j=1,\dots,l} S(M_\tau|\mathcal{A}_\sigma)\,\mathbf{1}(B_j) \\
&= \max_{j=1,\dots,l} S(M_\tau|\mathcal{A}_{b_j})\,\mathbf{1}(B_j) = \max_{j=1,\dots,l} S(M_\tau\,\mathbf{1}(B_j)|\mathcal{A}_{b_j}) \\
&= \max_{j=1,\dots,l} S(\max_{i=i(j),\dots,k} M_{a_i}\,\mathbf{1}(A_i)\,\mathbf{1}(B_j)|\mathcal{A}_{b_j}) \\
&= \max_{j=1,\dots,l}\max_{i=i(j),\dots,k} S(M_{a_i}\,\mathbf{1}(A_i)|\mathcal{A}_{b_j})\,\mathbf{1}(B_j).
\end{aligned} \tag{2.3.3}$$

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Now, $A_k = \big(\cup_{i=1}^{k-1} A_i\big)^c \in \mathcal{A}_{a_{k-1}}$, hence, if $i(j) \le k-1$, by the supermaxingale property

$$S(M_{a_k}\,\mathbf{1}(A_k)|\mathcal{A}_{b_j}) = S\big(S(M_{a_k}\,\mathbf{1}(A_k)|\mathcal{A}_{a_{k-1}})|\mathcal{A}_{b_j}\big)$$
$$= S\big(S(M_{a_k}|\mathcal{A}_{a_{k-1}})\,\mathbf{1}(A_k)|\mathcal{A}_{b_j}\big) \le S(M_{a_{k-1}}\,\mathbf{1}(A_k)|\mathcal{A}_{b_j}),$$

so

$$\max_{i=i(j),\dots,k} S(M_{a_i}\,\mathbf{1}(A_i)|\mathcal{A}_{b_j}) \le \max_{i=i(j),\dots,k-2} S(M_{a_i}\,\mathbf{1}(A_i)|\mathcal{A}_{b_j})$$
$$\vee\, S\big(M_{a_{k-1}}\,\mathbf{1}\big(A_k \bigcup A_{k-1}\big)|\mathcal{A}_{b_j}\big).$$

Next, $A_k \cup A_{k-1} = \big(\cup_{i=1}^{k-2} A_i\big)^c \in \mathcal{A}_{a_{k-2}}$, hence, in analogy with the above reasoning, if $i(j) \le k-2$,

$$S\big(M_{a_{k-1}}\,\mathbf{1}\big(A_k \bigcup A_{k-1}\big)|\mathcal{A}_{b_j}\big) \le S\big(M_{a_{k-2}}\,\mathbf{1}\big(A_k \bigcup A_{k-1}\big)|\mathcal{A}_{b_j}\big).$$

Carrying on like this, we obtain

$$\max_{i=i(j),\dots,k} S(M_{a_i}\,\mathbf{1}(A_i)|\mathcal{A}_{b_j}) \le S\big(M_{a_{i(j)}}\,\mathbf{1}\big(\bigcup_{i=i(j)}^{k} A_i\big)|\mathcal{A}_{b_j}\big)$$
$$\le M_{b_j}\,\mathbf{1}\big(\bigcup_{i=i(j)}^{k} A_i\big)$$

(for the latter inequality we used the fact that $\cup_{i=i(j)}^{k} A_i = \big(\cup_{i=1}^{i(j)-1} A_i\big)^c \in \mathcal{A}_{a_{i(j)-1}} \subset \mathcal{A}_{b_j}$). Thus, by (2.3.3)

$$S(M_\tau|\mathcal{A}_\sigma) \le \max_{j=1,\dots,l} M_{b_j}\,\mathbf{1}\big(\bigcup_{i=i(j)}^{k} A_i\big)\,\mathbf{1}(B_j)$$
$$= \max_{j=1,\dots,l} M_{b_j}\,\mathbf{1}(\tau \ge b_j)\,\mathbf{1}(\sigma = b_j) = M_\sigma\,\mathbf{1}(\tau \ge \sigma) = M_\sigma.$$

Hence, we have proved the required if τ and σ take on finitely many values and $\tau \ge \sigma$.

Now dropping the assumption that $\tau \ge \sigma$, we write, by Lemma 1.6.21, Lemma 2.1.11 and the part just proved

$$S(M_\tau|\mathcal{A}_\sigma) = S\big(M_\tau\,\mathbf{1}(\tau \ge \sigma)|\mathcal{A}_\sigma\big) \vee S\big(M_\tau\,\mathbf{1}(\tau < \sigma)|\mathcal{A}_\sigma\big)$$
$$= S(M_{\tau\vee\sigma}|\mathcal{A}_\sigma)\,\mathbf{1}(\tau \ge \sigma) \vee S(M_{\tau\wedge\sigma}|\mathcal{A}_\sigma)\,\mathbf{1}(\tau < \sigma)$$
$$\le M_\sigma\,\mathbf{1}(\tau \ge \sigma) \vee M_\tau\,\mathbf{1}(\tau < \sigma) = M_{\tau\wedge\sigma}. \quad (2.3.4)$$

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Let us assume now that τ takes on a finite number of values and σ is bounded. Let

$$\sigma_n = \min_{l=1,\dots,n} \frac{\lfloor l\sigma \rfloor + 1}{l}, \; n \in \mathbb{N}. \tag{2.3.5}$$

By Lemma 2.1.11 and Lemma 2.1.6 the σ_n are **A**-stopping times and, since they take on a finite number of values, by the part already proved $S(M_\tau|\mathcal{A}_{\sigma_n}) \le M_{\tau\wedge\sigma_n}$. Since M is right-continuous, $M_{\tau\wedge\sigma_n} \to M_{\tau\wedge\sigma}$ as $n \to \infty$. Also, by "Lévy" (part 1 of Lemma 1.6.23) $\lim_n S(M_\tau|\mathcal{A}_{\sigma_n}) = S(M_\tau| \cap_n \mathcal{A}_{\sigma_n})$ so that $S(M_\tau| \cap_n \mathcal{A}_{\sigma_n}) \le M_{\tau\wedge\sigma}$, which after conditioning on $\mathcal{A}_\sigma \subset \cap_n \mathcal{A}_{\sigma_n}$ yields $S(M_\tau|\mathcal{A}_\sigma) \le M_{\tau\wedge\sigma}$.

Let us consider now the case of general bounded τ and σ. We introduce in analogy with the σ_n

$$\tau_n = \min_{l=1,\dots,n} \frac{\lfloor l\tau \rfloor + 1}{l}, \; n \in \mathbb{N}. \tag{2.3.6}$$

Then the τ_n are **A**-stopping times, which assume a finite number of values. Therefore, by the part already proved $S(M_{\tau_n}|\mathcal{A}_\sigma) \le M_{\sigma\wedge\tau_n}$. The limit of the right-hand side as $n \to \infty$ is $M_{\sigma\wedge\tau}$. For the left-hand side by "Fatou" (part 1 of Lemma 1.6.22) $\liminf_n S(M_{\tau_n}|\mathcal{A}_\sigma) \ge S(M_\tau|\mathcal{A}_\sigma)$, proving the required inequality.

Let us assume now that σ and τ are arbitrary finite **A**-stopping times. Then since $\sigma \wedge n$ and $\tau \wedge m$, where $n, m \in \mathbb{N}$, are bounded, we have that $S(M_{\tau\wedge n}|\mathcal{A}_{\sigma\wedge m}) \le M_{\tau\wedge\sigma\wedge n\wedge m}$. Letting $n \to \infty$ yields by "Fatou" the inequality $S(M_\tau|\mathcal{A}_{\sigma\wedge m}) \le M_{\tau\wedge\sigma\wedge m}$. Therefore, in view of Lemma 2.3.7,

$$\begin{aligned} S(M_\tau|\mathcal{A}_\sigma) &= \big(S(M_\tau|\mathcal{A}_{\sigma\wedge m})\,\mathbf{1}(\sigma \le m)\big) \vee \big(S(M_\tau|\mathcal{A}_\sigma)\,\mathbf{1}(\sigma > m)\big) \\ &\le M_{\tau\wedge\sigma\wedge m} \vee \big(S(M_\tau|\mathcal{A}_\sigma)\,\mathbf{1}(\sigma > m)\big). \end{aligned} \tag{2.3.7}$$

The required follows since $S(M_\tau|\mathcal{A}_\sigma)\,\mathbf{1}(\sigma > m) = 0$ and $M_{\tau\wedge\sigma\wedge m} = M_{\tau\wedge\sigma}$ if m is large enough.

Now, let M be an exponential submaxingale. Let us show that if τ is such that the inequality $S(M_\tau|\mathcal{A}_\sigma) \ge M_{\tau\wedge\sigma}$ holds for all bounded σ, then it holds for arbitrary finite σ. Let σ be an arbitrary finite **A**-stopping time. Then for $n \in \mathbb{N}$ by hypothesis $S(M_\tau|\mathcal{A}_{\sigma\wedge n}) \ge M_{\tau\wedge\sigma\wedge n}$. Since by the equality in (2.3.7) we have that $S(M_\tau|\mathcal{A}_\sigma) \ge S(M_\tau|\mathcal{A}_{\sigma\wedge n})\,\mathbf{1}(\sigma \le n)$, it follows that $S(M_\tau|\mathcal{A}_\sigma) \ge M_{\sigma\wedge n\wedge\tau}\,\mathbf{1}(\sigma \le n)$; letting $n \to \infty$ proves the claim.

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Thus, we can assume that σ is a bounded $\mathbf{A}$-stopping time. An argument symmetric to the one used for exponential supermaxingales checks the required inequality for σ and τ taking on a finite number of values. Next, let σ be an arbitrary bounded $\mathbf{A}$-stopping time, τ assume a finite number of values, and σ_n be defined by (2.3.5). Then $S(M_\tau|\mathcal{A}_{\sigma_n}) \geq M_{\tau\wedge\sigma_n}$ for $n \in \mathbb{N}$. Letting $n \to \infty$ implies by "Lévy" (Lemma 1.6.23) that $S\big(M_\tau|\cap_{n=1}^\infty \mathcal{A}_{\sigma_n}\big) \geq M_{\tau\wedge\sigma}$; conditioning on $\mathcal{A}_\sigma \subset \cap_{n=1}^\infty \mathcal{A}_{\sigma_n}$ provides us with the required property under hypothesis (a).

Let M be stopping-time-right-continuous in idempotent probability, and τ and σ be bounded $\mathbf{A}$-stopping times. Let τ_n be defined by (2.3.6). Then by the part proved

$$S(M_{\tau_n}|\mathcal{A}_\sigma) \geq M_{\tau_n\wedge\sigma}. \tag{2.3.8}$$

We now show that the collection $\{M_{\tau_n},\, n \in \mathbb{N}\}$ is uniformly maximable. Let $T \in \mathbb{R}_+$ be an upper bound for τ. By the case just considered we have that $S(M_{T+1}|\mathcal{A}_{\tau_n}) \geq M_{\tau_n}$ so that for $a > 0$ we can write

$$\begin{aligned} S\big(M_{\tau_n}\,\mathbf{1}(M_{\tau_n} \geq a)\big) &\leq S\big(S(M_{T+1}|\mathcal{A}_{\tau_n})\,\mathbf{1}(M_{\tau_n} \geq a)\big) \\ &= S\big(M_{T+1}\,\mathbf{1}(M_{\tau_n} \geq a)\big). \end{aligned} \tag{2.3.9}$$

Also "by Chebyshev" $\Pi(M_{\tau_n} \geq a) \leq S(M_{\tau_n})/a \leq SM_{T+1}/a$ so $\lim_{a\to\infty}\sup_n \Pi(M_{\tau_n} \geq a) = 0$, and by (2.3.9), maximability of M_{T+1} and Corollary 1.4.12 the collection $\{M_{\tau_n},\, n \in \mathbb{N}\}$ is uniformly maximable. Since M is stopping-time-right-continuous in idempotent probability and $\tau_n \to \tau$ as $n \to \infty$ uniformly over $\omega \in \Omega$, we conclude that $M_{\tau_n} \to M_\tau$ in idempotent probability, so by Lemma 1.6.22 $S(M_{\tau_n}|\mathcal{A}_\sigma) \to S(M_\tau|\mathcal{A}_\sigma)$ as $n \to \infty$ in idempotent probability and hence Π-a.e. Therefore, by (2.3.8) $S(M_\tau|\mathcal{A}_\sigma) \geq M_{\tau\wedge\sigma}$.

Let M be stopping-time-right-continuous in idempotent probability and uniformly maximable, τ be a finite $\mathbf{A}$-stopping time, which is a proper idempotent variable, and σ be a bounded $\mathbf{A}$-stopping time. By the case just considered we have that $S(M_{\tau\wedge n}|\mathcal{A}_\sigma) \geq M_{\tau\wedge n\wedge\sigma}$. The sequence $\{M_{\tau\wedge n},\, n \in \mathbb{N}\}$ is uniformly maximable by uniform maximability of M and converges to M_τ in idempotent probability as $n \to \infty$ by the inequality $\Pi(M_{\tau\wedge n} \neq M_\tau) \leq \Pi(\tau > n)$ and the fact that τ is a proper idempotent variable. Therefore, by Lemma 1.6.22

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$S(M_{\tau\wedge n}|\mathcal{A}_\sigma) \to S(M_\tau|\mathcal{A}_\sigma)$ in idempotent probability and hence Π-a.e. completing the proof for the case of M being an exponential submaxingale.

Now let M be an exponential maxingale. Then the preceding results for exponential super- and submaxingales imply that $S(M_\tau|\mathcal{A}_\sigma) = M_{\tau\wedge\sigma}$ if τ assumes a finite number of values and σ is arbitrary. Let τ and σ be bounded $\mathbf{A}$-stopping times such that $\tau \geq \sigma$. Then τ_n defined by (2.3.6) assumes a finite number of values so $S(M_{\tau_n}|\mathcal{A}_\sigma) = M_\sigma$. For the same reason, $S(M_{\tau_n}|\mathcal{A}_\tau) = M_\tau$. Therefore,

$$M_\sigma = S(M_{\tau_n}|\mathcal{A}_\sigma) = S\big(S(M_{\tau_n}|\mathcal{A}_\tau)|\mathcal{A}_\sigma\big) = S(M_\tau|\mathcal{A}_\sigma).$$

Getting rid of the assumption $\tau \geq \sigma$ is carried out by the argument used in (2.3.4), where the inequality sign is replaced by an equality sign. Finally, the proof for M being uniformly maximable and τ being a proper idempotent variable repeats the argument used in the preceding paragraph for exponential submaxingales. □

Remark 2.3.9. *As the proof shows, if M is an exponential submaxingale, then the set $\{M_\tau\}$, where τ ranges in the set of uniformly bounded* $\mathbf{A}$*-stopping times, is uniformly maximable. Also if M is an exponential supermaxingale, then the set $\{M_\tau\}$, where τ ranges in the set of finite* $\mathbf{A}$*-stopping times, is uniformly maximable.*

Corollary 2.3.10. *Let $M = (M_t, t \in \mathbb{R}_+)$ be an $\mathcal{A}$-exponential maxingale. Then for every bounded $\mathcal{A}$-stopping times σ and τ such that $\tau \geq \sigma$*

$$S(M_\tau|\mathcal{A}_\sigma) = M_\sigma.$$

If M is uniformly maximable, the latter equality also holds for proper τ and σ.

Lemma 2.3.11. *1. Let $M = (M_t, t \in \mathbb{R}_+)$ be a right-continuous* $\mathbf{A}$*-exponential supermaxingale. Then for $a > 0$*

$$\Pi(\sup_t M_t \geq a) \leq \frac{SM_0}{a}.$$

2. Let $M = (M_t, t \in \mathbb{R}_+)$ be a right-continuous $\mathbf{A}$*-exponential submaxingale. Then for $a > 0$ and $T > 0$*

$$\Pi(\sup_{t\leq T} M_t \geq a) \leq \frac{SM_T}{a}.$$

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Proof. Let $T > 0$ and $\sigma = \inf\{t \in \mathbb{R}_+ : M_t \geq a\} \wedge T$. The function σ is an $\mathbf{A}$-stopping time by Lemma 2.2.18. If M is an exponential supermaxingale, by Theorem 2.3.8

$$SM_0 \geq SM_\sigma \geq S\big(M_\sigma \mathbf{1}(\sup_{t\leq T} M_t \geq a)\big) \geq a\Pi(\sup_{t\leq T} M_t \geq a).$$

The argument for M being an exponential submaxingale uses the inequality $SM_T \geq SM_\sigma$. □

Definition 2.3.12. *An $\mathbb{R}_+$–valued idempotent process $M = (M_t(\omega),\, t \in \mathbb{R}_+, \omega \in \Omega)$ is called an $\mathbf{A}$-local exponential maxingale if it is $\mathbf{A}$–adapted and there exists a sequence $\{\tau_n,\, n \in \mathbb{N}\}$ of $\mathbf{A}$–stopping times such that $\tau_n(\omega) \leq \tau_{n+1}(\omega)$, $\lim_{n\to\infty} \tau_n(\omega) = \infty$, $\omega \in \Omega$, and each $M^{\tau_n} = (M_{t\wedge\tau_n(\omega)}(\omega),\, t \in \mathbb{R}_+, \omega \in \Omega), n \in \mathbb{N}$, is a uniformly maximable $\mathbf{A}$-exponential maxingale. Any such sequence $\{\tau_n\}$ is called a localising sequence for M.*

Lemma 2.3.13. *Let $M = (M_t(\omega),\, t \in \mathbb{R}_+, \omega \in \Omega)$ be an $\mathbb{R}_+$-valued right-continuous idempotent process.*

1. *If M is an $\mathbf{A}$-exponential maxingale, it is an $\mathbf{A}$-local exponential maxingale.*

2. *If M is an $\mathbf{A}$-local exponential maxingale, then it admits a localising sequence consisting of bounded stopping times.*

3. *If M is an $\mathbf{A}$-local exponential maxingale and each $M_t(\omega), t \in \mathbb{R}_+$, is maximable, then M is an $\mathbf{A}$-exponential supermaxingale. If, in addition, the collection $\{M_t, t \in \mathbb{R}_+\}$ is uniformly maximable, then M is a uniformly maximable exponential maxingale.*

4. *If M is an $\mathbf{A}$-local exponential maxingale and τ is an $\mathbf{A}$-stopping time, then $M^\tau = (M_{t\wedge\tau}(\omega),\, t \in \mathbb{R}_+, \omega \in \Omega)$ is an $\mathbf{A}$-local exponential maxingale as well.*

5. *If M is an $\mathbf{A}$-local exponential maxingale, an increasing sequence $\{\tau_n\}$ of $\mathbf{A}$-stopping times is such that $\tau_n \to \infty$ as $n \to \infty$, and the $M^{\tau_n} = (M_{t\wedge\tau_n}(\omega),\, t \in \mathbb{R}_+, \omega \in \Omega)$ are uniformly maximable, then $\{\tau_n\}$ is a localising sequence for M.*

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Proof. In part 1 $\tau_n = n$ is a localising sequence by Theorem 2.3.8 and Remark 2.3.9. In part 2 if $\{\tau_n\}$ is a localising sequence, then by Theorem 2.3.8 and Remark 2.3.9 $\{\tau_n \wedge n\}$ is a localising sequence of bounded stopping times.

For part 3, let $\{\tau_n\}$ be a localising sequence for M. Then $S(M_{\tau_n \wedge t}|\mathcal{A}_s) = M_{\tau_n \wedge s}$ for $s \le t$. Since, given ω, $M_{\tau_n(\omega)\wedge t}(\omega) = M_t(\omega)$ and $M_{\tau_n(\omega)\wedge s}(\omega) = M_s(\omega)$ for all n large, by Lemma 1.6.22 $\liminf_n S(M_{\tau_n \wedge t}|\mathcal{A}_s) \ge S(M_t|\mathcal{A}_s)$ so that $S(M_t|\mathcal{A}_s) \le M_s$, proving the first claim. If $\{M_t, t \in \mathbb{R}_+\}$ is uniformly maximable, so is $\{M_{\tau_n \wedge t}, n \in \mathbb{N}\}$; hence, by Lemma 1.6.22 $\lim_n S(M_{\tau_n \wedge t}|\mathcal{A}_s) = S(M_t|\mathcal{A}_s)$.

We now prove part 4. Let $\{\tau_n\}$ be a localising sequence for M. By Theorem 2.3.8 $(M_{t\wedge\tau\wedge\tau_n}, t \in \mathbb{R}_+)$ is an **A**-exponential maxingale. It is uniformly maximable by Theorem 1.4.13 and the estimate $\sup_t M_{t\wedge\tau\wedge\tau_n} \le \sup_t M_{t\wedge\tau_n}$.

Part 5 follows by parts 3 and 4. ☐

The next lemma is an analogue of the Lenglart–Rebolledo domination property.

Lemma 2.3.14. *Let X and Y be positive right-continuous idempotent processes such that the idempotent process $(X_t/Y_t,\, t \in \mathbb{R}_+)$ is an* **A***-exponential supermaxingale and $X_0/Y_0 = 1$. Then for arbitrary $a > 0$, $b > 0$ and an* **A***-stopping time σ*

$$\Pi\bigl(\sup_{s\le\sigma} X_s \ge a\bigr) \le \frac{b}{a} \vee \Pi\bigl(\sup_{s\le\sigma} Y_s > b\bigr),$$

where $\sup_{s\le\infty} = \sup_{s\in\mathbb{R}_+}$.

Proof. Let $\tau = \inf\{s \in \mathbb{R}_+ : X_s \ge a\}$. It is an **A**-stopping time by Lemma 2.2.18. Assuming that $\sigma < \infty$ Π-a.e., we have

$$\begin{aligned}
&\Pi\bigl(\sup_{s\le\sigma} X_\sigma > a\bigr) \le \Pi(X_{\sigma\wedge\tau} \ge a)\\
&\le \Pi(Y_{\sigma\wedge\tau} > b) \vee \Pi(X_{\sigma\wedge\tau} \ge a,\, Y_{\sigma\wedge\tau} \le b)\\
&\le \Pi(Y_{\sigma\wedge\tau} > b) \vee \Pi\bigl(X_{\sigma\wedge\tau}/Y_{\sigma\wedge\tau} \ge a/b\bigr) \le \Pi(Y_{\sigma\wedge\tau} > b) \vee \frac{b}{a},
\end{aligned}$$

where the latter inequality follows by the Chebyshev inequality and hypotheses. The required follows since $\Pi\bigl(\sup_{s\le\sigma} X_\sigma \ge a\bigr) = \lim_{\epsilon\downarrow 0} \Pi\bigl(\sup_{s\le\sigma} X_\sigma > a - \epsilon\bigr)$.

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If $\Pi(\sigma = \infty) > 0$, then by the part of the lemma already proved, for arbitrary $\epsilon > 0$,

$$\Pi\big(\sup_{s\le\sigma} X_\sigma \ge a\big) \le \Pi\big(\sup_{s\le\sigma} X_\sigma > a-\epsilon\big)$$
$$= \lim_{M\to\infty} \Pi\big(\sup_{s\le\sigma\wedge M} X_\sigma > a-\epsilon\big) \le \frac{b}{a-\epsilon} \vee \Pi\big(\sup_{s\le\sigma} Y_\sigma > b\big).$$

□

It is often convenient to consider right-continuous τ-flows. The following simple observation shows that there is no loss of generality in assuming that the flow under consideration is right-continuous. We recall that the τ-flow $\mathbf{A}_+$ is defined in Remark 2.1.3.

Lemma 2.3.15. *Let M be a local exponential maxingale relative to a τ-flow $\mathbf{A}$. Then it is a local exponential maxingale relative to the τ-flow $\mathbf{A}_+$.*

Proof. We only need to check that the maxingale property holds for M relative to $\mathbf{A}_+$, which follows by Lemma 1.6.23. □

2.4 Wiener and Poisson idempotent processes

This section studies idempotent analogues of the Wiener and Poisson processes. We fix some notation to be used in this and following sections. We assume that the space $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$ of $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}_+$ is equipped with the locally uniform topology defined in Section 2.2. Let $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$ denote the discrete τ-algebra on $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$, i.e., $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d) = \mathcal{P}(\mathbb{C}(\mathbb{R}_+,\mathbb{R}^d))$, and $\mathcal{C}_t(\mathbb{R}_+,\mathbb{R}^d)$ denote the τ-algebra on $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$ generated by the maps $\mathbf{x} \to \mathbf{x}_s$, $s \le t$ (recall that $\mathbb{R}^d$ is equipped with the discrete τ-algebra). We also define the flow of τ-algebras $\mathbf{C}(\mathbb{R}_+,\mathbb{R}^d) = (\mathcal{C}_t(\mathbb{R}_+,\mathbb{R}^d),\, t\in\mathbb{R}_+)$. If the dimension d is understood we simply write $\mathbb{C}$, $\mathcal{C}$, $\mathcal{C}_t$, and $\mathbf{C}$ for $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$, $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$, $\mathcal{C}_t(\mathbb{R}_+,\mathbb{R}^d)$, and $\mathbf{C}(\mathbb{R}_+,\mathbb{R}^d)$, respectively. The atoms of $\mathcal{C}_t$ are of the form $[\mathbf{x}]_{\mathcal{C}_t} = \{\mathbf{y} : p_t\mathbf{y} = p_t\mathbf{x}\}$ (we recall that $p_t\mathbf{x} = (\mathbf{x}_{s\wedge t},\, s\in\mathbb{R}_+)$). Clearly, $\mathcal{C}_t = p_t^{-1}(\mathcal{C})$. We say that a function $\mathbf{x}\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$ is absolutely continuous if its component functions are absolutely continuous with respect to Lebesgue measure; $\dot{\mathbf{x}}_t$ denotes the Radon-Nikodym derivative at t (specified a.e.).

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Let (Ω, Π) be an idempotent probability space. We start with the one-dimensional Wiener idempotent process.

Definition 2.4.1. *We say that an idempotent probability Π^W on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ is a Wiener idempotent probability if it has density defined by*

$$\Pi^W(\mathbf{x}) = \begin{cases} \exp\Big(-\dfrac{1}{2}\displaystyle\int_0^\infty \dot{\mathbf{x}}(s)^2 ds\Big), & \text{if } \mathbf{x} \text{ is absolutely continuous and } \mathbf{x}_0 = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.1}$$

The Wiener idempotent process $W = (W_t, t \in \mathbb{R}_+)$ on (Ω, Π) is an idempotent process with paths from $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ and idempotent distribution Π^W, i.e., $\Pi(W = \mathbf{x}) = \Pi^W(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$. We call the canonical idempotent process on $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}), \Pi^W)$ the canonical Wiener idempotent process.

We give an analogue of Lévy's characterisation of the Wiener process. Let $\mathbf{A}^W = (\mathcal{A}_t^W, t \in \mathbb{R}_+)$, where $\mathcal{A}_t^W$ denotes the τ-algebra on Ω generated by the maps $\omega \to W_s(\omega)$ for $s \le t$.

Theorem 2.4.2. *Let W be an $\mathbb{R}$-valued idempotent process. The following statements are equivalent:*

1. *W is a Wiener idempotent process,*

2. *W is an idempotent process with independent increments, $W_0 = 0$ Π-a.e., and increments $W_t - W_s$ are idempotent Gaussian with parameters $(0, t-s)$ so that*

$$\Pi(W_t - W_s = x) = \exp\Big(-\frac{x^2}{2(t-s)}\Big), \; x \in \mathbb{R},$$

3. *the idempotent process $M(\lambda) = (M_t(\lambda), t \in \mathbb{R}_+)$ defined by*

$$M_t(\lambda) = \exp\Big(\lambda W_t - \frac{1}{2}\lambda^2 t\Big)$$

is an $\mathbf{A}^W$-exponential maxingale such that $M_0(\lambda) = 1$ Π-a.e. for every $\lambda \in \mathbb{R}$.

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Proof. We prove that part 1 implies part 3. Let W be a Wiener idempotent process. It is easy to see that $\mathcal{A}_t^W = W^{-1}(\mathcal{C}_t)$. Since also $\Pi^W = \Pi \circ W^{-1}$, by Lemma 1.6.27 we may assume that W is the canonical process so that it suffices to prove that the idempotent process $Y = (Y_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}))$ defined by

$$Y_t(\mathbf{x}) = \exp\bigl(\lambda \mathbf{x}_t - \frac{1}{2}\lambda^2 t\bigr) \tag{2.4.2}$$

is a $\mathbf{C}(\mathbb{R}_+,\mathbb{R})$-exponential maxingale on $(\mathbb{C}(\mathbb{R}_+,\mathbb{R}),\Pi^W)$.

We first observe that by (2.4.1)

$$S_{\Pi^W} Y_t(\mathbf{x}) = 1. \tag{2.4.3}$$

Let $\hat{\mathbf{x}} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R})$ be such that $\Pi^W(\hat{\mathbf{x}}) > 0$. In particular, $\hat{\mathbf{x}}$ is absolutely continuous and $\hat{\mathbf{x}}_0 = 0$. By the definition of conditional idempotent expectation

$$S_{\Pi^W}(Y_t|\mathcal{C}_s)(\hat{\mathbf{x}}) = \sup_{\mathbf{x}\in\mathbb{C}} Y_t(\mathbf{x})\Pi^W(\mathbf{x}|\mathcal{C}_s)(\hat{\mathbf{x}}). \tag{2.4.4}$$

The conditional idempotent probability $\Pi^W(\mathbf{x}|\mathcal{C}_s)(\hat{\mathbf{x}})$ is not equal to zero only if $\mathbf{x}$ and $\hat{\mathbf{x}}$ belong to the same atom of $\mathcal{C}_s$, i.e., $p_s\mathbf{x} = p_s\hat{\mathbf{x}}$. For these $\mathbf{x}$,

$$\Pi^W(\mathbf{x}|\mathcal{C}_s)(\hat{\mathbf{x}}) = \frac{\Pi^W(\mathbf{x})}{\Pi^W([\hat{\mathbf{x}}]_{\mathcal{C}_s})} = \frac{\Pi^W(\mathbf{x})}{\sup_{\mathbf{x}':\, p_s\mathbf{x}'=p_s\hat{\mathbf{x}}} \Pi^W(\mathbf{x}')}. \tag{2.4.5}$$

Easy calculations using (2.4.1) show that the latter supremum equals $\exp\Bigl(-\int_0^s \dot{\hat{\mathbf{x}}}_u^2 du/2\Bigr)$, so, assuming that $\Pi^W(\mathbf{x}) > 0$ and $\mathbf{x}$ is thus absolutely continuous and $\mathbf{x}_0 = 0$, by (2.4.5) and the equality $\mathbf{x}_u = \hat{\mathbf{x}}_u$, $u \le s$,

$$\Pi^W(\mathbf{x}|\mathcal{C}_s)(\hat{\mathbf{x}}) = \exp\Bigl(-\frac{1}{2}\int_s^\infty \dot{\mathbf{x}}_u^2 du\Bigr).$$

Recalling that $\theta_s\mathbf{x} = (\mathbf{x}_{s+u} - \hat{\mathbf{x}}_s,\, u \in \mathbb{R}_+)$, we thus have that if $p_s\mathbf{x} = p_s\hat{\mathbf{x}}$, then

$$\Pi^W(\mathbf{x}|\mathcal{C}_s)(\hat{\mathbf{x}}) = \Pi^W(\theta_s\mathbf{x}) \quad \Pi\text{-a.e.}$$

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Therefore, by the definition of $Y_t(\mathbf{x})$, (2.4.4) and (2.4.3)

$$S_{\Pi^W}(Y_t|\mathcal{C}_s)(\hat{\mathbf{x}}) = \sup_{\substack{\mathbf{x}\in\mathbb{C}:\\ p_s\mathbf{x}=p_s\hat{\mathbf{x}}}} Y_t(\mathbf{x})\Pi^W(\mathbf{x}|\mathcal{C}_s)(\hat{\mathbf{x}})$$
$$= \exp\Big(\lambda\hat{\mathbf{x}}_s - \frac{\lambda^2 s}{2}\Big)\sup_{\mathbf{x}\in\mathbb{C}}\Big[\exp\Big(\lambda(\mathbf{x}_t - \hat{\mathbf{x}}_s) - \frac{\lambda^2(t-s)}{2}\Big)\Pi^W(\theta_s\mathbf{x})\Big]$$
$$= Y_s(\hat{\mathbf{x}})\sup_{\mathbf{x}\in\mathbb{C}}\big(Y_{t-s}(\theta_s\mathbf{x})\Pi^W(\theta_s\mathbf{x})\big) = Y_s(\hat{\mathbf{x}})S_{\Pi^W}Y_{t-s}(\mathbf{x}) = Y_s(\hat{\mathbf{x}}).$$

Π^W-maximability of $Y_t(\mathbf{x})$ follows by Corollary 1.4.15 and the equalities

$$S_{\Pi^W}(Y_t^2) = \exp(\lambda^2 t)S_{\Pi^W}(2\lambda\mathbf{x}_t - 2\lambda^2 t) = \exp(\lambda^2 t),$$

where the latter equality follows by (2.4.2) and (2.4.3) with λ replaced by 2λ. Also, $\mathbf{x}_0 = 0$ Π^W-a.e. by the definition of Π^W. This ends the proof of the implication $1 \to 3$.

We prove that part 3 implies part 2. The maxingale property of Y easily implies that

$$S_{\Pi^W}(\exp(\lambda(\mathbf{x}_t - \mathbf{x}_s))|\mathcal{C}_s) = \exp\Big(\frac{1}{2}\lambda^2(t-s)\Big),$$

which yields the required by Definition 1.11.10, Corollary 1.11.9 and Lemma 1.11.12.

We prove that part 2 implies part 1. By independence of increments the finite-dimensional idempotent distributions of W are Gaussian so by Remark 1.11.16 they are deviabilities on the associated spaces. Therefore, by Corollary 2.2.6 the idempotent distribution of W is τ-smooth relative to the collection of closed subsets of $\mathbb{R}^{\mathbb{R}_+}$. Thus, for $\mathbf{x} \in \mathbb{R}^{\mathbb{R}_+}$ such that $\mathbf{x}_0 = 0$ by Theorem 2.2.2 and independence of increments of W

$$\Pi(\omega : W(\omega) = \mathbf{x}) = \inf_{t_1,\dots,t_k}\Pi(\omega : W_{t_i}(\omega) = \mathbf{x}_{t_i},\, i = 1,\dots,k)$$
$$= \inf_{t_1,\dots,t_k}\Pi(\omega : W_{t_1}(\omega) = \mathbf{x}_{t_1})\prod_{i=2}^k \Pi(\omega : W_{t_i}(\omega) - W_{t_{i-1}}(\omega)$$
$$= \mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}}) = \exp\Big(-\frac{1}{2}\sup_{t_1,\dots,t_k}\sum_{i=1}^k\frac{(\mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}})^2}{t_i - t_{i-1}}\Big).$$

The latter supremum equals $+\infty$ if $\mathbf{x}$ is not absolutely continuous and $\int_0^\infty \dot{\mathbf{x}}_t^2\,dt$ if $\mathbf{x}$ is absolutely continuous. □

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The next result shows that the Wiener idempotent process is Luzin-continuous.

Lemma 2.4.3. *The Wiener idempotent probability is a deviability, i.e., it is tight and τ-smooth relative to the collection of closed subsets of* $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$.

Proof. By Corollary 1.7.15 it would be sufficient to prove that $\Pi^W(\mathbf{x})$ is an upper compact function of $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$, which is an easy exercise. We give, however, a different "probabilistic" proof based on Theorem 2.4.2. Let X be the canonical idempotent process on $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}), \Pi^W)$. By Theorem 2.2.13 it suffices to prove that for arbitrary $T > 0$ and $\eta > 0$

$$\lim_{\delta \to 0} \sup_{\substack{s,t \in [0,T]: \\ |s-t| \le \delta}} \Pi^W\big(|X_t - X_s| > \eta\big) = 0.$$

By Theorem 2.4.2 and the Chebyshev inequality for $\lambda > 0$

$$\Pi^W\big(|X_t - X_s| > \eta\big) = \Pi^W\big(X_t - X_s > \eta\big) \vee \Pi^W\big(X_s - X_t > \eta\big)$$
$$\le \frac{S_{\Pi^W} \exp\big(\lambda(X_t - X_s)\big)}{\exp(\lambda\eta)} \vee \frac{S_{\Pi^W} \exp\big(\lambda(X_s - X_t)\big)}{\exp(\lambda\eta)}$$
$$= \frac{\exp\big(\lambda^2 |t-s|/2\big)}{\exp(\lambda\eta)},$$

which implies the required since λ is arbitrary. □

Definition 2.4.4. *We say that a continuous idempotent process W is Wiener relative to a τ-flow $\mathbf{A}$ (or an $\mathbf{A}$-Wiener idempotent process for short) if the idempotent process $M(\lambda)$ defined in the statement of Theorem 2.4.2 is an $\mathbf{A}$-exponential maxingale starting at* 1 *for every* $\lambda \in \mathbb{R}$.

Lemma 2.4.5. *A continuous idempotent process W is $\mathbf{A}$-Wiener if and only if the idempotent process $M(\lambda)$ defined in the statement of Theorem 2.4.2 is an $\mathbf{A}$-local exponential maxingale starting at* 1 *for every $\lambda \in \mathbb{R}$.*

Proof. Let $M(\lambda)$ be an $\mathbf{A}$-local exponential maxingale starting at 1 for every $\lambda \in \mathbb{R}$. By Lemma 2.3.13 we only need to prove that for

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every $s \in \mathbb{R}_+$ the process $(M_{t\wedge s}(\lambda),\, t \in \mathbb{R}_+)$ is uniformly maximable. As in the proof of Theorem 2.4.2

$$S\big(M_{t\wedge s}(\lambda)^2\big) \le S\big(M_{t\wedge s}(2\lambda)\big)\exp(\lambda^2 s) \le \exp(\lambda^2 s),$$

where the latter inequality follows by Lemma 2.3.13. The uniform maximability follows by Corollary 1.4.15. □

Obviously, if $M(\lambda)$ is an $\mathbf{A}$-exponential maxingale, then it is an $\mathbf{A}^W$-exponential maxingale. Thus, we have the following consequence of Theorem 2.4.2.

Corollary 2.4.6. *If W is an $\mathbf{A}$-Wiener idempotent process, then it has properties described in parts 1 and 2 of Theorem 2.4.2.*

Corollary 2.4.7. *If W is an $\mathbf{A}$-Wiener idempotent process, then $W_t - W_s$ is independent of $\mathcal{A}_s$ for $t \ge s$.*

We now consider the multi-dimensional case.

Definition 2.4.8. *An $\mathbb{R}^d$-valued idempotent process $W = (W^1, \dots, W^d)$ on (Ω, Π) is called a d-dimensional Wiener idempotent process if the processes $W^1, \dots, W^d$ are independent Wiener idempotent processes.*

The next theorem is proved similarly to Theorem 2.4.2. Let, as above, $\mathbf{A}^W$ denote the flow of τ-algebras generated by W and Π^W denote the idempotent distribution of W. Let $\mathbf{E}_d$ denote the $d \times d$ identity matrix.

Theorem 2.4.9. *Let W be an $\mathbb{R}^d$-valued idempotent process. The following conditions are equivalent:*

1. *W is a d-dimensional Wiener idempotent process,*
2. *the density of Π^W is given by ($\mathbf{x} \in \mathbb{R}^{\mathbb{R}_+}$)*

$$\Pi^W(\mathbf{x}) = \begin{cases} \exp\Big(-\dfrac{1}{2}\displaystyle\int_0^\infty |\dot{\mathbf{x}}_s|^2 ds\Big), & \text{if } \mathbf{x} \text{ is absolutely continuous and } \mathbf{x}_0 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

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3. *W is an idempotent process with independent increments, $W_0 = 0$ Π-a.e., and increments $W_t - W_s$ are idempotent Gaussian with parameters $(0, (t-s)\mathbf{E}_d)$ so that*

$$\Pi(W_t - W_s = x) = \exp\Big(-\frac{|x|^2}{2(t-s)}\Big), \ x \in \mathbb{R}^d,$$

4. *the idempotent process $M(\lambda) = (M_t(\lambda),\, t \in \mathbb{R}_+)$ defined by*

$$M_t(\lambda) = \exp\Big(\lambda \cdot W_t - \frac{1}{2}|\lambda|^2 t\Big),$$

is an $\mathbf{A}^W$-exponential maxingale such that $M_0(\lambda) = 1$ Π-a.e. for every $\lambda \in \mathbb{R}^d$.

Clearly, a d-dimensional idempotent Wiener process is Luzin-continuous. Also, since W has independent increments, which are idempotent Gaussian variables, we have the following corollary.

Corollary 2.4.10. *A d-dimensional idempotent Wiener process is an idempotent Gaussian process.*

Part 4 of Theorem 2.4.9 implies the following analogues of the properties of the Wiener process. Let $\mathbf{e} = (t,\, t \in \mathbb{R}_+)$. We recall that $\circ$ denotes the composition map.

Corollary 2.4.11. 1. *Let W be a one-dimensional idempotent Wiener process and $\alpha \in \mathbb{R}_+$. Then the idempotent process $W \circ (\alpha\mathbf{e})$ has the same idempotent distribution as $\alpha^{1/2}W$.*

2. *Let $W_1, \dots, W_k$ be independent d-dimensional idempotent Wiener processes and $\sigma_1, \dots, \sigma_k$ be $l \times d$ matrices. Then the idempotent distribution of $\sum_{i=1}^k \sigma_i W_i$ coincides with the idempotent distribution of $\big(\sum_{i=1}^k \sigma_i \sigma_i^T\big)^{1/2} W$, where W is an l-dimensional idempotent Wiener process.*

Definition 2.4.12. *An $\mathbb{R}^d$-valued continuous idempotent process W is called a d-dimensional Wiener idempotent process with respect to a τ-flow $\mathbf{A}$ (or $\mathbf{A}$-Wiener, for short) if the idempotent process $M(\lambda)$ defined in the statement of Theorem 2.4.9 is an $\mathbf{A}$-exponential maxingale starting at 1 for every $\lambda \in \mathbb{R}^d$.*

The proof of the following lemma is similar to the proof of Lemma 2.4.5.

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Lemma 2.4.13. *An $\mathbb{R}^d$-valued idempotent process W is a d-dimensional $\mathbf{A}$-Wiener idempotent process if and only if the idempotent process $M(\lambda)$ defined in the statement of Theorem 2.4.9 is an $\mathbf{A}$-local exponential maxingale starting at 1 for every $\lambda \in \mathbb{R}^d$.*

We define now the Poisson idempotent process. As above, we assume that $0 \ln 0 = 0$.

Definition 2.4.14. *We say that an idempotent probability $\Pi^{\mathcal{N}}$ on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ is the Poisson idempotent probability if it has density defined by*

$$\Pi^{\mathcal{N}}(\mathbf{x}) = \begin{cases} \exp\Big(-\int\limits_0^\infty (\dot{\mathbf{x}}_s \ln \dot{\mathbf{x}}_s - \dot{\mathbf{x}}_s + 1)ds\Big), & \text{if } \mathbf{x} \text{ is absolutely continuous, } \dot{\mathbf{x}}_s \in \mathbb{R}_+ \text{ a.e. and } \mathbf{x}_0 = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.6}$$

A Poisson idempotent process $\mathcal{N} = (\mathcal{N}_t,\, t \in \mathbb{R}_+)$ on (Ω, Π) is an idempotent process with paths from $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ and idempotent distribution $\Pi^{\mathcal{N}}$, i.e., $\Pi(\mathcal{N} = \mathbf{x}) = \Pi^{\mathcal{N}}(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$. We call $\mathcal{N}$ a canonical Poisson idempotent process if it is the canonical process on $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}), \Pi^{\mathcal{N}})$.

Remark 2.4.15. *According to the definition, the Poisson idempotent process has increasing paths Π-a.e.*

We now give a characterisation of the Poisson idempotent process in the spirit of Watanabe's characterisation of the Poisson process and analogous to that of the Wiener idempotent process. Let $\mathbf{A}^{\mathcal{N}} = (\mathcal{A}_t^{\mathcal{N}},\, t \in \mathbb{R}_+)$, where $\mathcal{A}_t^{\mathcal{N}}$ denotes the τ-algebra on Ω generated by the maps $\omega \to \mathcal{N}_s(\omega)$ for $s \le t$.

Theorem 2.4.16. *Let $\mathcal{N}$ be an $\mathbb{R}$-valued idempotent process. The following statements are equivalent:*

1. *$\mathcal{N}$ is a Poisson idempotent process,*

2. *$\mathcal{N}$ is an idempotent process with independent increments, $\mathcal{N}_0 = 0$ Π-a.e., and increments $\mathcal{N}_t - \mathcal{N}_s$ for $s < t$ are idempotent Poisson with parameters $t - s$, i.e.,*

$$\Pi(\mathcal{N}_t - \mathcal{N}_s = x) = \exp\Big(-x \ln \frac{x}{t-s} + x - (t-s)\Big),\ x \in \mathbb{R}_+,$$

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3. the idempotent process $M(\lambda) = (M_t(\lambda),\, t \in \mathbb{R}_+)$ *defined by*

$$M_t(\lambda) = \exp\Big(\lambda \mathcal{N}_t - (e^\lambda - 1)t\Big)$$

is an $\mathbf{A}^{\mathcal{N}}$*-exponential maxingale such that* $M_0(\lambda) = 1$ Π*-a.e. for every* $\lambda \in \mathbb{R}$.

Proof. The proof is analogous to the proof of Theorem 2.4.2. We prove that part 1 implies part 3. Let $\mathcal{N}$ be a Poisson idempotent process. As in the proof of Theorem 2.4.2 it is sufficient to prove that the idempotent process $Y = (Y_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}))$ defined by

$$Y_t(\mathbf{x}) = \exp\big(\lambda \mathbf{x}_t - (e^\lambda - 1)t\big) \tag{2.4.7}$$

is a $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$-exponential maxingale on $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}), \Pi^{\mathcal{N}})$.

We first note that by (2.4.6)

$$S_{\Pi^{\mathcal{N}}} Y_t(\mathbf{x}) = 1. \tag{2.4.8}$$

Let $\hat{\mathbf{x}} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ be such that $\Pi^{\mathcal{N}}(\hat{\mathbf{x}}) > 0$. In particular, $\hat{\mathbf{x}}$ is absolutely continuous, increasing and $\hat{\mathbf{x}}_0 = 0$. By the definition of conditional idempotent expectation

$$S_{\Pi^{\mathcal{N}}}(Y_t | \mathcal{C}_s)(\hat{\mathbf{x}}) = \sup_{\mathbf{x} \in \mathbb{C}} Y_t(\mathbf{x}) \Pi^{\mathcal{N}}(\mathbf{x} | \mathcal{C}_s)(\hat{\mathbf{x}}). \tag{2.4.9}$$

By the reasoning used in the proof of Theorem 2.4.2 we have that

$$\Pi^{\mathcal{N}}(\mathbf{x} | \mathcal{C}_s)(\hat{\mathbf{x}}) = \Pi^{\mathcal{N}}(\theta_s \mathbf{x}).$$

Therefore, by the definition of $Y_t(\mathbf{x})$, again repeating the argument of the proof of Theorem 2.4.2,

$$\begin{aligned} S_{\Pi^{\mathcal{N}}}(Y_t | \mathcal{C}_s)(\hat{\mathbf{x}}) &= \sup_{\substack{\mathbf{x} \in \mathbb{C}: \\ p_s \mathbf{x} = p_s \hat{\mathbf{x}}}} Y_t(\mathbf{x}) \Pi^{\mathcal{N}}(\mathbf{x} | \mathcal{C}_s)(\hat{\mathbf{x}}) \\ &= Y_s(\hat{\mathbf{x}}) \sup_{\mathbf{x} \in \mathbb{C}} Y_{t-s}(\theta_s \mathbf{x}) \Pi^{\mathcal{N}}(\theta_s \mathbf{x}) = Y_s(\hat{\mathbf{x}}) S_{\Pi^{\mathcal{N}}} Y_{t-s}(\mathbf{x}) = Y_s(\hat{\mathbf{x}}). \end{aligned}$$

$\Pi^{\mathcal{N}}$-maximability of $Y_t(\mathbf{x})$ follows by Corollary 1.4.15 and the equality

$$\begin{aligned} S_{\Pi^{\mathcal{N}}}(Y_t^2) = \exp\big((e^{2\lambda} - 2e^\lambda + 1)t\big) S_{\Pi^{\mathcal{N}}}\big(2\lambda \mathbf{x}_t - (e^{2\lambda} - 1)t\big) \\ = \exp\big((e^{2\lambda} - 2e^\lambda + 1)t\big), \end{aligned}$$

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where the latter equality follows by (2.4.7) and (2.4.8) with λ replaced by 2λ. Finally, $Y_0(\mathbf{x}) = 1$ $\Pi^{\mathcal{N}}$-a.e. since $\mathbf{x}_0 = 0$ Π-a.e. This ends the proof of the implication $1 \to 3$.

We prove that part 3 implies part 2. The maxingale property of Y yields

$$S_{\Pi^{\mathcal{N}}}(\exp(\lambda(\mathbf{x}_t - \mathbf{x}_s))|\mathcal{C}_s) = \exp\bigl((e^\lambda - 1)(t-s)\bigr),$$

which implies the required by Corollary 1.11.9 and Lemma 1.11.15.

To prove that part 2 implies part 1 we write for an increasing function $\mathbf{x} \in \mathbb{R}^{\mathbb{R}_+}$ such that $\mathbf{x}_0 = 0$ by Theorem 2.2.2 and independence of increments of $\mathcal{N}$

$$\begin{aligned}
\Pi(\omega : \mathcal{N}(\omega) = \mathbf{x}) &= \inf_{t_1,\dots,t_k} \Pi(\omega : \mathcal{N}_{t_i}(\omega) = \mathbf{x}_{t_i},\, i = 1,\dots,k) \\
&= \inf_{t_1,\dots,t_k} \Pi(\omega : \mathcal{N}_{t_1}(\omega) = \mathbf{x}_{t_1}) \prod_{i=2}^{k} \Pi(\omega : \mathcal{N}_{t_i}(\omega) - \mathcal{N}_{t_{i-1}}(\omega) \\
&= \mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}}) = \exp\Bigl(-\sup_{t_1,\dots,t_k} \sum_{i=1}^{k} \Bigl((\mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}}) \ln \frac{\mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}}}{t_i - t_{i-1}} \\
&\qquad - (\mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}}) + (t_i - t_{i-1})\Bigr)\Bigr).
\end{aligned}$$

The latter supremum equals $+\infty$ if $\mathbf{x}$ is not absolutely continuous and $\int_0^\infty (\dot{\mathbf{x}}_t \ln \dot{\mathbf{x}}_t - \dot{\mathbf{x}}_t + 1)\, dt$ if $\mathbf{x}$ is absolutely continuous. $\square$

The next result shows that $\mathcal{N}$ is a Luzin-continuous idempotent process.

Lemma 2.4.17. *The Poisson idempotent probability is a deviability, i.e., it is tight and τ-smooth relative to the collection of closed subsets of $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$.*

Proof. We give again a "probabilistic" proof. Let X be the canonical process on $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}), \Pi^{\mathcal{N}})$. By Theorem 2.2.13 it suffices to prove that for arbitrary $T > 0$ and $\eta > 0$

$$\lim_{\delta \to 0} \sup_{\substack{s,t \in [0,T]: \\ 0 \le t-s \le \delta}} \Pi^{\mathcal{N}}\bigl(X_t - X_s > \eta\bigr) = 0.$$

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By Theorem 2.4.16 and the Chebyshev inequality for $\lambda > 0$

$$\Pi^{\mathcal{N}}\big(X_t - X_s > \eta\big) \le \frac{S_{\Pi^{\mathcal{N}}} \exp\big(\lambda(X_t - X_s)\big)}{\exp(\lambda\eta)} = \frac{\exp\big((e^{\lambda} - 1)(t-s)\big)}{\exp(\lambda\eta)},$$

which implies the required since λ is arbitrary. □

Definition 2.4.18. *We say that a continuous idempotent process $\mathcal{N}$ is Poisson relative to a τ-flow $\mathbf{A}$ (or $\mathbf{A}$-Poisson idempotent process for short) if the idempotent process $M(\lambda)$ defined in the statement of Theorem 2.4.16 is an $\mathbf{A}$-exponential maxingale for every $\lambda \in \mathbb{R}$ such that $M_0(\lambda) = 1$ Π-a.e.*

Corollary 2.4.19. *If $\mathcal{N}$ is an $\mathbf{A}$-Poisson idempotent process, then it has properties described in parts 1 and 2 of Theorem 2.4.16. Also, $\mathcal{N}_t - \mathcal{N}_s$ is independent of $\mathcal{A}_s$ for $s \le t$.*

Lemma 2.4.20. *An $\mathbb{R}$-valued continuous idempotent process $\mathcal{N}$ is $\mathbf{A}$-Poisson if and only if the idempotent process $M(\lambda)$ defined in the statement of Theorem 2.4.16 is an $\mathbf{A}$-local exponential maxingale starting at 1 for every $\lambda \in \mathbb{R}$.*

2.5 Idempotent stochastic integrals

Let (Ω, Π) be an idempotent probability space with a flow of τ-algebras $\mathbf{A} = (\mathcal{A}_t, t \in \mathbb{R}_+)$.

Definition 2.5.1. *An $\mathbb{R}^d$-valued continuous $\mathbf{A}$-adapted idempotent process $M = (M_t,\, t \in \mathbb{R}_+)$ such that $M_0 = 0$ is called a local maxingale (maxingale or uniformly maximable maxingale, respectively) with a quadratic characteristic $\langle M \rangle$ relative to $\mathbf{A}$ (or an $\mathbf{A}$-local maxingale, $\mathbf{A}$-maxingale, or uniformly maximable $\mathbf{A}$-maxingale, respectively, for short) if there exists an $\mathbb{R}^{d\times d}$-valued continuous $\mathbf{A}$-adapted idempotent process $\langle M \rangle = (\langle M \rangle_t,\, t \in \mathbb{R}_+)$ such that $\langle M \rangle_0 = 0$, $\langle M \rangle_t - \langle M \rangle_s$ for $0 \le s \le t$ are positive semi-definite symmetric $d \times d$ matrices and the idempotent process $(\exp(\lambda \cdot M_t - \lambda \cdot \langle M \rangle_t \lambda/2), t \in \mathbb{R}_+)$ is an $\mathbf{A}$-local exponential maxingale (respectively, $\mathbf{A}$-exponential maxingale, uniformly maximable $\mathbf{A}$-exponential maxingale) for every $\lambda \in \mathbb{R}^d$.*

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By Theorem 2.4.9 an $\mathbb{R}^d$-valued Wiener idempotent process is a local maxingale with a quadratic characteristic $\mathbf{E}_d t$. Lemma 2.4.13 yields the following converse.

Corollary 2.5.2. *Let a continuous idempotent process M be an $\mathbb{R}^d$-valued $\mathbf{A}$-local maxingale with a quadratic characteristic $(\mathbf{E}_d t,\, t \in \mathbb{R}_+)$. Then M is a d-dimensional $\mathbf{A}$-Wiener idempotent process.*

The following consequence of Lemma 2.3.14 is also useful.

Lemma 2.5.3. *Let a continuous idempotent process M be an $\mathbb{R}^d$-valued $\mathbf{A}$-local maxingale with a quadratic characteristic $\langle M\rangle$. Then for $a > 0$, $b > 0$, $c > 0$, and finite $\mathbf{A}$-stopping times τ and σ such that $\tau \ge \sigma$*

$$\Pi(\sup_{\sigma\le t\le\tau} |M_t - M_\sigma| \ge a) \le e^{c(b-a)} \vee \Pi(\,\|\langle M\rangle_\tau - \langle M\rangle_\sigma\|\, > 2b/c).$$

Proof. By "the Doob stopping theorem" the idempotent process $\bigl(\exp\bigl(\lambda\cdot(M_{t+\sigma} - M_\sigma) - \lambda\cdot(\langle M\rangle_{t+\sigma} - \langle M\rangle_\sigma)\lambda/2\bigr),\, t \in \mathbb{R}_+\bigr)$, $\lambda \in \mathbb{R}^d$, is a supermaxingale relative to the τ-flow $(\mathcal{A}_{t+\sigma},\, t \in \mathbb{R}_+)$. Also, $\tau - \sigma$ is a stopping time relative to $(\mathcal{A}_{t+\sigma},\, t \in \mathbb{R}_+)$ by Lemma 2.1.11. Hence, by Lemma 2.3.14 and τ-maxitivity of Π

$$\begin{aligned}
\Pi(\sup_{\sigma\le t\le\tau} |M_t - M_\sigma| \ge a) &= \sup_{|\lambda|=1} \Pi(\sup_{\sigma\le t\le\tau} \lambda\cdot(M_t - M_\sigma) \ge a)\\
&\le e^{c(b-a)} \vee \sup_{|\lambda|=1} \Pi(\lambda\cdot(\langle M\rangle_\tau - \langle M\rangle_\sigma)\lambda > 2b/c)\\
&= e^{c(b-a)} \vee \Pi(\,\|\langle M\rangle_\tau - \langle M\rangle_\sigma\|\, > 2b/c).
\end{aligned}$$

□

Properties of the trajectories of $\langle M\rangle$ are often translated into the corresponding properties of M. The next result follows by Lemma 2.5.3 and Theorem 2.2.13.

Lemma 2.5.4. *Let a continuous idempotent process M be a local maxingale relative to the flow $\mathbf{A}$ with a quadratic characteristic $\langle M\rangle$.*

1. *If $\langle M\rangle$ is a proper idempotent process, then M is a proper idempotent process.*

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2. *If $\langle M\rangle$ is continuous (respectively, stopping-time-right-continuous) in idempotent probability, then M is continuous (respectively, stopping-time-right-continuous) in idempotent probability.*

3. *Let M be Luzin. If $\langle M\rangle$ is Luzin-continuous, then M is Luzin-continuous.*

We assume in the rest of the section that the τ-flow $\mathbf{A}$ is complete in the sense of the following definition.

Definition 2.5.5. *We say that a flow of τ-algebras on (Ω,Π) is complete if the τ-algebras in the flow are complete with respect to Π.*

Remark 2.5.6. *Clearly, if $\mathbf{A} = (\mathcal{A}_t,\, t\in\mathbb{R}_+)$ is a τ-flow, then $\mathbf{A}^\Pi = (\mathcal{A}_t^\Pi,\, t\in\mathbb{R}_+)$, where the $\mathcal{A}_t^\Pi$ are the completions of the $\mathcal{A}_t$ with respect to Π, is a complete τ-flow. We refer to it as the completion of $\mathbf{A}$ with respect to Π (or the Π-completion of $\mathbf{A}$).*

The next lemma extends Lemma 1.2.6.

Lemma 2.5.7. *Let $\mathbf{A}^\Pi$ be the completion of $\mathbf{A}$ with respect to Π. If σ is an $\mathbf{A}^\Pi$-stopping time, then there exists an $\mathbf{A}$-stopping time σ' such that $\sigma'=\sigma$ Π-a.e.*

Proof. We define σ' as follows: if $\Pi(\omega)>0$, then $\sigma'(\omega)=\sigma(\omega)$; if $\Pi(\omega)=0$ and there exists $\tilde\omega$ such that $\Pi(\tilde\omega)>0$ and $\omega\in[\tilde\omega]_{\mathcal{A}_{\sigma(\tilde\omega)}}$, then $\sigma'(\omega)=\sigma(\tilde\omega)$; if $\Pi(\omega)=0$ and no such $\tilde\omega$ exists, then $\sigma'(\omega)=\infty$. We first check that σ' is well defined. Indeed, suppose for some ω such that $\Pi(\omega)=0$ there exist $\tilde\omega$ and $\hat\omega$ such that $\Pi(\tilde\omega)>0$, $\Pi(\hat\omega)>0$, $\omega\in[\tilde\omega]_{\mathcal{A}_{\sigma(\tilde\omega)}}$, $\omega\in[\hat\omega]_{\mathcal{A}_{\sigma(\hat\omega)}}$, and $\sigma(\tilde\omega)\ge\sigma(\hat\omega)$; then $[\tilde\omega]_{\mathcal{A}_{\sigma(\tilde\omega)}} = [\omega]_{\mathcal{A}_{\sigma(\tilde\omega)}}\subset[\omega]_{\mathcal{A}_{\sigma(\hat\omega)}}=[\hat\omega]_{\mathcal{A}_{\sigma(\hat\omega)}}$; hence, $\tilde\omega\in[\hat\omega]_{\mathcal{A}^\Pi_{\sigma(\hat\omega)}}$ since $\Pi(\tilde\omega)>0$ and $\Pi(\hat\omega)>0$, so $\sigma(\tilde\omega)=\sigma(\hat\omega)$ by Lemma 2.1.6 proving the claim.

We check that σ' is an $\mathbf{A}$-stopping time. Let $\sigma'(\omega')=t$ and $\omega''\in[\omega']_{\mathcal{A}_t}$, where $t\in\mathbb{R}_+$. We have to check that $\sigma'(\omega'')=t$. If $\Pi(\omega')>0$ and $\Pi(\omega'')>0$, then $\sigma'(\omega')=\sigma(\omega')$, $\sigma'(\omega'')=\sigma(\omega'')$ and $\omega''\in[\omega']_{\mathcal{A}_t^\Pi}$; hence, $\sigma(\omega')=t$ so $\omega''\in[\omega']_{\mathcal{A}^\Pi_{\sigma(\omega')}}$ and $\sigma(\omega')=\sigma(\omega'')$ by Lemma 2.1.6. If $\Pi(\omega')>0$ and $\Pi(\omega'')=0$, then as above $\sigma(\omega')=t$, so $\omega''\in[\omega']_{\mathcal{A}_{\sigma(\omega')}}$ and by definition $\sigma'(\omega'')=\sigma(\omega')$. If $\Pi(\omega')=0$, then, since $\sigma'(\omega')$ is finite, there exists $\tilde\omega$ such that $\Pi(\tilde\omega)>0$, $\omega'\in[\tilde\omega]_{\mathcal{A}_{\sigma(\tilde\omega)}}$ and $\sigma'(\omega')=\sigma(\tilde\omega)$; hence, $\omega''\in[\tilde\omega]_{\mathcal{A}_{\sigma(\tilde\omega)}}$. If, in addition,

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$\Pi(\omega'') > 0$, then $\omega'' \in [\tilde{\omega}]_{\mathcal{A}^{\Pi}_{\sigma(\tilde{\omega})}}$ so by Lemma 2.1.6 $\sigma'(\omega'') = \sigma(\tilde{\omega})$; if $\Pi(\omega'') = 0$, then the latter equality holds by definition. □

We say that an $\mathbb{R}^{m\times d}$-valued continuous idempotent process X is absolutely continuous if the entry processes have Π-a.e. absolutely continuous with respect to Lebesgue measure trajectories; if X is, in addition, $\mathbf{A}$-adapted, then we denote by $\dot{X}$ an $\mathbf{A}$-progressively measurable idempotent process such that $\dot{X}(\omega)$ is a version of the Radon-Nikodym derivative of $X(\omega)$ with respect to Lebesgue measure for Π-almost all ω. (For instance, we could define $\dot{X}_t(\omega)$ as the left derivative of $X(\omega)$ at t if the latter exists and let $\dot{X}_t(\omega) = 0$ otherwise.) For a quadratic characteristic $\langle M\rangle$ to be absolutely continuous, it is actually sufficient that the diagonal entries are absolutely continuous. Note that if this is the case, then $\int_0^t \|\langle \dot{M}\rangle_s\|\, ds < \infty$, $t \in \mathbb{R}_+$.

The following lemma shows, in particular, that absolute continuity of the quadratic characteristic of a local maxingale implies absolute continuity of the local maxingale itself. It also lays the groundwork for the definition of idempotent integrals with respect to local maxingales. We recall that $\sigma^{\oplus}$ denotes the pseudo-inverse of a matrix σ.

Lemma 2.5.8. *Let an $\mathbb{R}^d$-valued continuous idempotent process M be an $\mathbf{A}$-local maxingale with an absolutely continuous quadratic characteristic $\langle M\rangle$. Then the following holds.*

1. *M is absolutely continuous.*

2. *$\dot{M}_s(\omega)$ belongs to the range of $\langle \dot{M}\rangle_s(\omega)$ for almost all s and Π-almost all ω, and*

$$S\Big[\exp\Big(\int\limits_0^\infty \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot\dot{M}_s - \frac{1}{2}\lambda\cdot\langle\dot{M}\rangle_s\lambda)\,ds\Big)\Big]$$
$$= S\Big[\exp\Big(\frac{1}{2}\int\limits_0^\infty \dot{M}_s\cdot\langle\dot{M}\rangle_s^{\oplus}\dot{M}_s\,ds\Big)\Big] \le 1,$$

in particular, $\int_0^\infty \dot{M}_s\cdot\langle\dot{M}\rangle_s^{\oplus}\dot{M}_s\,ds$ is a Π-a.e. finite proper idempotent variable.

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Let, in addition, $(\sigma_s(\omega),\ s \in \mathbb{R}_+, \omega \in \Omega)$ *be an* $\mathbb{R}^{m\times d}$*-valued* **A***-progressively measurable idempotent process such that*

$$\int_0^t \|\sigma_s \langle \dot{M}\rangle_s \sigma_s^T\|\ ds < \infty,\ t \in \mathbb{R}_+, \quad \Pi\text{-}a.e.$$

Then

$$\int_0^t |\sigma_s \dot{M}_s|\, ds < \infty,\ t \in \mathbb{R}_+, \quad \Pi\text{-}a.e.$$

Proof. Let, for $0 \le s_1 \le t_1 \le \ldots \le s_k \le t_k$ and $\lambda_i \in \mathbb{R}^d$, $i = 1, \ldots, k$,

$$Z = \exp\Bigl[\sum_{i=1}^k \lambda_i \cdot \bigl(M_{t_i} - M_{s_i}\bigr) - \frac{1}{2}\sum_{i=1}^k \lambda_i \cdot (\langle M\rangle_{t_i} - \langle M\rangle_{s_i})\lambda_i\Bigr]. \tag{2.5.1}$$

We show that

$$SZ \le 1. \tag{2.5.2}$$

Let

$$\tau_n = \inf\{t \in \mathbb{R}_+ :\ \|\langle M\rangle_t\|\ \ge n\}. \tag{2.5.3}$$

By Lemma 2.2.18 the τ_n are **A**-stopping times. Also by Lemma 2.3.13

$$\begin{aligned} &S\bigl(\exp(\lambda \cdot M_{t\wedge\tau_n} - \lambda \cdot \langle M\rangle_{t\wedge\tau_n}\lambda/2)^2\bigr) \\ &\le S\bigl(\exp(2\lambda \cdot M_{t\wedge\tau_n} - (2\lambda) \cdot \langle M\rangle_{t\wedge\tau_n}(2\lambda)/2)\bigr)\exp(n|\lambda|^2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \le \exp(n|\lambda|^2) \end{aligned}$$

so that by Lemma 2.3.13 $\{\tau_n\}$ is a localising sequence of stopping times for every local maxingale $\bigl(\exp(\lambda \cdot M_t - \lambda \cdot \langle M\rangle_t\lambda/2),\ t \in \mathbb{R}_+\bigr)$, $\lambda \in \mathbb{R}^d$.

Let

$$Y_n^i = \exp\bigl(\lambda_i \cdot \bigl(M_{t_i\wedge\tau_n} - M_{s_i\wedge\tau_n}\bigr) - \frac{1}{2}\lambda_i \cdot (\langle M\rangle_{t_i\wedge\tau_n} - \langle M\rangle_{s_i\wedge\tau_n})\lambda_i\bigr). \tag{2.5.4}$$

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Since $(\exp(\lambda_i \cdot M_{t\wedge\tau_n} - \lambda_i \cdot \langle M\rangle_{t\wedge\tau_n}\lambda_i/2),\, t \in \mathbb{R}_+)$ is an **A**-exponential maxingale,

$$S(Y_n^i|\mathcal{A}_{s_i}) = 1. \tag{2.5.5}$$

By (2.5.4) and (2.5.5) we have

$$S\Big(\prod_{i=1}^{k} Y_n^i\Big) = S\Big(\prod_{i=1}^{k-1} Y_n^i S(Y_n^k|\mathcal{A}_{s_k})\Big) = S\Big(\prod_{i=1}^{k-2} Y_n^i S(Y_n^{k-1}|\mathcal{A}_{s_{k-1}})\Big) = \ldots = SY_n^1 = 1.$$

Since $\tau_n \to \infty$, by (2.5.1) $Z = \lim_{n\to\infty}\prod_{i=1}^k Y_n^i$, and "the Fatou lemma" (see Theorem 1.4.19) yields (2.5.2). This inequality implies in view of the definition of idempotent expectation that

$$\sup_{\{(s_i,t_i)\},\{\lambda_i\}}\Big(\sum_{i=1}^k \lambda_i\cdot\big(M_{t_i}-M_{s_i}\big) - \frac{1}{2}\sum_{i=1}^k \lambda_i\cdot(\langle M\rangle_{t_i} - \langle M\rangle_{s_i})\lambda_i\Big) < \infty \quad \Pi\text{-a.e.} \tag{2.5.6}$$

Now let us suppose that there exists $\omega \in \Omega$ such that $\Pi(\omega) > 0$ and $T > 0$ such that $M_t(\omega)$ is not absolutely continuous on $[0,T]$. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exist non-overlapping subintervals $\{(s_i^\delta, t_i^\delta)\}$ of $[0,T]$ such that

$$\sum_i (t_i^\delta - s_i^\delta) < \delta \quad \text{and} \quad \sum_i |M_{t_i^\delta}(\omega) - M_{s_i^\delta}(\omega)| > \varepsilon. \tag{2.5.7}$$

Let λ_i^δ be such that $|\lambda_i^\delta| = 1$ and $\lambda_i^\delta \cdot (M_{t_i^\delta}(\omega) - M_{s_i^\delta}(\omega)) = |M_{t_i^\delta}(\omega) - M_{s_i^\delta}(\omega)|$. Given $A > 0$ we choose $\delta > 0$ such that

$$\sum_i \|\langle M\rangle_{t_i^\delta}(\omega) - \langle M\rangle_{s_i^\delta}(\omega)\| < \frac{1}{A^2},$$

which is possible in view of absolute continuity of $\langle M\rangle$. Then by (2.5.7)

$$\sum_{i=1}^k (A\lambda_i^\delta)\cdot\big(M_{t_i^\delta}(\omega) - M_{s_i^\delta}(\omega)\big) - \frac{1}{2}\sum_{i=1}^k (A\lambda_i^\delta)\cdot(\langle M\rangle_{t_i^\delta}(\omega) - \langle M\rangle_{s_i^\delta}(\omega))(A\lambda_i^\delta) \geq A\varepsilon - \frac{1}{2},$$

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which contradicts (2.5.6) since A is arbitrary. Part 1 is proved.

For part 2, we note that by properties of idempotent expectation and (2.5.2)

$$S\Big(\sup_{\{(s_i,t_i)\},\{\lambda_i\}} Z\Big) = \sup_{\{(s_i,t_i)\},\{\lambda_i\}} SZ \le 1.$$

By Lemma A.2 in Appendix A the supremum on the left-most side equals $\int_0^\infty \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot\dot{M}_s - \lambda\cdot\langle\dot{M}\rangle_s\lambda/2)\,ds$. We thus have

$$S\Big[\exp\Big(\int_0^\infty \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot\dot{M}_s - \frac{1}{2}\lambda\cdot\langle\dot{M}\rangle_s\lambda)\,ds\Big)\Big] \le 1.$$

The supremum in the integral is finite if and only if $\dot{M}_s$ is orthogonal to the nullspace of $\langle\dot{M}\rangle_s$, which is equivalent to $\dot{M}_s$ being in the range of $\langle\dot{M}\rangle_s$. It is then equal to $\dot{M}_s\cdot\langle\dot{M}\rangle_s^{\oplus}\dot{M}_s/2$, which proves the first claim of part 2. The second one is an obvious consequence of the first.

For the final assertion, we note that since $\dot{M}_s(\omega)$ belongs to the range of $\langle\dot{M}\rangle_s(\omega)$ for almost all s and Π-almost all ω, we have that $\dot{M}_s(\omega) = \langle\dot{M}\rangle_s(\omega)\langle\dot{M}\rangle_s^{\oplus}(\omega)\dot{M}_s(\omega)$ for almost all s and Π-almost all ω. Therefore, by the Cauchy-Schwarz inequality Π-a.e.

$$\int_0^t |\sigma_s\dot{M}_s|\,ds \le \Big(\int_0^t \|\sigma_s\langle\dot{M}\rangle_s\sigma_s^T\|\,ds\Big)^{1/2}\Big(\int_0^t \dot{M}_s\cdot\langle\dot{M}\rangle_s^{\oplus}\dot{M}_s\,ds\Big)^{1/2},$$

where the right-hand side is finite Π-a.e. by hypotheses and the part of the lemma already proved. ☐

Definition 2.5.9. *Let idempotent processes M and σ be as in the statement of Lemma 2.5.8. An idempotent process $X = (X_t(\omega),\ t \in \mathbb{R}_+, \omega\in\Omega)$ defined by*

$$X_t(\omega) = \begin{cases} \displaystyle\int_0^t \sigma_s(\omega)\dot{M}_s(\omega)\,ds, & \textit{if } \Pi(\omega) > 0, \\ \tilde{X}_t(\omega), & \textit{if } \Pi(\omega) = 0, \end{cases}$$

where $\tilde{X}_t(\omega)$ is a continuous idempotent process, is called an idempotent stochastic integral of σ with respect to M and denoted by $\sigma\diamond M = \big(\int_0^t \sigma_s\dot{M}_s\,ds,\ t\in\mathbb{R}_+\big)$.

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In particular, if M is a d-dimensional Wiener idempotent process W and $\int_0^t \|\sigma_s\|^2\,ds < \infty$, $t \in \mathbb{R}_+$, Π-a.e., the integral $\sigma \diamond W$ is called an idempotent Ito integral.

Clearly, an idempotent stochastic integral is a continuous $\mathbf{A}$-adapted process and is specified uniquely Π-a.e. We show that under certain conditions on the integrands idempotent stochastic integrals are local maxingales with quadratic characteristics. We begin with an approximation lemma.

Lemma 2.5.10. *Let an $\mathbb{R}^d$-valued continuous $\mathbf{A}$-adapted idempotent process M be an $\mathbf{A}$-local maxingale with an absolutely continuous quadratic characteristic $\langle M\rangle$. Let $(\sigma_s^k(\omega), s \in \mathbb{R}_+, \omega \in \Omega), k \in \mathbb{N}$, and $(\sigma_s(\omega), s \in \mathbb{R}_+, \omega \in \Omega)$ be $\mathbf{A}$-progressively measurable $\mathbb{R}^{m\times d}$-valued idempotent processes such that for $t \in \mathbb{R}_+$*

$$\int_0^t \|\sigma_s \langle \dot{M}\rangle_s \sigma_s^T\|\,ds < \infty, \qquad \int_0^t \|\sigma_s^k \langle \dot{M}\rangle_s (\sigma_s^k)^T\|\,ds < \infty,$$

$$\int_0^t \|(\sigma_s^k - \sigma_s)\langle \dot{M}\rangle_s (\sigma_s^k - \sigma_s)^T\|\,ds \xrightarrow{\Pi} 0 \text{ as } k \to \infty. \quad (2.5.8a)$$

If the idempotent processes $\sigma^k \diamond M$ are $\mathbf{A}$-local maxingales with the quadratic characteristics $\left(\int_0^t \sigma_s^k \langle \dot{M}\rangle_s (\sigma_s^k)^T\,ds,\ t \in \mathbb{R}_+\right)$, then the idempotent process $\sigma \diamond M$ is an $\mathbf{A}$-local maxingale with the quadratic characteristic $\left(\int_0^t \sigma_s \langle \dot{M}\rangle_s \sigma_s^T\,ds,\ t \in \mathbb{R}_+\right)$.

Proof. The idempotent processes $\sigma \diamond M$ and $\sigma^k \diamond M$ are well defined by Lemma 2.5.8. The idempotent process $\sigma \diamond M$ is $\mathbf{A}$-adapted by Lemma 2.2.17 and completeness of $\mathbf{A}$. We introduce, for $\lambda \in \mathbb{R}^m$,

$$Z_t(\lambda) = \exp\Big(\lambda \cdot (\sigma \diamond M)_t - \frac{1}{2}\int_0^t \lambda \cdot \sigma_s \langle \dot{M}\rangle_s \sigma_s^T \lambda\,ds\Big), \quad (2.5.9a)$$

$$Z_t^k(\lambda) = \exp\Big(\lambda \cdot (\sigma^k \diamond M)_t - \frac{1}{2}\int_0^t \lambda \cdot \sigma_s^k \langle \dot{M}\rangle_s (\sigma_s^k)^T \lambda\,ds\Big). \quad (2.5.9b)$$

The idempotent processes $(Z_t^k(\lambda),\ t \in \mathbb{R}_+)$ are $\mathbf{A}$-local exponential maxingales by hypotheses. We have to prove that the idempotent

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process $(Z_t(\lambda),\ t \in \mathbb{R}_+)$ is an **A**-local exponential maxingale. Let us note that by Lemma 2.5.8

$$SZ_t(\lambda) \le 1,\ SZ_t^k(\lambda) \le 1,\ \lambda \in \mathbb{R}^m. \tag{2.5.10}$$

Let us introduce for $n \in \mathbb{N}$

$$\tau_n = \inf\{t \in \mathbb{R}_+ :\ \int_0^t \|\sigma_s \langle \dot{M} \rangle_s \sigma_s^T\|\ ds \ge n\}, \tag{2.5.11a}$$

$$\tau_n^k = \inf\{t \in \mathbb{R}_+ :\ \int_0^t \|\sigma_s^k \langle \dot{M} \rangle_s (\sigma_s^k)^T\|\ ds \ge n+1\} \wedge \tau_n. \tag{2.5.11b}$$

We show that the idempotent processes $(Z_{t\wedge\tau_n}(\lambda), t \in \mathbb{R}_+)$ are uniformly maximable. By (2.5.9a), (2.5.11a) and (2.5.10)

$$S\big(Z_{t\wedge\tau_n}(\lambda)^2\big) = S\Big[Z_{t\wedge\tau_n}(2\lambda)\exp\Big(\int_0^{t\wedge\tau_n} \lambda \cdot \sigma_s \langle \dot{M} \rangle_s \sigma_s^T \lambda\, ds\Big)\Big]$$
$$\le \exp\big(|\lambda|^2 n\big).$$

Thus,

$$\sup_{t\in\mathbb{R}_+} S\big(Z_{t\wedge\tau_n}(\lambda)^2\big) < \infty, \tag{2.5.12}$$

proving uniform maximability of $(Z_{t\wedge\tau_n}(\lambda), t \in \mathbb{R}_+)$ by Corollary 1.4.15. A similar argument shows that $S\big(Z^k_{t\wedge\tau_n^k}(\lambda)^2\big) \le \exp\big(|\lambda|^2(n+1)\big)$. Therefore, the collection $\{Z^k_{t\wedge\tau_n^k}(\lambda), k \in \mathbb{N},\ t \in \mathbb{R}_+\}$ is uniformly maximable; in particular, the $(Z^k_{t\wedge\tau_n^k}(\lambda),\ t \in \mathbb{R}_+),\ k \in \mathbb{N}$, are uniformly maximable exponential maxingales. By Lemma 1.6.22 it thus suffices to prove that $Z^k_{t\wedge\tau_n^k}(\lambda) \xrightarrow{\Pi} Z_{t\wedge\tau_n}(\lambda)$ as $k \to \infty$. Since (2.5.12) implies that $Z_{t\wedge\tau_n}(\lambda)$ is a proper idempotent variable, it follows by the definitions (2.5.9a) and (2.5.9b) that the required is a consequence of the convergences as $k \to \infty$

$$\lambda \cdot (\sigma^k \diamond M)_{t\wedge\tau_n^k} \xrightarrow{\Pi} \lambda \cdot (\sigma \diamond M)_{t\wedge\tau_n}, \tag{2.5.13a}$$

$$\int_0^{t\wedge\tau_n^k} \lambda \cdot \sigma_s^k \langle \dot{M} \rangle_s (\sigma_s^k)^T \lambda\, ds \xrightarrow{\Pi} \int_0^{t\wedge\tau_n} \lambda \cdot \sigma_s \langle \dot{M} \rangle_s \sigma_s^T \lambda\, ds. \tag{2.5.13b}$$

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Let us first note that in view of (2.5.11a), (2.5.11b) and (2.5.8a) we have that

$$\lim_{k\to\infty} \Pi\big(t\wedge\tau_n^k \neq t\wedge\tau_n\big) = 0. \tag{2.5.14}$$

Limit (2.5.13a) follows now by the inequalities

$$\Pi\big(|\lambda\cdot(\sigma^k\diamond M)_{t\wedge\tau_n^k} - \lambda\cdot(\sigma\diamond M)_{t\wedge\tau_n}| > \epsilon\big) \leq \Pi\big(t\wedge\tau_n^k \neq t\wedge\tau_n\big) + \Pi\Big(\int_0^t |\lambda\cdot(\sigma_s^k - \sigma_s)\dot{M}_s(\omega)|\,ds > \varepsilon\Big),$$

$$\int_0^t |\lambda\cdot(\sigma_s^k-\sigma_s)\dot{M}_s|\,ds \leq \Big(\int_0^t \lambda\cdot(\sigma_s^k-\sigma_s)\langle\dot{M}\rangle_s(\sigma_s^k-\sigma_s)^T\lambda\,ds\Big)^{1/2} \Big(\int_0^t \dot{M}_s\cdot\langle\dot{M}\rangle_s^{\oplus}\dot{M}_s\,ds\Big)^{1/2},$$

$$\Pi\Big(\int_0^t \dot{M}_s\cdot\langle\dot{M}\rangle_s^{\oplus}\dot{M}_s\,ds \geq A\Big) \leq \exp(-A/2),$$

and (2.5.8a). Similarly, limit (2.5.13b) follows by (2.5.14), (2.5.8a), (2.5.11a), and the inequality

$$\Big|\int_0^{t\wedge\tau_n} \lambda\cdot\sigma_s^k\langle\dot{M}\rangle_s(\sigma_s^k)^T\lambda\,ds - \int_0^{t\wedge\tau_n} \lambda\cdot\sigma_s\langle\dot{M}\rangle_s\sigma_s^T\lambda\,ds\Big|$$
$$\leq 2\Big(\int_0^{t\wedge\tau_n} \lambda\cdot(\sigma_s^k-\sigma_s)\langle\dot{M}\rangle_s(\sigma_s^k-\sigma_s)^T\lambda\,ds\Big)^{1/2} \Big(\int_0^{t\wedge\tau_n} \lambda\cdot\sigma_s\langle\dot{M}\rangle_s\sigma_s^T\lambda\,ds\Big)^{1/2}$$
$$+ \int_0^{t\wedge\tau_n} \lambda\cdot(\sigma_s^k-\sigma_s)\langle\dot{M}\rangle_s(\sigma_s^k-\sigma_s)^T\lambda\,ds.$$

□

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Theorem 2.5.11. *Let an $\mathbb{R}^d$-valued continuous $\mathbf{A}$-adapted idempotent process M be an $\mathbf{A}$-local maxingale with an absolutely continuous quadratic characteristic $\langle M \rangle$, which is a proper idempotent process. Let $(\sigma_s(\omega),\ s \in \mathbb{R}_+, \omega \in \Omega)$ be an $\mathbb{R}^{m\times d}$-valued $\mathbf{A}$-progressively measurable idempotent process such that for $t \in \mathbb{R}_+$*

$$\int_0^t \|\sigma_s \langle \dot{M} \rangle_s \sigma_s^T\|\, ds \quad < \quad \infty,$$

$$\int_0^t \|\sigma_s \langle \dot{M} \rangle_s \sigma_s^T\|\, \mathbf{1}(\,\|\sigma_s\| > A)\, ds \ \stackrel{\Pi}{\to} 0 \quad \text{as } A \to \infty.$$

Let there exist functions $n_A : \mathbb{R}_+ \to [0,1]$, where $A \in \mathbb{R}_+$, such that $n_A(x) = 1$ if $x \le A$, $n_A(x) \le A/x$ if $x \ge A$, and for all A large enough

$$\lim_{\delta \to 0} \Pi\Big(\int_0^t \|\big(\sigma_s n_A(\,\|\sigma_s\|\,) - \sigma_{s-\delta} n_A(\,\|\sigma_{s-\delta}\|\,)\big) \langle \dot{M} \rangle_s \big(\sigma_s n_A(\,\|\sigma_s\|\,) - \sigma_{s-\delta} n_A(\,\|\sigma_{s-\delta}\|\,)\big)^T \|\, ds > \eta \Big) = 0, \qquad t \in \mathbb{R}_+, \eta > 0,$$

where $\sigma_s(\omega) = 0$ for $s < 0$. Then the idempotent stochastic integral $\sigma \diamond M$ is an $\mathbf{A}$-local maxingale with the quadratic characteristic

$$\langle \sigma \diamond M \rangle_t = \int_0^t \sigma_s \langle \dot{M} \rangle_s \sigma_s^T\, ds.$$

The latter is a proper idempotent process.

Proof. We have to prove that $Z(\lambda) = (Z_t(\lambda), t \in \mathbb{R}_+)$, $\lambda \in \mathbb{R}^m$, defined as in (2.5.9a) is an $\mathbf{A}$-local exponential maxingale. It is $\mathbf{A}$-adapted by Lemma 2.2.17 and completeness of the τ-flow $\mathbf{A}$. Let us first consider the case that

$$\sigma_s(\omega) = \sum_{i=1}^k f_i(\omega)\, \mathbf{1}_{(t_{i-1}, t_i]}(s),$$

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where $0 = t_0 < t_1 < \ldots < t_k$ and the f_i are $\mathcal{A}_{t_{i-1}}$-measurable and bounded $\mathbb{R}^{m\times d}$-valued idempotent variables. Then

$$Z_t(\lambda) = \exp\Big[\sum_{i=1}^k\Big(\lambda\cdot f_i\big(M_{t_i\wedge t} - M_{t_{i-1}\wedge t}\big) - \frac{1}{2}\lambda\cdot f_i(\langle M\rangle_{t_i\wedge t} - \langle M\rangle_{t_{i-1}\wedge t})f_i^T\lambda\Big)\Big]. \quad (2.5.15)$$

Let τ_n be defined by (2.5.3). By Lemma 2.5.8 $S[\exp(\lambda\cdot M_{t\wedge\tau_n} - \lambda\cdot\langle M\rangle_{t\wedge\tau_n}\lambda/2)] \le 1$, which implies as in the proof of Lemma 2.5.10 that the idempotent process $(\exp(\lambda\cdot M_{t\wedge\tau_n} - \lambda\cdot\langle M\rangle_{t\wedge\tau_n}\lambda/2),\ t\in\mathbb{R}_+)$ is uniformly maximable. Hence, by Lemma 2.3.13 $(\exp(\lambda\cdot M_{t\wedge\tau_n} - \lambda\cdot\langle M\rangle_{t\wedge\tau_n}\lambda/2),\ t\in\mathbb{R}_+)$ is a uniformly maximable **A**-exponential maxingale.

Let

$$Y_t^i = \exp\Big(\lambda\cdot f_i\big(M_{t_i\wedge t\wedge\tau_n} - M_{t_{i-1}\wedge t\wedge\tau_n}\big) - \frac{1}{2}\lambda\cdot f_i(\langle M\rangle_{t_i\wedge t\wedge\tau_n} - \langle M\rangle_{t_{i-1}\wedge t\wedge\tau_n})f_i^T\lambda\Big). \quad (2.5.16)$$

Since f_i is $\mathcal{A}_{t_{i-1}}$-measurable, Lemma 1.6.21 implies that

$$S(Y_t^i|\mathcal{A}_{t_{i-1}}) = 1. \quad (2.5.17)$$

By (2.5.15), (2.5.16), and (2.5.17) we have

$$SZ_{t\wedge\tau_n}(\lambda) = S\Big(\prod_{i=1}^k Y_t^i\Big) = S\Big(\prod_{i=1}^{k-1} Y_t^i S(Y_t^k|\mathcal{A}_{t_{k-1}})\Big) = S\Big(\prod_{i=1}^{k-2} Y_t^i S(Y_t^{k-1}|\mathcal{A}_{t_{k-2}})\Big) = \ldots = SY_t^1 = 1.$$

Since the latter holds for all $\lambda\in\mathbb{R}^m$, we have by (2.5.15) and (2.5.3) that

$$S\big(Z_{t\wedge\tau_n}(\lambda)^2\big) \le S\big(Z_{t\wedge\tau_n}(2\lambda)\big)\exp(|\lambda|^2Bnk) = \exp(|\lambda|^2Bnk),$$

where B is an upper bound for the $\|f_i\|^2$. By Corollary 1.4.15 we conclude that the idempotent process $(Z_{t\wedge\tau_n}(\lambda),\ t\in\mathbb{R}_+)$ is uniformly maximable.

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The maxingale property of $(Z_{t\wedge\tau_n}(\lambda),\, t \in \mathbb{R}_+)$ is checked similarly. Let $s \le t$. If $s \ge t_k$, then $Z_{s\wedge\tau_n}(\lambda) = Z_{t\wedge\tau_n}(\lambda)$ and the maxingale property trivially holds. Let $t_{i-1} < s \le t_i$ for some i. Since by the argument used above $S(Z_{t\wedge\tau_n}(\lambda)|\mathcal{A}_{t_i}) = Z_{t_i\wedge\tau_n}(\lambda)$, it follows that

$$S(Z_{t\wedge\tau_n}(\lambda)|\mathcal{A}_s) = S(S(Z_{t\wedge\tau_n}(\lambda)|\mathcal{A}_{t_i})|\mathcal{A}_s) = S(Z_{t_i\wedge\tau_n}(\lambda)|\mathcal{A}_s) \\ = Z_{t_{i-1}\wedge\tau_n}(\lambda) S(Y^i_{t_i}|\mathcal{A}_s).$$

By (2.5.16), since f_i is $\mathcal{A}_s$-measurable,

$$\begin{aligned}
&S(Y^i_{t_i}|\mathcal{A}_s)\\
&= \exp\Bigl(\lambda \cdot f_i(M_{s\wedge\tau_n} - M_{t_{i-1}\wedge\tau_n})\\
&- \frac{1}{2}\lambda \cdot f_i(\langle M\rangle_{s\wedge\tau_n} - \langle M\rangle_{t_{i-1}\wedge\tau_n}) f_i^T \lambda\Bigr)\\
&S\Bigl(\exp\bigl(\lambda \cdot f_i(M_{t_i\wedge\tau_n} - M_{s\wedge\tau_n})\\
&- \frac{1}{2}\lambda \cdot f_i(\langle M\rangle_{t_i\wedge\tau_n} - \langle M\rangle_{s\wedge\tau_n}) f_i^T \lambda\bigr)|\mathcal{A}_s\Bigr)\\
&= \exp\Bigl(\lambda \cdot f_i(M_{s\wedge\tau_n} - M_{t_{i-1}\wedge\tau_n})\\
&\qquad\qquad - \frac{1}{2}\lambda \cdot f_i(\langle M\rangle_{s\wedge\tau_n} - \langle M\rangle_{t_{i-1}\wedge\tau_n}) f_i^T \lambda\Bigr),
\end{aligned}$$

where the latter equality follows by the maxingale property of $\bigl(\exp(\lambda \cdot M_{t\wedge\tau_n} - \lambda \cdot \langle M\rangle_{t\wedge\tau_n}\lambda/2)\bigr)$ and Lemma 1.6.21. Putting everything together, we conclude that

$$S(Z_{t\wedge\tau_n}(\lambda)|\mathcal{A}_s) = Z_{s\wedge\tau_n}(\lambda).$$

Let us now assume that σ_s is bounded and locally continuous in s uniformly on Ω, i.e.,

$$w_T(\delta) = \sup_{\omega\in\Omega} \sup_{\substack{s,t\le T\\ |s-t|\le\delta}} \|\sigma_s(\omega) - \sigma_t(\omega)\| \to 0 \quad \text{as } \delta \to 0,\, T > 0.$$

Let us first note that $\int_0^t \|\langle \dot{M}\rangle_s\|\, ds$ is a proper idempotent variable since the integral is not greater than the sum of the diagonal entries of $\langle M\rangle_t$ and the latter is a proper idempotent variable. Let

$$\sigma^k_s = \sum_{i=1}^{k^2} \sigma_{(i-1)/k}\, \mathbf{1}(s \in ((i-1)/k, i/k]).$$

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Then by the part just proved the $Z^k(\lambda)$, defined as $Z(\lambda)$ with σ_s changed to σ_s^k, are $\mathbf{A}$-local exponential maxingales. We also have that for $k \geq t$

$$\int_0^t \|(\sigma_s - \sigma_s^k)\langle \dot{M}\rangle_s(\sigma_s - \sigma_s^k)^T\| \, ds \leq w_t(1/k)^2 \int_0^t \|\langle \dot{M}\rangle_s\| \, ds$$

so that by Lemma 2.5.10 and the fact that $\int_0^t \|\langle \dot{M}\rangle_s\| \, ds$ is a proper idempotent variable we conclude that $Z(\lambda)$ is an $\mathbf{A}$-local exponential maxingale.

Let us assume now that σ_s in the statement of the theorem is bounded. We introduce the Steklov functions

$$\sigma_s^k = k \int_{s-1/k}^{s} \sigma_u \, du.$$

Since $\|\sigma_s^k(\omega) - \sigma_t^k(\omega)\| \leq 2k \sup_{u,\omega} \|\sigma_u(\omega)\| \, |t-s|$, the functions σ_s^k are continuous in s uniformly over ω. They are also bounded and properly measurable so that by the part just proved the associated idempotent processes $Z^k(\lambda)$ are $\mathbf{A}$-local exponential maxingales. We again apply Lemma 2.5.10 to deduce that $Z(\lambda)$ is an $\mathbf{A}$-local exponential maxingale. We have

$$\begin{aligned}
&\int_0^t \|(\sigma_s - \sigma_s^k)\langle \dot{M}\rangle_s(\sigma_s - \sigma_s^k)^T\| \, ds \\
&= \int_0^t \Big\| \Big(k \int_0^{1/k} (\sigma_s - \sigma_{s-u}) \, du\Big) \langle \dot{M}\rangle_s \Big(k \int_0^{1/k} (\sigma_s - \sigma_{s-u})^T \, du\Big) \Big\| \, ds \\
&\leq \int_0^t k \int_0^{1/k} \|(\sigma_s - \sigma_{s-u})\langle \dot{M}\rangle_s(\sigma_s - \sigma_{s-u})^T\| \, du \, ds \\
&\qquad\qquad \leq \sup_{0\leq u\leq 1/k} \int_0^t \|(\sigma_s - \sigma_{s-u})\langle \dot{M}\rangle_s(\sigma_s - \sigma_{s-u})^T\| \, ds.
\end{aligned}$$

The latter supremum converges in idempotent probability to 0 as $k \to \infty$ by hypotheses.

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Finally, if σ_s in the statement of the theorem is not bounded, we define $\sigma_s^k = n_k(\|\sigma_s\|)\sigma_s$. Then by hypotheses

$$\int_0^t \|(\sigma_s - \sigma_s^k)\langle \dot{M}\rangle_s (\sigma_s - \sigma_s^k)^T\| \, ds$$

$$\leq \int_0^t \|\sigma_s \langle \dot{M}\rangle_s \sigma_s^T\| \, \mathbf{1}(\|\sigma_s\| \geq k) \, ds \stackrel{\Pi}{\to} 0$$

as $k \to \infty$, and since associated with the σ^k idempotent processes $Z^k(\lambda)$ are $\mathbf{A}$-local exponential maxingales by the part already proved, Lemma 2.5.10 implies that $Z(\lambda)$ is an $\mathbf{A}$-local exponential maxingale.

The fact that $\langle \sigma \diamond M \rangle$ is a proper idempotent process is obvious. □

For idempotent Ito integrals we have the following corollary.

Theorem 2.5.12. *Let W be an $\mathbb{R}^d$-valued $\mathbf{A}$-Wiener idempotent process. Let $(\sigma_s(\omega),\ s \in \mathbb{R}_+, \omega \in \Omega)$ be an $\mathbb{R}^{m\times d}$-valued $\mathbf{A}$-progressively measurable idempotent process such that $\int_0^t \|\sigma_s\|^2 \, ds < \infty$ and for a function n_A as in Theorem 2.5.11*

$$\int_0^t \|\sigma_s n_A(\|\sigma_s\|) - \sigma_{s+\delta} n_A(\|\sigma_{s+\delta}\|)\|^2 \, ds \quad \stackrel{\Pi}{\to} \quad 0 \quad \text{as } \delta \to 0, \quad t \in \mathbb{R}_+,$$

for all A large enough,

$$\int_0^t \|\sigma_s\|^2 \, \mathbf{1}(\|\sigma_s\| > A) \, ds \quad \stackrel{\Pi}{\to} \quad 0 \quad \text{as } A \to \infty, \quad t \in \mathbb{R}_+.$$

Then $\sigma \diamond W$ is an $\mathbf{A}$-local maxingale with the quadratic characteristic

$$\langle \sigma \diamond W \rangle_t = \int_0^t \sigma_s \sigma_s^T \, ds,$$

which is a proper process.

Remark 2.5.13. *The convergence conditions in Theorem 2.5.12 imply that $\int_0^t \|\sigma_s - \sigma_{s+\delta}\|^2 \, ds \stackrel{\Pi}{\to} 0$ as $\delta \to 0$. Therefore, by M.*

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Riesz's criterion for relative compactness in L_2, see, e.g., Kantorovich and Akilov [70], under the hypotheses the idempotent distribution of $\omega \to (\sigma_s(\omega),\ s \le t)$ is a deviability in $L_2([0,t], \mathbb{R}^{m\times d})$ for all $t \in \mathbb{R}_+$.

We now prove that in analogy with stochastic calculus under certain regularity conditions local maxingales with quadratic characteristics are idempotent Ito integrals. We adapt notation (1.7.2) to denote

$$K_\Pi(a) = \{\omega \in \Omega : \Pi(\omega) \ge a\},\ a \in (0,1].$$

Theorem 2.5.14. *Let an $\mathbb{R}^d$-valued continuous $\mathbf{A}$-adapted idempotent process M be an $\mathbf{A}$-local maxingale with an absolutely continuous quadratic characteristic $(\int_0^t \sigma_s\sigma_s^T\, ds,\ t \in \mathbb{R}_+)$, which is a proper idempotent process, where the σ_s are $d \times d$ matrices. If for $t \in \mathbb{R}_+$ and $a \in (0,1]$*

$$\int_0^t \|\sigma_s - \sigma_{s+\delta}\|^2\, ds \xrightarrow{\Pi} 0 \quad as\ \delta \to 0,$$

$$\inf_{\omega \in K_\Pi(a)} \inf_{s \le t} \inf_{\substack{\lambda \in \mathbb{R}^d: \\ |\lambda|=1}} \lambda \cdot \sigma_s(\omega)\sigma_s^T(\omega)\lambda > 0,$$

then there exists a d-dimensional $\mathbf{A}$-Wiener idempotent process W such that $M = \sigma \diamond W$.

Proof. We define a continuous idempotent process $W = \sigma^{-1} \diamond M$, where the right-hand side is well defined by Lemma 2.5.8. Clearly, $M = \sigma \diamond W$. Let $n_A(x) = \mathbf{1}(x \le A)$, where $x \in \mathbb{R}_+$ and $A \in \mathbb{R}_+$. Then, denoting $\sigma_s^{-1} = \sigma_s = 0$ for $s < 0$, we have

$$\int_0^t \|\big(\sigma_{s-\delta}^{-1} n_A(\|\sigma_{s-\delta}^{-1}\|) - \sigma_s^{-1} n_A(\|\sigma_s^{-1}\|)\big)\sigma_s\sigma_s^T$$
$$\big(\sigma_{s-\delta}^{-1} n_A(\|\sigma_{s-\delta}^{-1}\|) - \sigma_s^{-1} n_A(\|\sigma_s^{-1}\|)\big)^T\|\, ds$$
$$= \int_0^t \|\sigma_{s-\delta}^{-1}\big(\sigma_s n_A(\|\sigma_{s-\delta}^{-1}\|) - \sigma_{s-\delta} n_A(\|\sigma_s^{-1}\|)\big)$$
$$\big(\sigma_s n_A(\|\sigma_{s-\delta}^{-1}\|) - \sigma_{s-\delta} n_A(\|\sigma_s^{-1}\|)\big)^T (\sigma_{s-\delta}^T)^{-1}\|\, ds$$

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$$\leq \sup_{s\leq t} \|\sigma_s\sigma_s^T\|^{-1} \int_0^t \|\sigma_s n_A(\|\sigma_{s-\delta}^{-1}\|) - \sigma_{s-\delta} n_A(\|\sigma_s^{-1}\|)\|^2 \, ds$$

$$\leq \sup_{s\leq t} \|\sigma_s\sigma_s^T\|^{-1}\Big(15 \int_0^t \|\sigma_s - \sigma_{s-\delta}\|^2 \, ds$$

$$+ 10 \int_0^t \|\sigma_s\|^2 \, \mathbf{1}(\|\sigma_s^{-1}\| > A) \, ds\Big).$$

By hypotheses the latter converges in idempotent probability to 0 as $\delta \to 0$ for all large A. Hence, by Theorem 2.5.11 W is an $\mathbf{A}$-local maxingale with the quadratic characteristic $(\mathbf{E}_d\, t,\, t \in \mathbb{R}_+)$; hence, W is a Wiener idempotent process by Corollary 2.5.2. □

Remark 2.5.15. *One can replace the infimum over s above by the essential infimum with respect to Lebesgue measure. This remark also concerns other conditions of a similar sort.*

We now consider versions for strictly Luzin idempotent processes defined on Hausdorff topological spaces with deviabilities. We start with a lemma on properties of trajectories.

Lemma 2.5.16. *Let Ω be a Hausdorff topological space and Π be a deviability on Ω. Let an $\mathbb{R}^d$-valued continuous $\mathbf{A}$-adapted strictly Luzin idempotent process M be an $\mathbf{A}$-local maxingale on (Ω, Π) with an absolutely continuous quadratic characteristic $\langle M \rangle$ such that $(\langle \dot{M} \rangle_s,\, s \in \mathbb{R}_+)$ is a strictly Luzin idempotent process and for $t \in \mathbb{R}_+$*

$$\int_0^t \|\langle \dot{M}\rangle_s\| \, \mathbf{1}(\|\langle \dot{M}\rangle_s\| > A) \, ds \xrightarrow{\Pi} 0 \quad \text{as } A \to \infty.$$

Let $(\sigma_s(\omega),\, s \in \mathbb{R}_+, \omega \in \Omega)$ be an $\mathbb{R}^{m\times d}$-valued $\mathbf{A}$-progressively measurable strictly Luzin idempotent process such that $\int_0^t \|\sigma_s \langle \dot{M}\rangle_s \sigma_s^T\| \, ds < \infty$ and

$$\int_0^t \|\sigma_s \langle \dot{M}\rangle_s \sigma_s^T\| \, \mathbf{1}(\|\sigma_s\| > A) \, ds \xrightarrow{\Pi} 0 \quad \text{as } A \to \infty, \quad t \in \mathbb{R}_+.$$

Then both $\sigma \diamond M$ and $(\int_0^t \sigma_s \langle \dot{M}\rangle_s \sigma_s^T \, ds,\, t \in \mathbb{R}_+)$ are strictly Luzin-continuous idempotent processes.

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Proof. We first check that $\sigma \diamond M$ is strictly Luzin. Let $\omega_\phi \in K_\Pi(a)$, where $a \in (0,1]$, and $\omega_\phi \to \omega'$ as $\phi \in \Phi$. We write for $t \in \mathbb{R}_+$ and $A \in \mathbb{R}_+$, denoting $k_A(\gamma) = \gamma\big((A+1-\|\gamma\|)^+ \wedge 1\big)$ for a matrix γ,

$$\begin{aligned}
&\|\sigma \diamond M_t(\omega_\phi) - \sigma \diamond M_t(\omega')\| \\
&\le 2 \sup_{\omega \in K_\Pi(a)} \Big\| \int_0^t \big(\sigma_s(\omega) - k_A(\sigma_s(\omega))\big) \dot{M}_s(\omega)\, ds \Big\| \\
&+ \Big\| \int_0^t k_A(\sigma_s(\omega_\phi)) \dot{M}_s(\omega_\phi)\, ds - \int_0^t k_A(\sigma_s(\omega')) \dot{M}_s(\omega_\phi)\, ds \Big\| \\
&\qquad + \Big\| \int_0^t k_A(\sigma_s(\omega')) \dot{M}_s(\omega_\phi)\, ds - \int_0^t k_A(\sigma_s(\omega')) \dot{M}_s(\omega')\, ds \Big\|.
\end{aligned} \tag{2.5.18}$$

We estimate the first term on the right-hand side of (2.5.18) as

$$\begin{aligned}
&\Big\| \int_0^t \big(\sigma_s(\omega) - k_A(\sigma_s(\omega))\big) \dot{M}_s(\omega)\, ds \Big\| \\
&\le \Big(\int_0^t \|\sigma_s(\omega) \langle \dot{M} \rangle_s(\omega) \sigma_s(\omega)^T\| \, \mathbf{1}(\|\sigma_s(\omega)\| > A)\, ds \Big)^{1/2} \\
&\qquad\qquad \Big(\int_0^t \dot{M}_s(\omega) \cdot \langle \dot{M} \rangle_s(\omega)^{\oplus} \dot{M}_s(\omega)\, ds \Big)^{1/2} ds,
\end{aligned}$$

which converges to 0 as $A \to \infty$ uniformly over $\omega \in K_\Pi(a)$ by Lemma 2.5.8 and the hypotheses.

For the second term on the right-hand side of (2.5.18) we write

$$\Big\| \int_0^t k_A(\sigma_s(\omega_\phi)) \dot{M}_s(\omega_\phi)\, ds - \int_0^t k_A(\sigma_s(\omega')) \dot{M}_s(\omega_\phi)\, ds \Big\|$$

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$$\leq \Big(\int_0^t \big\| \big(k_A(\sigma_s(\omega_\phi)) - k_A(\sigma_s(\omega'))\big) \langle \dot{M}\rangle_s(\omega_\phi)$$

$$\big(k_A(\sigma_s(\omega_\phi)) - k_A(\sigma_s(\omega'))\big)^T \big\| \, ds\Big)^{1/2}$$

$$\Big(\int_0^t \dot{M}_s(\omega_\phi) \cdot \langle \dot{M}\rangle_s(\omega_\phi)^{\oplus} \dot{M}_s(\omega_\phi) \, ds\Big)^{1/2}.$$

The second multiplier on the right is bounded in ϕ by Lemma 2.5.8. The integrand in the first one can be estimated for $B \in \mathbb{R}_+$ as

$$\big\| \big(k_A(\sigma_s(\omega_\phi)) - k_A(\sigma_s(\omega'))\big) \langle \dot{M}\rangle_s(\omega_\phi)$$
$$\big(k_A(\sigma_s(\omega_\phi)) - k_A(\sigma_s(\omega'))\big)^T \big\|$$
$$\leq B \big\| k_A(\sigma_s(\omega_\phi)) - k_A(\sigma_s(\omega')) \|^2$$
$$+ 4(A+1)^2 \, \|\langle \dot{M}\rangle_s(\omega_\phi)\| \; \mathbf{1}(\, \|\langle \dot{M}\rangle_s(\omega_\phi)\| \, > B),$$

which implies by hypotheses and Lebesgue's dominated convergence theorem that the second term on the right-hand side of (2.5.18) converges to 0 as $\phi \in \Phi$.

Finally, the third term on the right-hand side of (2.5.18) converges to 0 as $\phi \in \Phi$ by the following argument: since $\langle M\rangle_s(\omega_\phi) \to \langle M\rangle_s(\omega')$, it follows that $\int_0^t f_s \dot{M}_s(\omega_\phi)\, ds \to \int_0^t f_s \dot{M}_s(\omega')\, ds$ for step functions f_s; since the $\int_0^t \dot{M}_s(\omega_\phi) \cdot \langle \dot{M}\rangle_s(\omega_\phi)^{\oplus} \dot{M}_s(\omega_\phi)\, ds$ are uniformly bounded and $\int_0^t \dot{M}_s(\omega') \cdot \langle \dot{M}\rangle_s(\omega')^{\oplus} \dot{M}_s(\omega')\, ds$ is finite, by the Cauchy-Shwarz inequality the class of functions f_s for which the latter convergence holds is closed under bounded pointwise convergence; so a monotone class argument shows that it contains all bounded Borel-measurable functions f_s (see the proof of (2.7.28) in the proof of Lemma 2.7.17 below for a more detailed argument of this sort).

Now, in order to check that $\sigma \diamond M$ is strictly Luzin-continuous it is sufficient to show that uniformly over $\omega \in K_\Pi(a)$, where $a \in (0,1]$, the functions $(\sigma \diamond M_t(\omega),\, t \in \mathbb{R}_+)$ are uniformly continuous in $t \in$

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$[0, T]$ for every $T \in \mathbb{R}_+$. We have for $s, t \in [0, T]$

$$\|\sigma \diamond M_t(\omega) - \sigma \diamond M_s(\omega)\|^2$$
$$\leq \int_s^t \|\sigma_u(\omega)\langle \dot{M}\rangle_u(\omega)\sigma_u(\omega)^T\| \, du$$
$$\int_s^t \dot{M}_u(\omega) \cdot \langle \dot{M}\rangle_u(\omega)^{\oplus} \dot{M}_u(\omega) \, du.$$

The second term on the right is bounded on $K_\Pi(a)$ by Lemma 2.5.8. The first term is not greater for $A \in \mathbb{R}_+$ and $B \in \mathbb{R}_+$ than

$$\int_s^t \|\sigma_u(\omega)\langle \dot{M}\rangle_u(\omega)\sigma_u(\omega)^T\| \, \mathbf{1}(\, \|\sigma_u(\omega)\| \, > A) \, du$$
$$+ A^2 \int_s^t \|\langle \dot{M}\rangle_u(\omega)\| \, \mathbf{1}(\, \|\langle \dot{M}\rangle_u(\omega)\| \, > B) \, du + A^2 B(t - s),$$

which implies the required in view of the hypotheses.

The proof of $(\int_0^t \sigma_s \langle \dot{M}\rangle_s \sigma_s^T \, ds, \, t \in \mathbb{R}_+)$ being strictly Luzin-continuous uses similar ideas. Let $\omega_\phi \to \omega'$, where $\omega_\phi \in K_\Pi(a)$. Then by hypotheses and Lebesgue's dominated convergence theorem for $t \in \mathbb{R}_+$, $A \in \mathbb{R}_+$, $B \in \mathbb{R}_+$, and $\lambda \in \mathbb{R}^d$

$$\lim_\phi \int_0^t \lambda \cdot k_A(\sigma_s(\omega_\phi)) k_B(\langle \dot{M}\rangle_s(\omega_\phi)) k_A(\sigma_s^T(\omega_\phi))\lambda \, ds$$
$$= \int_0^t \lambda \cdot k_A(\sigma_s(\omega')) k_B(\langle \dot{M}\rangle_s(\omega')) k_A(\sigma_s^T(\omega'))\lambda \, ds.$$

Since

$$\int_0^t \lambda \cdot \sigma_s(\omega_\phi)\langle \dot{M}\rangle_s(\omega_\phi)\sigma_s^T(\omega_\phi)\lambda \, ds$$
$$\leq \int_0^t \lambda \cdot k_A(\sigma_s(\omega_\phi)) k_B(\langle \dot{M}\rangle_s(\omega_\phi)) k_A(\sigma_s^T(\omega_\phi))\lambda \, ds$$

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$$+ |\lambda|^2 \int_0^t \|\sigma_s(\omega_\phi)\langle \dot M\rangle_s(\omega_\phi)\sigma_s^T(\omega_\phi)\| \,\mathbf{1}(\,\|\sigma_s(\omega_\phi)\| > A)\,ds$$

$$+ |\lambda|^2 A^2 \int_0^t \|\langle \dot M\rangle_s(\omega_\phi)\| \,\mathbf{1}(\,\|\langle \dot M\rangle_s(\omega_\phi)\| > B)\,ds$$

and by hypotheses

$$\lim_{A\to\infty} \sup_\phi \int_0^t \|\sigma_s(\omega_\phi)\langle \dot M\rangle_s(\omega_\phi)\sigma_s^T(\omega_\phi)\| \,\mathbf{1}(\,\|\sigma_s(\omega_\phi)\| > A)\,ds = 0,$$

$$\lim_{B\to\infty} \sup_\phi \int_0^t \|\langle \dot M\rangle_s(\omega_\phi)\| \,\mathbf{1}(\,\|\langle \dot M\rangle_s(\omega_\phi)\| > B)\,ds = 0,$$

we conclude that

$$\limsup_\phi \int_0^t \lambda\cdot\sigma_s(\omega_\phi)\langle \dot M\rangle_s(\omega_\phi)\sigma_s^T(\omega_\phi)\lambda\,ds \le \int_0^t \lambda\cdot\sigma_s(\omega')\langle \dot M\rangle_s(\omega')\sigma_s^T(\omega')\lambda\,ds.$$

Fatou's lemma provides the reverse inequality. Thus, $(\int_0^t \sigma_s\langle \dot M\rangle_s\sigma_s^T\,ds,\ t\in\mathbb{R}_+)$ is strictly Luzin. Next, for $s,t\in[0,T]$

$$\Big\|\int_0^t \sigma_u\langle \dot M\rangle_u\sigma_u^T\,du - \int_0^s \sigma_u\langle \dot M\rangle_u\sigma_u^T\,du\Big\|$$

$$\le \int_0^T \|\sigma_u\langle \dot M\rangle_u\sigma_u^T\| \,\mathbf{1}(\,\|\sigma_u\| > A)\,du$$

$$+ A^2\int_0^T \|\langle \dot M\rangle_u\| \,\mathbf{1}(\,\|\langle \dot M\rangle_u\| > B)\,du + A^2B|t-s|,$$

which implies uniform continuity of $(\int_0^t \sigma_s\langle \dot M\rangle_s\sigma_s^T\,ds,\ t\in\mathbb{R}_+)$ on $[0,T]$ uniformly over $\omega\in K_\Pi(a)$. □

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Remark 2.5.17. *The hypotheses imply that both M and $\langle M\rangle$ are also strictly Luzin-continuous.*

Theorem 2.5.18. *Let Ω be a Hausdorff topological space and Π be a deviability on Ω. Let an $\mathbb{R}^d$-valued, continuous, strictly Luzin $\mathbf{A}$-adapted idempotent process M be an $\mathbf{A}$-local maxingale on (Ω,Π) with an absolutely continuous quadratic characteristic $\langle M\rangle$ such that $(\langle \dot{M}\rangle_s,\ s\in\mathbb{R}_+)$ is a strictly Luzin idempotent process and*

$$\sup_{s\le t}\ \sup_{\omega\in K_\Pi(a)} \|\langle \dot{M}\rangle_s(\omega)\| < \infty,\ t\in\mathbb{R}_+,\ a\in(0,1].$$

Let $(\sigma_s(\omega),\ s\in\mathbb{R}_+,\omega\in\Omega)$ be an $\mathbb{R}^{m\times d}$-valued $\mathbf{A}$-progressively measurable strictly Luzin idempotent process such that $\int_0^t \|\sigma_s\|^2\,ds < \infty,\ t\in\mathbb{R}_+$, and

$$\int_0^t \|\sigma_s\|^2\,\mathbf{1}(\|\sigma_s\| > A)\,ds \overset{\Pi}{\to} 0 \quad \text{as } A\to\infty, \quad t\in\mathbb{R}_+.$$

Then the idempotent process $\sigma\diamond M$ is an $\mathbf{A}$-local maxingale with the quadratic characteristic

$$\langle \sigma\diamond M\rangle_t = \int_0^t \sigma_s\langle \dot{M}\rangle_s\sigma_s^T\,ds.$$

Both $\sigma\diamond M$ and $\langle\sigma\diamond M\rangle$ are strictly Luzin-continuous idempotent processes.

Proof. We check the local maxingale property for $\sigma\diamond M$. Taking in the hypotheses of Theorem 2.5.11 $n_A(x) = (A+1-x)^+\wedge 1$, we have by hypotheses that $\omega\to(\sigma_s(\omega)n_A(\|\sigma_s(\omega)\|),\ s\in[0,t])$ is a continuous mapping from $K_\Pi(a)$ to $L_2([0,t],\mathbb{R}^{m\times d})$. Since

$$\int_0^t \|\big(\sigma_s n_A(\|\sigma_s\|) - \sigma_{s-\delta}n_A(\|\sigma_{s-\delta}\|)\big)\langle \dot{M}\rangle_s$$
$$\big(\sigma_s n_A(\|\sigma_s\|) - \sigma_{s-\delta}n_A(\|\sigma_{s-\delta}\|)\big)^T\|\,ds$$
$$\le \sup_{s\le t}\|\langle\dot{M}\rangle_s\|\int_0^t \|\sigma_s n_A(\|\sigma_s\|) - \sigma_{s-\delta}n_A(\|\sigma_{s-\delta}\|)\|^2\,ds \tag{2.5.19}$$

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and by M. Riesz's criterion for relative compactness in L_2 the right-hand side of (2.5.19) converges to 0 as $\delta \to 0$ uniformly over $\omega \in K_\Pi(a)$, we conclude by Theorem 2.5.11 that $\sigma \diamond M$ is an $\mathbf{A}$-local maxingale with the quadratic characteristic in the statement of the theorem. Both $\sigma \diamond M$ and $\langle \sigma \diamond M \rangle$ are strictly Luzin-continuous by Lemma 2.5.16. □

The following version of Theorem 2.5.12 is proved along the lines of the proof of Theorem 2.5.18.

Theorem 2.5.19. *Let W be an $\mathbb{R}^m$-valued $\mathbf{A}$-Wiener idempotent process and X be an $\mathbb{R}^d$-valued $\mathbf{A}$-adapted Luzin-continuous idempotent process with idempotent distribution Π^X. Let $(\sigma_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d))$ be an $\mathbb{R}^{k\times m}$-valued $\mathbf{C}(\mathbb{R}_+, \mathbb{R}^d)$-progressively measurable strictly Luzin idempotent process on $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d), \Pi^X)$ such that $\int_0^t \|\sigma_s(\mathbf{x})\|^2\, ds < \infty$, $t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$, and*

$$\int_0^t \|\sigma_s(\mathbf{x})\|^2\, \mathbf{1}(\|\sigma_s(\mathbf{x})\| > A)\, ds \overset{\Pi^X}{\to} 0 \quad \text{as } A \to \infty, \quad t \in \mathbb{R}_+.$$

Then the idempotent process $\sigma(X) \diamond W$ is an $\mathbf{A}$-local maxingale with the quadratic characteristic

$$\langle \sigma(X) \diamond W \rangle_t = \int_0^t \sigma_s(X)\sigma_s(X)^T\, ds,$$

which is a Luzin-continuous idempotent processes.

Remark 2.5.20. *For the convergence condition in the hypotheses to hold it is sufficient that for every compact $K \subset \mathbb{C}$ and $t \in \mathbb{R}_+$*

$$\int_0^t \sup_{\mathbf{x}\in K} \|\sigma_s(\mathbf{x})\|^2\, ds < \infty;$$

in particular, it is sufficient that $(\sigma_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d))$ satisfies the linear-growth condition

$$\|\sigma_t(\mathbf{x})\|^2 \le l_t(1 + \sup_{s\le t} |\mathbf{x}_s|^2),\ \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d), t \in \mathbb{R}_+,$$

where l_t is Lebesgue measurable and $\int_0^t l_s\, ds < \infty$, $t \in \mathbb{R}_+$.

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Remark 2.5.21. *Note that the maxingale property in Theorem 2.5.19 does not generally hold for discontinuous* $\sigma_s(\mathbf{x})$. *For example,* $\left(\int_0^t \operatorname{sign}(W_s)\dot{W}_s\,ds, t \in \mathbb{R}_+\right)$, *where* W *is an* $\mathbb{R}$*-valued idempotent Wiener process, is not a local maxingale, since* Π*-a.e.* $\int_0^t \operatorname{sign}(W_s)\dot{W}_s\,ds = |W_t|$.

The next consequence of Theorem 2.5.14 considers strictly Luzin processes on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$.

Theorem 2.5.22. *Let* Π *be a deviability on* $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ *and* $\mathbf{C}^\Pi(\mathbb{R}_+, \mathbb{R}^d)$ *denote the* Π*-completion of the* τ*-flow* $\mathbf{C}(\mathbb{R}_+, \mathbb{R}^d)$. *Let* $M = (M_t(\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d))$ *be a strictly Luzin idempotent process, which is a* $\mathbf{C}^\Pi(\mathbb{R}_+, \mathbb{R}^d)$*-local maxingale on* $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d), \Pi)$ *with quadratic characteristic* $(\int_0^t \sigma_s(\mathbf{x})\sigma_s(\mathbf{x})^T\,ds,\ t \in \mathbb{R}_+)$, *where* $(\sigma_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d))$ *is an* $\mathbb{R}^{d\times d}$*-valued* $\mathbf{C}^\Pi(\mathbb{R}_+, \mathbb{R}^d)$*-progressively measurable strictly Luzin idempotent process on* $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d), \Pi)$ *such that* $\int_0^t \|\sigma_s(\mathbf{x})\|^2\,ds < \infty$, $t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$,

$$\int_0^t \|\sigma_s(\mathbf{x})\|^2\,\mathbf{1}(\|\sigma_s(\mathbf{x})\| > A)\,ds \xrightarrow{\Pi} 0 \text{ as } A \to \infty,\ t \in \mathbb{R}_+,$$

and

$$\inf_{\mathbf{x} \in K_\Pi(a)} \inf_{s \le t} \inf_{\substack{\lambda \in \mathbb{R}^d: \\ |\lambda| = 1}} \lambda \cdot \sigma_s(\mathbf{x})\sigma_s(\mathbf{x})^T \lambda > 0,\ a \in (0,1].$$

Then there exists a strictly Luzin-continuous d-dimensional $\mathbf{C}^\Pi(\mathbb{R}_+, \mathbb{R}^d)$*-Wiener idempotent process* W *on* $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d), \Pi)$ *such that* $M = \sigma \diamond W$.

Proof. The hypotheses and M. Riesz's criterion for compactness in L_2 imply that

$$\int_0^t \|\sigma_s - \sigma_{s+\delta}\|^2\,ds \xrightarrow{\Pi} 0$$

as $\delta \to 0$. Thus, the hypotheses of Theorem 2.5.14 hold so that, according to the proof of the theorem, $W = \sigma^{-1} \diamond M$ is a Wiener idempotent process. It is strictly Luzin-continuous by Lemma 2.5.16. $\square$

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Theorem 2.5.23. *Let the τ-flow $\mathbf{A}$ be right-continuous. Let an $\mathbb{R}$-valued continuous $\mathbf{A}$-adapted idempotent process M be an $\mathbf{A}$-local maxingale with a continuous quadratic characteristic $\langle M\rangle$ such that $\lim_{t\to\infty}\langle M\rangle_t = \infty$ in idempotent probability. Then there exists a Wiener idempotent process W such that $M_s = W_{\langle M\rangle_s}$ Π-a.e.*

Proof. Let $\tau_s = \inf\{t \in \mathbb{R}_+ : \langle M\rangle_t > s\}$. The function τ_s is an $\mathbf{A}$-stopping time by Lemma 2.2.18 and is a proper idempotent variable. Since $\langle M\rangle_{\tau_s} = s$, Lemma 2.3.13 implies that $(\exp(\lambda M_{t\wedge\tau_s} - \lambda^2\langle M\rangle_{t\wedge\tau_s}/2),\ t \in \mathbb{R}_+)$ is a uniformly maximable $\mathbf{A}$-exponential maxingale so that by Theorem 2.3.8 the idempotent process $(\exp(\lambda M_{\tau_s} - \lambda^2 s/2),\ s \in \mathbb{R}_+)$ is an exponential maxingale relative to the τ-flow $\tilde{\mathbf{A}} = (\tilde{\mathcal{A}}_s,\ s \in \mathbb{R}_+)$, where $\tilde{\mathcal{A}}_s = \mathcal{A}_{\tau_s}$ and $M_\infty = 0$ for definiteness. Since the idempotent process $W_s = M_{\tau_s}$ also is Π-a.e. continuous by Lemma 2.5.4 and $\tilde{\mathbf{A}}$-adapted by Lemma 2.2.19, it is an $\tilde{\mathbf{A}}$-Wiener idempotent process by Theorem 2.4.2. The proof is complete if we show that $M_{\tau_{\langle M\rangle_s}} = M_s$ Π-a.e. Since $\{\tau_{\langle M\rangle_s} < t\} = \{\langle M\rangle_t > \langle M\rangle_s\}$, $\tau_{\langle M\rangle_s} \geq s$ and $\mathbf{A}$ is right-continuous, it follows that $\tau_{\langle M\rangle_s}$ is an $\mathbf{A}$-stopping time. Therefore, Lemma 2.5.3 implies that for $a > 0$, $b > 0$ and $c > 0$

$$\Pi(\sup_{s\leq t\leq \tau_{\langle M\rangle_s}} |M_t - M_s| \geq a) \leq e^{c(b-a)} \vee \Pi(\langle M\rangle_{\tau_{\langle M\rangle_s}} - \langle M\rangle_s > 2b/c).$$

Since $\langle M\rangle_{\tau_{\langle M\rangle_s}} = \langle M\rangle_s$ Π-a.e., we conclude that $\Pi(\sup_{s\leq t\leq\tau_{\langle M\rangle_s}}|M_t - M_s| > 0) = 0$. $\square$

Remark 2.5.24. *Note that the τ-flow $\tilde{\mathbf{A}}$ is also right-continuous by Lemma 2.1.11 and the fact that τ_s is right-continuous.*

We now consider analogues of Girsanov's theorem.

Theorem 2.5.25. *Let W be an $\mathbb{R}^m$-valued $\mathbf{A}$-Wiener idempotent process on (Ω, Π). Let $(b_s(\omega),\ s \in \mathbb{R}_+, \omega \in \Omega)$ be an $\mathbb{R}^m$-valued $\mathbf{A}$-progressively measurable idempotent process such that the idempotent processes $\big(\exp\big(\int_0^t(-b_s + \lambda)\cdot \dot{W}_s\,ds - \int_0^t|-b_s+\lambda|^2\,ds/2\big),\ t \in \mathbb{R}_+\big)$ are well-defined $\mathbf{A}$-exponential maxingales under Π for all $\lambda \in \mathbb{R}^m$. Let*

$$M_t = \exp\Big(-\int_0^t b_s\cdot\dot{W}_s\,ds - \frac{1}{2}\int_0^t |b_s|^2\,ds\Big).$$

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If there exists an idempotent probability Π' on Ω such that its restrictions Π'_t to the τ-algebras $\mathcal{A}_t$ are expressed as $d\Pi'_t = M_t\, d\Pi$, $t \in \mathbb{R}_+$, then the idempotent process $X = (X_t,\, t \in \mathbb{R}_+)$, defined by

$$X_t = \int_0^t b_s\, ds + W_t,$$

is an $\mathbf{A}$*-Wiener idempotent process under* Π'.

Proof. We first note that since $M_t > 0$, the sets of Π'-idempotent probability 0 have Π-idempotent probability 0; hence, the flow $\mathbf{A}$ is Π'-complete.

For $s \le t$ and $\lambda \in \mathbb{R}^m$ in view of Lemma 1.6.35

$$S_{\Pi'}\Bigl(\exp\bigl(\lambda \cdot X_t - \frac{1}{2}|\lambda|^2 t\bigr)|\mathcal{A}_s\Bigr) = \frac{S_\Pi\Bigl(M_t \exp\bigl(\lambda \cdot X_t - \frac{1}{2}|\lambda|^2 t\bigr)|\mathcal{A}_s\Bigr)}{S_\Pi(M_t|\mathcal{A}_s)}. \quad (2.5.20)$$

By the definition of X and M, and the maxingale property of $\bigl(\exp\bigl(\int_0^t(-b_s+\lambda)\cdot \dot{W}_s\, ds - \int_0^t|-b_s+\lambda|^2\, ds/2\bigr),\, t \in \mathbb{R}_+\bigr)$

$$\begin{aligned}
&S_\Pi\Bigl(M_t \exp\bigl(\lambda \cdot X_t - \frac{1}{2}|\lambda|^2 t\bigr)|\mathcal{A}_s\Bigr)\\
&= S_\Pi\Bigl(\exp\bigl(\int_0^t (-b_r+\lambda)\cdot \dot{W}_r\, dr - \frac{1}{2}\int_0^t |-b_r+\lambda|^2\, dr\bigr)|\mathcal{A}_s\Bigr)\\
&= \exp\Bigl(\int_0^s (-b_r+\lambda)\cdot \dot{W}_r\, dr - \frac{1}{2}\int_0^s |-b_r+\lambda|^2\, dr\Bigr)\\
&\qquad\qquad = M_s \exp\bigl(\lambda \cdot X_s - \frac{1}{2}|\lambda|^2 s\bigr).
\end{aligned}$$

Also $S_\Pi(M_t|\mathcal{A}_s) = M_s$ by the maxingale property of M_t. Thus, by (2.5.20)

$$S_{\Pi'}\Bigl(\exp\bigl(\lambda\cdot X_t - \frac{1}{2}|\lambda|^2 t\bigr)|\mathcal{A}_s\Bigr) = \exp\bigl(\lambda\cdot X_s - \frac{1}{2}|\lambda|^2 s\bigr).$$

The Π'-maximability property is obvious. □

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We now give a version for the canonical setting. We denote $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ by $\mathbb{C}$.

Theorem 2.5.26. *Let space $\mathbb{C}\times\mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ be endowed with a deviability Π such that the idempotent process W defined by $W_t(\mathbf{x}, w) = w_t$ is an $\mathbb{R}^m$-valued Wiener idempotent process. Let Y be defined by $Y_t(\mathbf{x}, w) = \mathbf{x}_t$ and Π^Y denote the idempotent distribution of Y. Let $(b_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ be an $\mathbb{R}^m$-valued $\mathbb{C}$-progressively measurable bounded strictly Luzin idempotent process on $(\mathbb{C}, \Pi^Y)$. Let M and X be defined as in Theorem 2.5.25. Then there exists an idempotent probability Π' on $\mathbb{C}$ such that its restrictions Π'_t to the τ-algebras $\mathcal{C}_t \otimes \mathcal{C}_t(\mathbb{R}_+, \mathbb{R}^m)$ are expressed as $d\Pi'_t = M_t\, d\Pi$, $t \in \mathbb{R}_+$. The idempotent process X is a Wiener idempotent process under Π'.*

Proof. Let $\mathbf{A}$ denote the natural τ-flow on $\mathbb{C}\times\mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ completed with respect to Π. Theorem 2.5.19 implies that the idempotent process $\bigl(\exp\bigl(\int_0^t(-b_s + \lambda)\cdot \dot{W}_s\, ds - \int_0^t|-b_s + \lambda|^2\, ds/2\bigr),\ t \in \mathbb{R}_+\bigr)$ is an $\mathbf{A}$-local exponential maxingale under Π for arbitrary $\lambda \in \mathbb{R}^m$; it is actually an exponential maxingale, which is derived from the fact that $\int_0^t|-b_s + \lambda|^2\, ds$ is a bounded idempotent variable by a standard argument (cf. the proof of Lemma 2.5.10).

We now check existence of Π'. We first show that the idempotent probabilities Π'_t defined by $d\Pi'_t = M_t\, d\Pi$ are deviabilities on $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$. Since M is an exponential maxingale on $(\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m), \Pi)$, by Lemma 1.7.20 it is enough to check that $M(\mathbf{x}, w)$ is strictly Luzin on $(\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m), \Pi)$, which follows by Lemma 2.5.16 and the fact that $b_s(\mathbf{x})$ is strictly Luzin and bounded. Let Π''_t denote the deviabilities on the spaces $\mathbb{C}([0, t], \mathbb{R}^d \times \mathbb{R}^m)$ of continuous $\mathbb{R}^d \times \mathbb{R}^m$-valued functions on $[0, t]$ that are the images of Π'_t under the mappings $\mathbf{x} \to (\mathbf{x}_s,\ s \in [0, t])$. The maxingale property of M implies that $\bigl(\Pi''_t, \mathbb{C}([0, t], \mathbb{R}^d \times \mathbb{R}^m)\bigr)$ is a projective system so that by Lemma 1.8.3 and the fact that the projective limit of the $\mathbb{C}([0, t], \mathbb{R}^d \times \mathbb{R}^m)$, $t \in \mathbb{R}_+$, is homeomorphic to $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ we conclude that there exists a deviability Π' on $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ that extends the Π'_t. The fact that X is idempotent Wiener follows by Theorem 2.5.25. □

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2.6 Idempotent Ito differential equations

This section studies idempotent analogues of Ito differential equations. We fix space $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$, which we denote throughout the section by $\mathbb{C}$; the associated τ-flow $\mathbf{C}(\mathbb{R}_+, \mathbb{R}^d) = (\mathcal{C}_t(\mathbb{R}_+, \mathbb{R}^d),\, t \in \mathbb{R}_+)$ is denoted as $\mathbf{C} = (\mathcal{C}_t,\, t \in \mathbb{R}_+)$. Given space $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ we denote by $\overline{\mathcal{C}}_t^W(\mathbb{R}_+, \mathbb{R}^m)$ the completion of $\mathcal{C}_t(\mathbb{R}_+, \mathbb{R}^m)$ with respect to the Wiener idempotent probability Π^W on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$.

Let $b_t(\mathbf{x})$, $t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}$, and $\sigma_t(\mathbf{x})$, $t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}$, be respective $\mathbb{R}^d$-valued and $\mathbb{R}^{d\times m}$-valued functions, which are continuous in $\mathbf{x}$ for every t, $\mathbf{C}$-progressively measurable in $(t, \mathbf{x})$, and

$$\int_0^t |b_s(\mathbf{x})|\, ds < \infty,\ \int_0^t \|\sigma_s(\mathbf{x})\|^2\, ds < \infty,\ t \in \mathbb{R}_+.$$

We introduce the equation

$$X_t = X_0 + \int_0^t b_s(X)\, ds + \int_0^t \sigma_s(X)\dot{W}_s\, ds, \tag{2.6.1}$$

where $W = (W_s,\, s \in \mathbb{R}_+)$ is an $\mathbb{R}^m$-valued Wiener idempotent process, $X = (X_s,\, s \in \mathbb{R}_+)$ is an $\mathbb{R}^d$-valued continuous idempotent process, and the second integral on the right is an Ito idempotent integral. The flow $\mathbf{A}$ in the following definitions is assumed to be complete with respect to the associated idempotent probability.

Definition 2.6.1. *We say that equation* (2.6.1) *with an initial idempotent distribution μ on $\mathbb{R}^d$ has a solution if there exist an idempotent probability space (Ω, Π) with a τ-flow $\mathbf{A}$, an $\mathbb{R}^d$-valued continuous $\mathbf{A}$-adapted idempotent process $X = (X_s,\, s \in \mathbb{R}_+)$ and an $\mathbb{R}^m$-valued $\mathbf{A}$-Wiener idempotent process $W = (W_s,\, s \in \mathbb{R}_+)$ on (Ω, Π) such that X_0 has the idempotent distribution μ, and* (2.6.1) *holds for all $t \in \mathbb{R}_+$ Π-a.e. in $\omega \in \Omega$. The pair (X, W) is then called a solution to the equation. We say that existence holds for* (2.6.1) *if a solution exists for every μ.*

Definition 2.6.2. *We say that equation* (2.6.1) *with an initial deviability distribution μ on $\mathbb{R}^d$ has a Luzin solution if there exist an idempotent probability space (Ω, Π) with a τ-flow $\mathbf{A}$, an $\mathbb{R}^d$-valued*

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continuous $\mathbf{A}$*-adapted idempotent process* $X = (X_s,\, s \in \mathbb{R}_+)$ *and an* $\mathbb{R}^m$*-valued* $\mathbf{A}$*-Wiener idempotent process* $W = (W_s,\, s \in \mathbb{R}_+)$ *on* (Ω, Π) *such that* (X, W) *is a Luzin-continuous idempotent process on* $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$*,* X_0 *has the idempotent distribution* μ*, and* (2.6.1) *holds for all* $t \in \mathbb{R}_+$ Π*-a.e. in* $\omega \in \Omega$*. The pair* (X, W) *is then called a Luzin solution to the equation with initial deviability distribution* μ*. We say that Luzin existence holds for* (2.6.1) *if a Luzin solution exists for every deviability* μ*.*

Remark 2.6.3. *We sometimes loosely refer to* X *alone as a solution (respectively, Luzin solution).*

Remark 2.6.4. *We recall that by definition* (X, W) *is a Luzin-continuous idempotent process if the idempotent distribution of* (X, W) *is a deviability on* $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$*.*

Remark 2.6.5. *Let* (X, W) *be a solution (respectively, Luzin solution) on some idempotent probability space and* $\Pi^{X,W}$ *denote the idempotent distribution of* (X, W)*. Then by the transitivity property of conditional idempotent expectations the canonical idempotent process* $(\mathbf{x}, w)$ *on* $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ *with the natural* τ*-flow completed with respect to* $\Pi^{X,W}$ *is a solution (respectively, Luzin solution) under* $\Pi^{X,W}$*. Thus, a solution (respectively, Luzin solution) can always be implemented on the canonical space. Therefore, we occasionally refer to the idempotent distribution of* (X, W) *as a solution (respectively, Luzin solution) as well. We denote by* Π_μ *(respectively,* Π_x*) the idempotent distribution of* (X, W) *associated with an initial idempotent distribution* μ *(respectively, with an initial condition* $X_0 = x \in \mathbb{R}^d$*).*

Definition 2.6.6. *We say that uniqueness (respectively, Luzin uniqueness) holds for* (2.6.1) *if for every two solutions (respectively, Luzin solutions)* (X, W) *and* (X', W') *such that the idempotent distributions of* X_0 *and* X'_0 *coincide the idempotent distributions of* (X, W) *and* (X', W') *also coincide.*

Definition 2.6.7. *We say that strong existence holds for equation* (2.6.1) *if there exists a function* $F : \mathbb{R}^d \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m) \to \mathbb{C}$ *such that the function* $w \to F(x, w)$ *is* $\overline{\mathcal{C}}_t^W(\mathbb{R}_+, \mathbb{R}^m)/\mathcal{C}_t$*-measurable for every* $x \in \mathbb{R}^d$ *and* $t \in \mathbb{R}_+$ *and, given an* $\mathcal{A}_0$*-measurable idempotent variable* $f \in \mathbb{R}^d$ *and an* $\mathbb{R}^m$*-valued* $\mathbf{A}$*-Wiener idempotent process* $W = (W_s,\, s \in \mathbb{R}_+)$ *both defined on an idempotent probability space*

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(Ω, Π) *with a* τ*-flow* $\mathbf{A}$, *the idempotent process* $X = F(f, W)$ *satisfies* (2.6.1) *for all* $t \in \mathbb{R}_+$ Π*-a.e., and* $X_0 = f$ Π*-a.e. The idempotent process* X *is then called a strong solution to the equation with an initial condition* f. *If the function* $F(x, w)$ *is, in addition, continuous on* $\mathbb{R}^d \times K_{\Pi^W}(a)$, $a \in (0, 1]$, *then we say that Luzin strong existence holds, and* X *is called a Luzin strong solution.*

Remark 2.6.8. *We recall that* $K_{\Pi^W}(a) = \{w \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m) : \Pi^W(w) \geq a\}$.

Remark 2.6.9. *We note that a strong solution* X *is* $\mathbf{A}$*-adapted since* W *is* $\mathbf{A}$*-adapted,* $\mathbf{A}$ *is complete and* $w \to F(x, w)$ *is* $\overline{\mathcal{C}}_t^W(\mathbb{R}_+, \mathbb{R}^m)/\mathcal{C}_t$*-measurable.*

Definition 2.6.10. *We say that* (2.6.1) *has a unique strong solution (respectively, Luzin strong solution) if strong existence (respectively, Luzin strong existence) holds and if, given a solution (respectively, Luzin solution)* (X, W) *with an initial condition* X_0 *on an idempotent probability space* (Ω, Π) *with a* τ*-flow* $\mathbf{A}$, *we have that* $X = F(X_0, W)$ Π*-a.e., where* F *is the function in the definition of a strong solution (respectively, Luzin strong solution).*

Definition 2.6.11. *We say that pathwise uniqueness holds for* (2.6.1) *if for every two solutions* (X, W) *and* (X', W'), *which are defined on the same idempotent probability space* (Ω, Π) *with the same* τ*-flow* $\mathbf{A}$, *we have* $X = X'$ Π*-a.e. provided* $X_0 = X_0'$ *and* $W = W'$ Π*-a.e.*

We refer to (2.6.1) as an idempotent Ito differential equation and to X as an idempotent diffusion. The functions $b_t(\mathbf{x})$ and $\sigma_t(\mathbf{x})\sigma_t(\mathbf{x})^T$ are occasionally referred to as infinitesimal drift and diffusion coefficients, respectively. We also use the following short-hand notation for (2.6.1)

$$\dot{X}_t = b_t(X) + \sigma_t(X)\dot{W}_t.$$

It is clear that uniqueness implies Luzin uniqueness, Luzin strong existence implies strong existence, and strong existence implies existence. We study some other relationships between the introduced concepts. We first discuss the role of initial conditions in the above definitions. The next lemma follows by Lemma 1.5.5 and Theorem 1.8.9. It implies, in particular, that if Luzin existence holds, then existence holds.

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Lemma 2.6.12. *1. If for every initial condition $x \in \mathbb{R}^d$ there exists a solution Π_x, then existence holds. Specifically, given an initial idempotent distribution μ, the idempotent distribution defined by $\Pi_\mu(\mathbf{x}, w) = \sup_{x\in\mathbb{R}^d} \Pi_x(\mathbf{x}, w)\mu(x)$ is a solution to* (2.6.1).

2. If for every initial condition $x \in \mathbb{R}^d$ there exists a Luzin solution Π_x, which is a deviability transition kernel from $\mathbb{R}^d$ into $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$, then Luzin existence holds. More specifically, given an initial deviability μ, the idempotent probability defined as in part 1 is a Luzin solution with initial deviability distribution μ.

Lemma 2.6.13. *If the idempotent distributions of every two solutions (respectively, Luzin solutions) (X, W) and (X', W') with initial condition x coincide for every $x \in \mathbb{R}^d$, then uniqueness (respectively, Luzin uniqueness) holds.*

Proof. Let X be a solution with an initial idempotent distribution μ defined on an idempotent probability space (Ω, Π) with a τ-flow $\mathbf{A}$. For $x \in \mathbb{R}^d$ such that $\mu(x) > 0$, let $\Pi_x(A) = \Pi(A|X_0 = x)$, $A \subset \Omega$. Then it follows from the definition of conditional idempotent probability that $X_0 = x$ Π_x-a.e.; since W is independent of $\mathcal{A}_0$ by the definition of an $\mathbf{A}$-Wiener idempotent process, it is independent of X_0, so that W is an $\mathbf{A}$-Wiener idempotent process on (Ω, Π_x); (2.6.1) holds Π_x-a.e. since it holds Π-a.e. Thus, X is a solution of (2.6.1) with initial condition x on the space (Ω, Π_x) for the $\mathbf{A}$-Wiener idempotent process W; hence, the idempotent distribution of (X, W) under Π_x is specified uniquely. By Theorem 1.6.12 we have for $A \subset \mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ that

$$\Pi((X, W) \in A) = \sup_{x\in\mathbb{R}^d} \Pi_x((X, W) \in A)\Pi(X_0 = x)$$
$$= \sup_{x\in\mathbb{R}^d} \Pi_x((X, W) \in A)\mu(x), \quad (2.6.2)$$

which implies that the idempotent distribution of (X, W) under Π is specified uniquely.

We now turn to Luzin uniqueness. We assume the canonical setting so that $\Omega = \mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$, Π is a deviability on Ω, and (X, W) is the canonical idempotent process. Let μ be an initial

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deviability distribution on $\mathbb{R}^d$. Then Π_x defined as in the preceding proof is a deviability on Ω by Theorem 1.6.12 and the fact that the set $\{X_0 = x\}$ belongs to the collection of closed subsets of Ω. Since (X, W) is a Luzin solution such that $X_0 = x$ Π_x-a.e., deviability Π_x is specified uniquely. Since (2.6.2) holds for this case as well, Π is specified uniquely. $\square$

The next lemma follows by similar arguments and the definitions.

Lemma 2.6.14. *1. If there exists a function $F(x, w)$ in the definition of a strong solution (respectively, Luzin strong solution) such that the idempotent process $X = F(x, W)$ is a strong solution (respectively, Luzin strong solution) for an initial condition x, then strong existence (respectively, Luzin strong existence) holds.*

2. If pathwise uniqueness holds for solutions with initial condition x for every $x \in \mathbb{R}^d$, then pathwise uniqueness holds.

Let us associate with (2.6.1) a collection of ordinary differential equations depending on absolutely continuous functions $w = (w_t,\, t \in \mathbb{R}_+) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ such that $\int_0^\infty |\dot{w}_t|^2\, dt < \infty$:

$$\dot{\mathbf{x}}_t = b_t(\mathbf{x}) + \sigma_t(\mathbf{x})\dot{w}_t \text{ a.e. in } t,\ \mathbf{x}_0 = x \in \mathbb{R}^d, \tag{2.6.3}$$

where the $\mathbf{x} = (\mathbf{x}_t,\, t \in \mathbb{R}_+) \in \mathbb{C}$ are absolutely continuous functions.

Definition 2.6.15. *We say that the extension condition holds for equation* (2.6.3) *if for every $x \in \mathbb{R}^d$ and $w \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ such that $\int_0^\infty |\dot{w}_t|^2\, dt < \infty$ the following holds: if a function $(\mathbf{x}_t)$, defined on an interval $[0, T]$, satisfies* (2.6.3) *for $t \in [0, T]$, then it can be extended to a solution of* (2.6.3) *on $\mathbb{R}_+$.*

Remark 2.6.16. *The extension condition implies existence of solutions for every equation* (2.6.3)*.*

Lemma 2.6.17. *1. If the extension condition holds for* (2.6.3)*, then existence holds for the idempotent Ito differential equation* (2.6.1)*. If existence holds for* (2.6.1)*, then every ordinary differential equation* (2.6.3) *has a solution.*

2. If every ordinary differential equation (2.6.3) *has at most one solution, then pathwise uniqueness and uniqueness hold for* (2.6.1)*.*

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3. *Luzin strong existence implies Luzin existence.*
4. *If strong existence (respectively, Luzin strong existence) and pathwise uniqueness hold for* (2.6.1)*, then there is a unique strong solution (respectively, Luzin strong solution).*

Proof. Let the extension condition hold for (2.6.3). We define an idempotent distribution $\mathbf{\Pi}_x$ on $\mathbb{C}\times\mathbb{C}(\mathbb{R}_+,\mathbb{R}^m)$ by $\mathbf{\Pi}_x(\mathbf{x},w) = \Pi^W(w)$ if $\mathbf{x}$ and w satisfy (2.6.3) and $\mathbf{x}_0 = x$, and $\mathbf{\Pi}_x(\mathbf{x},w) = 0$ otherwise. Let (X,W) denote the canonical idempotent process on $\mathbb{C}\times\mathbb{C}(\mathbb{R}_+,\mathbb{R}^m)$ and $\mathbf{A} = \{\mathcal{A}_t,\, t\in\mathbb{R}_+\}$ be the natural τ-flow completed with respect to $\mathbf{\Pi}_x$. Then (X,W) satisfies (2.6.1) for the initial condition x $\mathbf{\Pi}_x$-a.e. We show that W is an $\mathbf{A}$-Wiener idempotent process. It is sufficient to check that W has $\mathbf{A}$-independent increments, i.e., $\mathbf{\Pi}_x(\theta_t w'|\mathcal{A}_t) = \mathbf{\Pi}_x(\theta_t w')$ $\mathbf{\Pi}_x$-a.e., where $t\in\mathbb{R}_+$ and $w'\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}^m)$ is such that $\Pi^W(w')>0$. Let $(\mathbf{x}'',w'')$ be such that $\mathbf{\Pi}_x\big(p_t^{-1}\circ p_t(\mathbf{x}'',w'')\big)>0$. Then

$$\mathbf{\Pi}_x(\theta_t w'|\mathcal{A}_t)(\mathbf{x}'',w'') = \frac{\mathbf{\Pi}_x\big((\mathbf{x},w):\,\theta_t w=\theta_t w',p_t(\mathbf{x},w)=p_t(\mathbf{x}'',w'')\big)}{\mathbf{\Pi}_x\big((\mathbf{x},w):\,p_t(\mathbf{x},w)=p_t(\mathbf{x}'',w'')\big)}.$$

Since $\mathbf{\Pi}_x\big(p_t^{-1}\circ p_t(\mathbf{x}'',w'')\big)>0$, the pair $(\mathbf{x}'',w'')$ satisfies (2.6.3) on $[0,t]$. The extension condition implies that for every w such that $p_t w = p_t w''$ and $\theta_t w=\theta_t w'$ there exists a solution to (2.6.3) on $\mathbb{R}_+$ that coincides with $\mathbf{x}''$ on $[0,t]$. Therefore, by the definition of $\mathbf{\Pi}_x$ we have that

$$\mathbf{\Pi}_x\big((\mathbf{x},w):\,\theta_t w=\theta_t w',p_t(\mathbf{x},w)=p_t(\mathbf{x}'',w'')\big) = \Pi^W(w:\,\theta_t w=\theta_t w',p_t w=p_t w'').$$

By a similar reasoning

$$\mathbf{\Pi}_x\big((\mathbf{x},w):\,p_t(\mathbf{x},w)=p_t(\mathbf{x}'',w'')\big)=\Pi^W(w:\,p_t w=p_t w'').$$

Thus, by independence of increments of W

$$\mathbf{\Pi}_x(\theta_t w'|\mathcal{A}_t)(\mathbf{x}'',w'') = \frac{\Pi^W(w:\,\theta_t w=\theta_t w',p_t w=p_t w'')}{\Pi^W(w:\,p_t w=p_t w'')} = \Pi^W(\theta_t w')$$

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as required. Thus, (X, W) is a solution of (2.6.1) on $(\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m), \boldsymbol{\Pi}_x)$ with τ-flow $\mathbf{A}$ for every $x \in \mathbb{R}^d$. By Lemma 2.6.12 existence holds. Conversely, let existence hold. Let Π_x be an idempotent distribution of (X, W) on $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ for an initial condition x. Since $\sup_{\mathbf{x}} \Pi_x(\mathbf{x}, w) = \Pi^W(w)$, it follows that if $\Pi^W(w) > 0$, then there exists $\mathbf{x}$ such that $\Pi_x(\mathbf{x}, w) > 0$ so that $(\mathbf{x}, w)$ satisfy (2.6.3) and $\mathbf{x}_0 = x$. This ends the proof of part 1.

We now prove part 2. The fact that pathwise uniqueness holds if every differential equation (2.6.3) has at most one solution is obvious. Let us assume that Π_x is a solution on $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ with an initial condition x. By definition if $\Pi_x(\mathbf{x}, w) > 0$, then $\mathbf{x}$ solves (2.6.3) with $\mathbf{x}_0 = x$. By uniqueness for (2.6.3) $\mathbf{x}$ is a unique solution for given w and x. Since we must have that $\sup_{\mathbf{x}} \Pi_x(\mathbf{x}, w) = \Pi^W(w)$ it follows that $\Pi_x(\mathbf{x}, w) = \Pi^W(w)$ so that Π_x coincides with the solution $\boldsymbol{\Pi}_x$ defined in the proof of part 1.

Luzin strong existence implies Luzin existence by Theorem 1.7.11. Part 4 is obvious. □

Remark 2.6.18. *Note that under the hypotheses of part 2 we have pathwise uniqueness even for two solutions (X, W) and (X', W) that are not necessarily associated with the same τ-flow. The latter is also true if there is a unique strong solution.*

The following lemma takes advantage of the proof of Lemma 2.6.17 to indicate a candidate for a solution of (2.6.1). The proof also relies on Lemma 2.6.12.

Lemma 2.6.19. *Let the extension condition hold for* (2.6.3). *Then the idempotent distribution*

$$\boldsymbol{\Pi}_\mu(\mathbf{x}, w) = \sup_{x \in \mathbb{R}^d} \boldsymbol{\Pi}_x(\mathbf{x}, w)\mu(x),$$

where

$$\boldsymbol{\Pi}_x(\mathbf{x}, w) = \begin{cases} \Pi^W(w), & \text{if } \dot{\mathbf{x}}_t = b_t(\mathbf{x}) + \sigma_t(\mathbf{x})\dot{w}_t \text{ a.e.} \\ & \qquad \text{and } \mathbf{x}_0 = x, \\ 0, & \text{otherwise}, \end{cases}$$

is a solution for an initial idempotent distribution μ.

If $\boldsymbol{\Pi}_x(\mathbf{x}, w)$ is a deviability transition kernel from $\mathbb{R}^d$ into $\mathbb{C} \times \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$, then $\boldsymbol{\Pi}_\mu$ is a Luzin solution for an initial deviability μ.

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Remark 2.6.20. *Easy calculations show that* $\mathbf{\Pi}^X_x(\mathbf{x}) = \sup_w \mathbf{\Pi}_x(\mathbf{x}, w)$ *is given by*

$$\mathbf{\Pi}^X_x(\mathbf{x}) = \exp\Big(-\frac{1}{2}\int_0^\infty (\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\cdot\big(\sigma_t(\mathbf{x})\sigma_t(\mathbf{x})^T\big)^{\oplus}(\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\,dt\Big)$$

if $\mathbf{x}_0 = x$, $\mathbf{x}$ *is absolutely continuous and* $\dot{\mathbf{x}}_t - b_t(\mathbf{x})$ *is in the range of* $\sigma_t(\mathbf{x})$ *a.e., and* $\mathbf{\Pi}^X_x(\mathbf{x}) = 0$ *otherwise. We also note that the range of* $\sigma_t(\mathbf{x})$ *coincides with the range of* $\sigma_t(\mathbf{x})\sigma_t(\mathbf{x})^T$.

Theorem 2.6.21.
1. *If pathwise uniqueness holds, then uniqueness holds.*
2. *Let pathwise uniqueness hold. If existence (respectively, Luzin existence) holds, then strong existence (respectively, Luzin strong existence) holds so that there exists a unique strong solution (respectively, Luzin strong solution).*

Proof. Let (X, W) and (X', W') be two solutions of (2.6.1) with an initial condition $x \in \mathbb{R}^d$ on respective idempotent probability spaces (Ω, Π) and (Ω', Π') with respective τ-flows $\mathbf{A}$ and $\mathbf{A}'$. Let us introduce the conditional idempotent distributions $\Pi_w(A) = \Pi(X \in A|W = w)$ and $\Pi'_w(A) = \Pi'(X' \in A|W' = w)$. We show that for $\mathbf{x} \in \mathbb{C}$ and $t \in \mathbb{R}_+$

$$\Pi_w\big(p_t^{-1}(p_t\mathbf{x})\big) = \Pi(p_tX = p_t\mathbf{x}|p_tW = p_tw) \quad \text{for } \Pi^W\text{–almost all } w, \tag{2.6.4}$$

i.e., the left-hand side depends only on the piece of w up to t. (Of course, a similar relation holds for Π'.) Recalling the notation $\theta_t w_s = w_{t+s} - w_t$, $s \in \mathbb{R}_+, w \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$, we have, for w such that $\Pi^W(w) > 0$, in view of the fact that p_tX and p_tW are $\mathcal{A}_t$-measurable, and θ_tW is independent of $\mathcal{A}_t$, that

$$\begin{aligned}\Pi_w\big(p_t^{-1}(p_t\mathbf{x})\big) &= \frac{\Pi(p_tX = p_t\mathbf{x},\, W = w)}{\Pi(W = w)}\\
&= \frac{\Pi(p_tX = p_t\mathbf{x},\, p_tW = p_tw, \theta_tW = \theta_tw)}{\Pi(p_tW = p_tw, \theta_tW = \theta_tw)}\\
&= \frac{\Pi(p_tX = p_t\mathbf{x},\, p_tW = p_tw)\Pi(\theta_tW = \theta_tw)}{\Pi(p_tW = p_tw)\Pi(\theta_tW = \theta_tw)}\\
&= \Pi(p_tX = p_t\mathbf{x}|p_tW = p_tw).\end{aligned}$$

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The claim has been proved.

We define an idempotent probability $\tilde{\Pi}$ on $\tilde{\Omega} = \mathbb{C}\times\mathbb{C}\times\mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ by

$$\tilde{\Pi}(\mathbf{x}, \mathbf{x}', w) = \Pi_w(\mathbf{x})\Pi'_w(\mathbf{x}')\Pi^W(w). \tag{2.6.5}$$

Clearly, $\tilde{\Pi}\big(\{\mathbf{x}\} \times \mathbb{C} \times \{w\}\big) = \Pi(X = \mathbf{x}, W = w)$ and $\tilde{\Pi}\big(\mathbb{C} \times \{\mathbf{x}'\} \times \{w\}\big) = \Pi'(X' = \mathbf{x}', W' = w)$. Let $\tilde{\mathcal{C}}_t$ be the completion of $\mathcal{C}_t \otimes \mathcal{C}_t \otimes \mathcal{C}_t(\mathbb{R}_+, \mathbb{R}^m)$ with respect to $\tilde{\Pi}$ and $\tilde{\mathbf{C}} = (\tilde{\mathcal{C}}_t,\, t \in \mathbb{R}_+)$. We check that the canonical idempotent process $(w_t, t \in \mathbb{R}_+)$ is a $\tilde{\mathbf{C}}$-Wiener idempotent process on $(\tilde{\Omega}, \tilde{\Pi})$. It obviously has idempotent distribution Π^W. By Theorem 2.4.9 it is sufficient to check that $\theta_t w$ is $\tilde{\Pi}$-independent of $\tilde{\mathcal{C}}_t$. We have, for $\hat{\mathbf{x}}, \hat{\mathbf{x}}' \in \mathbb{C}$ and $\hat{w}, \tilde{w} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$, in view of (2.6.5),

$$\begin{aligned}
&\tilde{\Pi}\big((\mathbf{x}, \mathbf{x}', w) : p_t(\mathbf{x}, \mathbf{x}', w) = p_t(\hat{\mathbf{x}}, \hat{\mathbf{x}}', \hat{w}), \theta_t w = \tilde{w}\big) \\
&= \sup_{(\mathbf{x},\mathbf{x}',w)\in\tilde{\Omega}} \mathbf{1}\big(p_t(\mathbf{x}, \mathbf{x}', w) = p_t(\hat{\mathbf{x}}, \hat{\mathbf{x}}', \hat{w}), \theta_t w = \tilde{w}\big) \\
&\Pi_w(\mathbf{x})\Pi'_w(\mathbf{x}')\Pi^W(w) \\
&= \sup_w \mathbf{1}\big(p_t w = p_t \hat{w}, \theta_t w = \tilde{w}\big)\Pi_w(p_t^{-1}(p_t\hat{\mathbf{x}})) \\
&\Pi'_w(p_t^{-1}(p_t\hat{\mathbf{x}}'))\Pi^W(w) \\
&= \big[\sup_w \mathbf{1}(p_t w = p_t\hat{w})\Pi_w(p_t^{-1}(p_t\hat{\mathbf{x}}))\Pi'_w(p_t^{-1}(p_t\hat{\mathbf{x}}'))\Pi^W(w)\big] \\
&\big[\sup_w \mathbf{1}(\theta_t w = \tilde{w})\Pi^W(w)\big] \\
&\qquad\qquad = \tilde{\Pi}\big(p_t(\mathbf{x}, \mathbf{x}', w) = p_t(\hat{\mathbf{x}}, \hat{\mathbf{x}}', \hat{w})\big)\tilde{\Pi}(\theta_t w = \tilde{w}),
\end{aligned}$$

where the equality before the last one follows by (2.6.4) and the fact that $\theta_t w$ is independent of $p_t w$ under Π^W.

Thus, $(\mathbf{x}, w)$ and $(\mathbf{x}', w)$ are two solutions to (2.6.1) on the same idempotent probability space and adapted to the same τ-flow. By pathwise uniqueness we conclude that $\mathbf{x} = \mathbf{x}'$ $\tilde{\Pi}$-a.e. so

$$\sup_{(\mathbf{x},\mathbf{x}',w)\in\tilde{\Omega}} \mathbf{1}(\mathbf{x} \neq \mathbf{x}')\tilde{\Pi}(\mathbf{x}, \mathbf{x}', w) = 0. \tag{2.6.6}$$

Therefore,

$$\begin{aligned}
\Pi((X, W) = (\mathbf{x}, w)) &= \tilde{\Pi}(\{\mathbf{x}\} \times \mathbb{C} \times \{w\}) = \tilde{\Pi}((\mathbf{x}, \mathbf{x}, w)) \\
&= \tilde{\Pi}(\mathbb{C} \times \{\mathbf{x}\} \times \{w\}) = \Pi'((X', W') = (\mathbf{x}, w))
\end{aligned}$$

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so that uniqueness holds.

Next, by (2.6.6) and (2.6.5) $\sup_{\mathbf{x},\mathbf{x}'} \mathbf{1}(\mathbf{x} \neq \mathbf{x}')\Pi_w(\mathbf{x})\Pi'_w(\mathbf{x}') = 0$ for Π^W-almost all w. Fixing w such that $\Pi^W(w) > 0$ and picking $\tilde{\mathbf{x}}'$ such that $\Pi'_w(\tilde{\mathbf{x}}') > 0$ we define $F(x,w) = \tilde{\mathbf{x}}'$. Since $\mathbf{x} = \tilde{\mathbf{x}}'$ for Π_w-almost all $\mathbf{x}$, we have that $\mathbf{x} = F(x,w)$ whenever $\Pi(X = \mathbf{x}, W = w) > 0$. If $\Pi^W(w) = 0$, we define $F(x,w)$ arbitrarily. By construction, $X = F(x,W)$ Π-a.e.

We prove that $F(x,w)$ is $\overline{\mathcal{C}}_t^W(\mathbb{R}_+,\mathbb{R}^m)/\mathcal{C}_t$-measurable in w for every $x \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$. Since $X = F(x,W)$ Π-a.e., we have that $\Pi(X = \mathbf{x}|W = w) = \mathbf{1}(F(x,w) = \mathbf{x})$ if $\Pi(W = w) > 0$. Therefore, by (2.6.4) for Π^W-almost all w

$$\begin{aligned}\mathbf{1}(F(x,w) \in p_t^{-1}(p_t\mathbf{x})) &= \Pi(p_tX = p_t\mathbf{x}|W = w) \\ &= \Pi(p_tX = p_t\mathbf{x}|p_tW = p_tw).\end{aligned}$$

Since the right-most side, for fixed $\mathbf{x}$, is a function of p_tw, we conclude that $\{w : F(x,w) \in p_t^{-1}(p_t\mathbf{x})\} \in \overline{\mathcal{C}}_t^W(\mathbb{R}_+,\mathbb{R}^m)$. Thus, strong existence holds, and by part 4 of Lemma 2.6.17 there exists a unique strong solution.

For the part concerned with Luzin solutions, we need to check, in addition, that if Luzin existence holds, then $F(x,w)$ is continuous in (x,w) on $\mathbb{R}^d \times K_{\Pi^W}(a)$, $a \in (0,1]$. Let $(x^n,w^n) \in \mathbb{R}^d \times K_{\Pi^W}(a)$, $a \in (0,1]$, converge to $(\tilde{x},\tilde{w})$ and $\mathbf{x}^n = F(x^n,w^n)$. Let $(\hat{X},\hat{W})$ be a Luzin solution on an idempotent probability space $(\hat{\Omega},\hat{\Pi})$ with an initial condition $\hat{X}_0$ such that $\hat{\Pi}(\hat{X}_0 = x^n) = 1$, $n \in \mathbb{N}$, and $\hat{\Pi}(\hat{X}_0 = \tilde{x}) = 1$. Since $\hat{X} = F(\hat{X}_0,\hat{W})$ $\hat{\Pi}$-a.e., $\hat{\Pi}((\hat{X},\hat{X}_0,\hat{W}) = (\mathbf{x}^n,x^n,w^n)) = \hat{\Pi}(\hat{X}_0 = x^n)\Pi^W(w^n) \geq a$. Since $(\hat{X},\hat{W})$ is a Luzin solution, the set $\{(\mathbf{x},x,w) : \hat{\Pi}((\hat{X},\hat{X}_0,\hat{W}) = (\mathbf{x},x,w)) \geq a\}$ is compact, which implies that the sequence $\{(\mathbf{x}^n,x^n,w^n),\ n \in \mathbb{N}\}$ is relatively compact and every accumulation point $(\tilde{\mathbf{x}},\tilde{x},\tilde{w})$ is such that $\hat{\Pi}\big((\hat{X},\hat{X}_0,\hat{W}) = (\tilde{\mathbf{x}},\tilde{x},\tilde{w})\big) \geq a > 0$. Hence, $\tilde{\mathbf{x}} = F(\tilde{x},\tilde{w})$. □

According to Lemma 2.6.17 existence and uniqueness issues for (2.6.1) and (2.6.3) are closely related. Thus, the methods of the theory of ordinary differential equations apply to the study of existence and pathwise uniqueness. We recall that $b_t(\mathbf{x})$ and $\sigma_t(\mathbf{x})$ are assumed to be continuous in $\mathbf{x}$.

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Theorem 2.6.22. *1. Let $b_t(\mathbf{x})$ and $\sigma_t(\mathbf{x})$ satisfy the linear-growth conditions*

$$|b_t(\mathbf{x})| \le l_t(1 + \sup_{s\le t}|\mathbf{x}_s|), \quad \|\sigma_t(\mathbf{x})\|^2 \le l_t(1 + \sup_{s\le t}|\mathbf{x}_s|^2),$$
$$t \in \mathbb{R}_+,\ \mathbf{x} \in \mathbb{C},$$

where l_t is Lebesgue measurable and $\int_0^t l_s\,ds < \infty$, $t \in \mathbb{R}_+$. Then existence holds for (2.6.1).

2. Let $b_t(\mathbf{x})$ and $\sigma_t(\mathbf{x})$ be locally Lipshitz continuous, i.e., for every $a \in \mathbb{R}_+$, $t \in \mathbb{R}_+$ and $\mathbf{x}, \mathbf{y} \in \mathbb{C}$, such that $\sup_{s\le t}|\mathbf{x}_s| \le a$ and $\sup_{s\le t}|\mathbf{y}_s| \le a$ we have

$$|b_t(\mathbf{x}) - b_t(\mathbf{y})| \le k_t^a \sup_{s\le t}|\mathbf{x}_s - \mathbf{y}_s|,$$
$$\|\sigma_t(\mathbf{x}) - \sigma_t(\mathbf{y})\|^2 \le k_t^a \sup_{s\le t}|\mathbf{x}_s - \mathbf{y}_s|^2,$$

where k_t^a is Lebesgue measurable and $\int_0^t k_s^a\,ds < \infty$, $t \in \mathbb{R}_+$. Then pathwise uniqueness holds for (2.6.1).

Proof. For the existence part we have to check the extension condition for (2.6.3). We use the method of successive approximations. Standard details are omitted. Let $x \in \mathbb{R}^d$, $w \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^m)$ be such that $\int_0^\infty |\dot{w}_s|^2\,ds < \infty$, and a function $\hat{\mathbf{x}}_t$, $t \in [0,T]$, satisfy (2.6.3) on $[0,T]$. Let u_t^n be successive approximations defined by

$$u_t^{n+1} = x + \int_0^t b_s(u^n)\,ds + \int_0^t \sigma_s(u^n)\dot{w}_s\,ds,\ t \in \mathbb{R}_+, \tag{2.6.7}$$

where $u_t^0 = \hat{\mathbf{x}}_t$ for $t \in [0,T]$ and $u_t^0 = \hat{\mathbf{x}}_T$ for $t \ge T$. Then by the Cauchy-Schwarz inequality and the linear-growth conditions

$$|u_t^{n+1}|^2 \le 3|x|^2 + 3\Big(\int_0^t |b_s(u^n)|\,ds\Big)^2$$

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$$+ 3\int_0^t \|\sigma_s(u^n)\|^2\,ds \int_0^t |\dot{w}_s|^2\,ds$$

$$\le 3|x|^2 + 3\Big(2\int_0^t l_s\,ds + \int_0^t |\dot{w}_s|^2\,ds\Big)\int_0^t l_s\,ds$$

$$+ 3\Big(2\int_0^t l_s\,ds + \int_0^t |\dot{w}_s|^2\,ds\Big)\int_0^t l_s \sup_{r\le s}|u_r^n|^2\,ds.$$

This string of inequalities shows in particular that the right-hand side of (2.6.7) is well defined. Denoting $f_t^n = \sup_{s\le t}|u_s^n|^2$, we conclude that, given $T>0$, there exist constants A_1 and A_2 such that for all $t\le T$

$$f_t^{n+1} \le A_1 + A_2\int_0^t l_s f_s^n\,ds,$$

which implies by a version of Gronwall's inequality that

$$f_t^n \le A_1 \exp\Big(A_2\int_0^t l_s\,ds\Big),\ t\le T.$$

Therefore, $\sup_n \sup_{s\le t}|u_s^n| < \infty$, which easily implies by (2.6.7) and the linear-growth conditions that the sequence $\{(u_t^n, t\in\mathbb{R}_+), n\in\mathbb{N}\}$ is locally equicontinuous, so that by Arzelà-Ascoli's theorem it is relatively compact in $\mathbb{C}$. In a standard way, by using continuity of $b_t(\mathbf{x})$ and $\sigma_t(\mathbf{x})$ in $\mathbf{x}$ and the linear-growth conditions, it follows that every accumulation point of $\{(u_t^n, t\in\mathbb{R}_+)\}$ solves (2.6.3). It coincides with $\hat{\mathbf{x}}$ on $[0,T]$ since $u_t^n = \hat{\mathbf{x}}_t$ for $t\in[0,T]$.

The uniqueness part is also proved by a standard argument. Let u and v be two solutions of (2.6.3) such that $u_0=v_0=x$. Let $\tau^a(\mathbf{x}) = \inf\{t\in\mathbb{R}_+ : |\mathbf{x}_t|\ge a\}$, $\mathbf{x}\in\mathbb{C}$. Then denoting $u_t^a = u_{t\wedge\tau^a(u)\wedge\tau^a(v)}$ and $v_t^a = v_{t\wedge\tau^a(v)\wedge\tau^a(u)}$, by the Cauchy-Schwarz inequality and Lipshitz

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continuity conditions

$$|u_t^a - v_t^a|^2 \le 2 \int_0^t k_s^a \, ds \int_0^t k_s^a \sup_{r \le s} |u_r^a - v_r^a|^2 \, ds \\ + 2 \int_0^t |\dot{w}_s|^2 \, ds \int_0^t k_s^a \sup_{r \le s} |u_r^a - v_r^a|^2 \, ds,$$

so that $u^a = v^a$ by Gronwall's inequality. Hence, $\tau^a(u) = \tau^a(v)$ and $u_t = v_t$ for $t \le \tau_u^a$. Letting $a \to \infty$ completes the proof. □

Remark 2.6.23. *Under the hypotheses of part 2 the stronger version of pathwise uniqueness of Remark 2.6.18 holds.*

We now strengthen part 1 of Theorem 2.6.22.

Theorem 2.6.24. *Under the hypotheses of part 1 of Theorem 2.6.22 Luzin existence holds for equation* (2.6.1)*.*

Proof. Since according to the proof of Theorem 2.6.22 the extension condition holds for (2.6.3) under the hypotheses, by Lemma 2.6.19 it suffices to check that $\mathbf{\Pi}_x(\mathbf{x}, w)$ defined in the statement of the lemma is a deviability transition kernel. By Lemma 1.8.12 this can be done by proving that $\mathbf{\Pi}_x(\mathbf{x}, w)$ is upper semi-continuous in $(x, \mathbf{x}, w)$ and the sets $\{(\mathbf{x}, w) : \sup_{|x| \le A} \mathbf{\Pi}_x(\mathbf{x}, w) \ge a\}$ are relatively compact for every $A \in \mathbb{R}_+$ and $a \in (0, 1]$.

We consider the upper semi-continuity first. Let $(x^n, \mathbf{x}^n, w^n) \to (\tilde{x}, \tilde{\mathbf{x}}, \tilde{w})$ as $n \to \infty$. We can obviously assume that the $(x^n, \mathbf{x}^n, w^n)$ satisfy equation (2.6.3) for otherwise $\mathbf{\Pi}_{x^n}(\mathbf{x}^n, w^n) = 0$. In addition, since by definition $\mathbf{\Pi}_{x^n}(\mathbf{x}^n, w^n) = \Pi^W(w^n)$, we may assume that $\Pi^W(w^n) \ge a > 0$. We check that $\tilde{\mathbf{x}}$ is a solution to (2.6.3) associated with $\tilde{x}$ and $\tilde{w}$. The linear-growth conditions, continuity of $b_s(\mathbf{x})$ and $\sigma_s(\mathbf{x})$ in $\mathbf{x}$, Lebesgue's dominated convergence theorem, and Lemma 2.5.16 yield

$$\lim_{n \to \infty} \int_0^t b_s(\mathbf{x}^n) \, ds = \int_0^t b_s(\tilde{\mathbf{x}}) \, ds,$$

$$\lim_{n \to \infty} \int_0^t \sigma_s(\mathbf{x}^n) \dot{w}_s^n \, ds = \int_0^t \sigma_s(\tilde{\mathbf{x}}) \dot{\tilde{w}}_s \, ds,$$

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proving the claim. Thus, $\mathbf{\Pi}_{\tilde{x}}(\tilde{\mathbf{x}},\tilde{w}) = \Pi^W(\tilde{w})$. Since by upper semi-continuity of $\Pi^W(w)$ we have that $\limsup_{n\to\infty}\mathbf{\Pi}_{x^n}(\mathbf{x}^n,w^n) = \limsup_{n\to\infty}\Pi^W(w^n) \le \Pi^W(\tilde{w})$, the required follows.

Let us check that the set $\{(\mathbf{x},w) : \sup_{|x|\le A}\mathbf{\Pi}_x(\mathbf{x},w) \ge a\}$ is relatively compact. Let a sequence $\{(x^n,\mathbf{x}^n,w^n),\, n\in\mathbb{N}\}$ be such that $|x^n| \le A$ and $\mathbf{\Pi}_{x^n}(\mathbf{x}^n,w^n) \ge a(1-1/n)$. Since $\mathbf{\Pi}_{x^n}(\mathbf{x}^n,w^n) = \Pi^W(w^n)$ and $\Pi^W(w)$ is upper compact, we may assume that the x^n and w^n converge to some $\tilde{x}$ and $\tilde{w}$, respectively. Since the $\mathbf{x}^n$ are solutions of (2.6.3),

$$|\mathbf{x}^n_t|^2 \le 3(x^n)^2 + 3\Bigl(\int\limits_0^t |b_s(\mathbf{x}^n)|\,ds\Bigr)^2 + 3\int\limits_0^t |\dot{w}^n_s|^2\,ds \int\limits_0^t \|\sigma_s(\mathbf{x}^n)\|^2\,ds,$$

$$|\mathbf{x}^n_t - \mathbf{x}^n_s|^2 \le 2\Bigl(\int\limits_s^t |b_r(\mathbf{x}^n)|\,dr\Bigr)^2 + 2\int\limits_s^t |\dot{w}^n_r|^2\,dr \int\limits_s^t \|\sigma_r(\mathbf{x}^n)\|^2\,dr.$$

Since also $\inf_n \Pi^W(w^n) > 0$, we conclude in analogy with the proof of Theorem 2.6.22 that the sequence $\{\mathbf{x}^n\}$ is relatively compact in $\mathbb{C}$. □

Remark 2.6.25. *One can also show that in the hypotheses of part 1 of Theorem 2.6.22 the idempotent process X defined for $(\mathbf{x},w)\in\mathbb{C}\times\mathbb{C}(\mathbb{R}_+,\mathbb{R}^m)$ by $X_t(\mathbf{x},w) = \mathbf{x}_t - x - \int_0^t b_s(\mathbf{x})\,ds$ is a local maxingale on $\bigl(\mathbb{C}\times\mathbb{C}(\mathbb{R}_+,\mathbb{R}^m), \mathbf{C}\otimes\mathbf{C}(\mathbb{R}_+,\mathbb{R}^m), \mathbf{\Pi}_x\bigr)$ with quadratic characteristic $\bigl(\int_0^t \sigma_s(\mathbf{x})\sigma_s^T(\mathbf{x})\,ds,\, t\in\mathbb{R}_+\bigr)$.*

Combining Lemma 2.6.19, Theorem 2.6.21, Theorem 2.6.22, and Theorem 2.6.24 we obtain the following existence and uniqueness result.

Theorem 2.6.26. *Let $b_t(\mathbf{x})$ and $\sigma_t(\mathbf{x})$ be locally Lipshitz-continuous and satisfy the linear-growth conditions. Then the equation*

$$\dot{X}_t = b_t(X) + \sigma_t(X)\dot{W}_t,\; X_0 = x,$$

has a unique Luzin solution, which is also a strong Luzin solution. The deviability distribution of X is given by

$$\Pi^X_x(\mathbf{x}) = \exp\Bigl(-\frac{1}{2}\int\limits_0^\infty (\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\cdot\bigl(\sigma_t(\mathbf{x})\sigma_t(\mathbf{x})^T\bigr)^{\oplus}(\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\,dt\Bigr)$$

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if $\mathbf{x}_0 = x$, $\mathbf{x}$ *is absolutely continuous and* $\dot{\mathbf{x}}_t - b_t(\mathbf{x})$ *is in the range of* $\sigma_t(\mathbf{x})$ *a.e., and* $\Pi_x^X(\mathbf{x}) = 0$ *otherwise.*

Remark 2.6.27. *Existence of a strong solution under the hypotheses can also be proved directly by using a version of the method of successive approximations.*

As a consequence of "the Girsanov theorem" (Theorem 2.5.26) we have the following existence and uniqueness result.

Theorem 2.6.28. *Let* $(\alpha_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ *be an* $\mathbb{R}^m$*-valued* $\mathbf{C}(\mathbb{R}_+, \mathbb{R}^m)$*-progressively measurable bounded function such that* $\alpha_s(\mathbf{x})$ *is continuous in* $\mathbf{x}$ *for* $s \in \mathbb{R}_+$*. Then Luzin existence and uniqueness hold for the equation*

$$\dot{X}_t = b_t(X) + \sigma_t(X)\dot{W}_t,\ X_0 = x, \tag{2.6.8}$$

if and only if Luzin existence and uniqueness hold for the equation

$$\dot{X}_t = \big(b_t(X) - \sigma_t(X)\alpha_t(X)\big) + \sigma_t(X)\dot{W}_t,\ X_0 = x, \tag{2.6.9}$$

where W *is an* $\mathbb{R}^m$*-valued* $\mathbf{A}$*-Wiener idempotent process.*

Corollary 2.6.29. *Let* $(b_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ *be bounded. Then Luzin existence and uniqueness hold for the equation*

$$\dot{X}_t = b_t(X) + \dot{W}_t,\ X_0 = x,$$

where W *is an* $\mathbb{R}^d$*-valued* $\mathbf{A}$*-Wiener idempotent process.*

Let Π^X *denote the idempotent distribution of* X*. Then* $\Pi^X(\mathbf{x}) = 0$ *unless* $\mathbf{x}_0 = 0$ *and* $\mathbf{x}$ *is absolutely continuous. For these* $\mathbf{x}$

$$\Pi^X(\mathbf{x}) = \exp\Big(-\frac{1}{2}\int_0^\infty |\dot{\mathbf{x}}_t - b_t(\mathbf{x})|^2\,dt\Big).$$

We now outline another approach, which is analogous to the martingale problem approach and which we will explore in detail later in the text. We state the result for Luzin solutions, which are our main concern below. Given a deviability Π on $\mathbb{C}$, we denote by $\mathbf{C}^\Pi$ the Π-completion of the τ-flow $\mathbf{C}$.

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Theorem 2.6.30. *Let the matrix $\sigma_s(\mathbf{x})$ have size $d \times d$ and for every compact $K \subset \mathbb{C}$ and $t \in \mathbb{R}_+$*

$$
\begin{aligned}
\lim_{a\to\infty} \sup_{\mathbf{x}\in K} \int_0^t \|\sigma_s(\mathbf{x})\|^2 \mathbf{1}(\|\sigma_s(\mathbf{x})\| > a)\, ds &= 0,\\
\inf_{\mathbf{x}\in K} \inf_{s\le t} \inf_{\substack{\lambda\in\mathbb{R}^d:\\ |\lambda|=1}} \lambda \cdot \sigma_s(\mathbf{x})\sigma_s(\mathbf{x})^T \lambda &> 0,\\
\lim_{a\to\infty} \sup_{\mathbf{x}\in K} \int_0^t |b_s(\mathbf{x})|\, \mathbf{1}(|b_s(\mathbf{x})| > a)\, ds &= 0.
\end{aligned}
$$

Then the equation

$$\dot{X}_t = b_t(X) + \sigma_t(X)\dot{W}_t,\ X_0 = x,$$

has a Luzin solution (X, W) if and only if there exists a deviability Π on $\mathbb{C}$ such that $\mathbf{x}_0 = x$ Π-a.e., and the idempotent process $M_t(\mathbf{x}) = \mathbf{x}_t - x - \int_0^t b_s(\mathbf{x})\, ds$ is a $\mathbf{C}^\Pi$-local maxingale on $(\mathbb{C}, \Pi)$ with the quadratic characteristic $\langle M\rangle_t(\mathbf{x}) = \int_0^t \sigma_s(\mathbf{x})\sigma_s(\mathbf{x})^T\, ds$ and is strictly Luzin. The solution (X, W) is unique if and only if the deviability Π is unique. The idempotent distribution of X then coincides with Π.

Proof. Let (X, W) be a Luzin solution on an idempotent probability space $(\Omega, \tilde{\Pi})$ with a τ-flow $\mathbf{A}$. Then by Theorem 2.5.19 $(\int_0^t \sigma_s(X)\dot{W}_s\, ds,\ t \in \mathbb{R}_+)$ is an $\mathbf{A}$-local maxingale with the quadratic characteristic $(\int_0^t \sigma_s(X)\sigma_s(X)^T\, ds,\ t \in \mathbb{R}_+)$. The idempotent distribution of X is the required deviability Π.

Conversely, let $M_t(\mathbf{x}) = \mathbf{x}_t - x - \int_0^t b_s(\mathbf{x})\, ds$ be a strictly Luzin idempotent process, which is a $\mathbf{C}^\Pi$-local maxingale with the quadratic characteristic $\langle M\rangle_t(\mathbf{x}) = \int_0^t \sigma_s(\mathbf{x})\sigma_s(\mathbf{x})^T\, ds$ on $(\mathbb{C}, \Pi)$. Then by Theorem 2.5.22 there exists a strictly Luzin-continuous $\mathbf{C}^\Pi$-Wiener idempotent process W on $(\mathbb{C}, \Pi)$ such that $M_t = \int_0^t \sigma_s \dot{W}_s\, ds$, which implies that the canonical process on $(\mathbb{C}, \Pi)$ and W make up a Luzin solution to the equation. ☐

Remark 2.6.31. *Under the hypotheses, both M and $\langle M\rangle$ are also strictly Luzin-continuous on $(\mathbb{C}, \Pi)$.*

We conclude the section with an existence result for an equation with respect to Poisson idempotent processes, which will be used

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in a queueing application later on. We confine ourselves to Luzin solutions.

Let $(u_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}))$ and $(v_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}))$ be $\mathbf{C}(\mathbb{R}_+,\mathbb{R})$-progressively measurable $\mathbb{R}_+$-valued functions, which are continuous in $\mathbf{x}$ and such that

$$\int_0^t u_s(\mathbf{x})\,ds < \infty,\ \int_0^t u_s(\mathbf{x})\,ds < \infty,\ \ t \in \mathbb{R}_+,\ \mathbf{x} \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}).$$

Let us consider the equation

$$X_t = x + \mathcal{N}_1\Big(\int_0^t u_s(X)\,ds\Big) - \mathcal{N}_2\Big(\int_0^t v_s(X)\,ds\Big), \tag{2.6.10}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent Poisson idempotent processes and $x \in \mathbb{R}$.

Definition 2.6.32. *We say that equation* (2.6.10) *has a Luzin solution if there exist an idempotent probability space* (Ω,Π) *equipped with a* τ*-flow* $\mathbf{A}$ *and* $\mathbb{R}$*-valued continuous idempotent processes* $X = (X_s,\ s \in \mathbb{R}_+)$, $\mathcal{N}_1 = (\mathcal{N}_1(s),\ s \in \mathbb{R}_+)$ *and* $\mathcal{N}_2 = (\mathcal{N}_2(s),\ s \in \mathbb{R}_+)$ *on* (Ω,Π) *such that the following holds*

1. X, $\big(\mathcal{N}_1(\int_0^t u_s(X)\,ds),\ t \in \mathbb{R}_+\big)$ *and* $\big(\mathcal{N}_2(\int_0^t v_s(X)\,ds),\ t \in \mathbb{R}_+\big)$ *are* $\mathbf{A}$*-adapted,*
2. *the idempotent processes* $\big(\mathcal{N}_1(r + \int_0^t u_s(X)\,ds) - \mathcal{N}_1(\int_0^t u_s(X)\,ds),\ r \in \mathbb{R}_+\big)$ *and* $\big(\mathcal{N}_2(r + \int_0^t v_s(X)\,ds) - \mathcal{N}_1(\int_0^t v_s(X)\,ds),\ r \in \mathbb{R}_+\big)$, *when conditioned on* $\mathcal{A}_t$, *where* $t \in \mathbb{R}_+$, *are independent Poisson idempotent processes,*
3. $(X,\mathcal{N}_1,\mathcal{N}_2)$ *is a Luzin-continuous idempotent process,*
4. (2.6.10) *holds for* $t \in \mathbb{R}_+$ Π*-a.e. in* $\omega \in \Omega$.

The triplet $(X,\mathcal{N}_1,\mathcal{N}_2)$ *is then called a Luzin solution to the equation with initial condition* x.

The next existence result is an analogue of Theorem 2.6.24 and is proved along the same lines.

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Theorem 2.6.33. *Let $u_s(\mathbf{x})$ and $v_s(\mathbf{x})$, in addition to the above conditions, satisfy the linear-growth condition $u_s(\mathbf{x})+v_s(\mathbf{x}) \le l_s(1+\sup_{t\le s}|\mathbf{x}_t|)$, where l_s is locally integrable. Then equation (2.6.10) has a Luzin solution $(X,\mathcal{N}_1,\mathcal{N}_2)$ on an idempotent probability space (Ω,Π) with a τ-flow $\mathbf{A}$ such that the idempotent distribution of X has density*

$$\Pi_x^X(\mathbf{x}) = \exp\Bigl(-\int_0^\infty \sup_{\lambda\in\mathbb{R}}\bigl(\lambda\dot{\mathbf{x}}_t-(e^\lambda-1)u_t(\mathbf{x})-(e^{-\lambda}-1)v_t(\mathbf{x})\bigr)\,dt\Bigr)$$

if $\mathbf{x}$ is absolutely continuous and $\mathbf{x}_0 = x$, and $\Pi_x^X(\mathbf{x}) = 0$ otherwise.

Proof. We first prove an analogue of the extension property for the equation

$$\mathbf{x}_t = x+n_1\Bigl(\int_0^t u_s(\mathbf{x})\,ds\Bigr)-n_2\Bigl(\int_0^t v_s(\mathbf{x})\,ds\Bigr),\ t\in\mathbb{R}_+. \qquad (2.6.11)$$

More specifically, we prove that given $n_1 \in \mathbb{C}(\mathbb{R}_+,\mathbb{R})$ and $n_2 \in \mathbb{C}(\mathbb{R}_+,\mathbb{R})$ such that $\Pi^{\mathcal{N}}(n_1)\Pi^{\mathcal{N}}(n_2) > 0$, where $\Pi^{\mathcal{N}}$ is the Poisson idempotent probability, every solution of (2.6.11) on an interval $[0,T]$ can be extended to a solution on $\mathbb{R}_+$. Since by properties of the idempotent Poisson process for $n\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})$, $A\in\mathbb{R}_+$ and $t\in\mathbb{R}_+$

$$\Pi^{\mathcal{N}}\Bigl(\frac{n(t)}{1+t} > A\Bigr) \le \frac{S_{\Pi^{\mathcal{N}}}e^{n(t)}}{e^{A(1+t)}} = \frac{e^{(e-1)t}}{e^{A(1+t)}}$$

and the latter ratio is less than $\Pi^{\mathcal{N}}(n_1)$ for all A large enough, we conclude that $n_1(t)\le A(1+t)$ for all $t\in\mathbb{R}_+$ if A is large. The same fact holds for n_2. Since also n_1 and n_2 are continuous, the claim follows by a successive approximation argument as in the proof of Theorem 2.6.22.

We define idempotent probability $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}$ on $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^3)$ by $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\mathbf{x},n_1,n_2) = \Pi^{\mathcal{N}}(n_1)\Pi^{\mathcal{N}}(n_2)$ if $\mathbf{x}_t = x+n_1\bigl(\int_0^t u_s(\mathbf{x})\,ds\bigr) - n_2\bigl(\int_0^t v_s(\mathbf{x})\,ds\bigr)$, $t\in\mathbb{R}_+$, and $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\mathbf{x},n_1,n_2) = 0$ otherwise. Let $(X,\mathcal{N}_1,\mathcal{N}_2)$ be the canonical process on $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^3)$. It clearly satisfies (2.6.10) $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}$-a.e. We check that $(X,\mathcal{N}_1,\mathcal{N}_2)$ is Luzin-continuous, i.e., that $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}$ is a deviability by showing that the sets $K(a) = \{(\mathbf{x},n_1,n_2)\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}^3) : \Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\mathbf{x},n_1,n_2)\ge a\}$ are

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compact for all $a \in (0,1]$, which is carried out as in the proof of Theorem 2.6.24. In some more detail, let $(\mathbf{x}_k, n_{1,k}, n_{2,k}) \in K(a)$, $k \in \mathbb{N}$. Since $\mathcal{N}$ is a Luzin-continuous idempotent process by Lemma 2.4.17, we have by Theorem 2.2.13 that for $T \in \mathbb{R}_+$ and $\epsilon > 0$

$$\lim_{\delta \to 0} \Pi^{\mathcal{N}}\big(\sup_{\substack{s,t \in [0,T]:\\ |t-s| \le \delta}} |n(t) - n(s)| > \epsilon \big) = 0.$$

Since $\Pi^{\mathcal{N}}(n_{1,k}) \ge a > 0$ and $\Pi^{\mathcal{N}}(n_{2,k}) \ge a > 0$, it follows that the functions $n_{1,k}$ and $n_{2,k}$, $k \in \mathbb{N}$, are locally uniformly equicontinuous. Besides in analogy with the above argument there exists $B > 0$ such that $n_{1,k}(t) + n_{2,k}(t) \le B(1+t)$ for $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$. Since also the $(\mathbf{x}_k, n_{1,k}, n_{2,k})$ satisfy (2.6.11), a standard argument shows that the $\mathbf{x}_k$ are uniformly bounded on bounded intervals and locally uniformly equicontinuous as well. Arzelà-Ascoli's theorem implies that the sequence $\{(\mathbf{x}_k, n_{1,k}, n_{2,k}),\ k \in \mathbb{N}\}$ is relatively compact in $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^3)$. Let $(\tilde{\mathbf{x}}, \tilde{n}_1, \tilde{n}_2)$ be an accumulation point. It clearly satisfies (2.6.11). Therefore, $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\tilde{\mathbf{x}}, \tilde{n}_1, \tilde{n}_2) = \Pi^{\mathcal{N}}(\tilde{n}_1)\Pi^{\mathcal{N}}(\tilde{n}_2) \ge a$, where the latter inequality follows since $(\tilde{n}_1, \tilde{n}_2)$ is an accumulation point of $(n_{1,k}, n_{2,k})$, $\Pi^{\mathcal{N}}(n_1)\Pi^{\mathcal{N}}(n_2) \ge a$, and the function $\Pi^{\mathcal{N}}(n)$ is upper semi-continuous.

Let $\mathcal{A}_t$ be the τ-algebra generated by the atoms $p_t\mathbf{x}$, $p_{\int_0^t u_s(\mathbf{x})\,ds} n_1$ and $p_{\int_0^t v_s(\mathbf{x})\,ds} n_2$, where $(\mathbf{x}, n_1, n_2) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^3)$. Defining $\mathbf{A} = \{\mathbf{A}_t,\ t \in \mathbb{R}_+\}$, we see that the idempotent processes X, $\big(\mathcal{N}_1(\int_0^t u_s(X)\,ds),\ t \in \mathbb{R}_+\big)$, and $\big(\mathcal{N}_2(\int_0^t v_s(X)\,ds),\ t \in \mathbb{R}_+\big)$ are $\mathbf{A}$-adapted. We now show that the idempotent processes $\big(\mathcal{N}_1(r + \int_0^t u_s(X)\,ds) - \mathcal{N}_1(\int_0^t u_s(X)\,ds),\ r \in \mathbb{R}_+\big)$ and $\big(\mathcal{N}_2(r + \int_0^t v_s(X)\,ds) - \mathcal{N}_1(\int_0^t v_s(X)\,ds),\ r \in \mathbb{R}_+\big)$, when conditioned on $\mathcal{A}_t$, are independent Poisson idempotent processes. Equivalently, we have to prove that for $n_1', n_2' \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ the following holds $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}$-a.e.

$$\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}\big(\theta_{\int_0^t u_s(X)\,ds}\mathcal{N}_1 = n_1', \theta_{\int_0^t v_s(X)\,ds}\mathcal{N}_2 = n_2' | \mathcal{A}_t\big) \\ = \Pi^{\mathcal{N}}(n_1')\Pi^{\mathcal{N}}(n_2'). \quad (2.6.12)$$

Let $n_1'', n_2'', \mathbf{x}'' \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ be such that $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\mathbf{x}'', n_1'', n_2'') > 0$. Then by the definition of $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}$, the extension property and the

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properties of the Poisson idempotent probability

$$\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}\big(\theta_{\int_0^t u_s(X)\,ds}\mathcal{N}_1 = n_1', \theta_{\int_0^t v_s(X)\,ds}\mathcal{N}_2 = n_2', p_tX = p_t\mathbf{x}'',$$
$$p_{\int_0^t u_s(X)\,ds}\mathcal{N}_1 = p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1'', p_{\int_0^t v_s(X)\,ds}\mathcal{N}_2 = p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2''\big)$$
$$= \sup_{(\mathbf{x},n_1,n_2)\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}^3)} \mathbf{1}\big(\theta_{\int_0^t u_s(\mathbf{x})\,ds}n_1 = n_1', \theta_{\int_0^t v_s(\mathbf{x})\,ds}n_2 = n_2',$$
$$p_t\mathbf{x} = p_t\mathbf{x}'', p_{\int_0^t u_s(\mathbf{x})\,ds}n_1 = p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1'',$$
$$p_{\int_0^t v_s(\mathbf{x})\,ds}n_2 = p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2''\big)\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\mathbf{x},n_1,n_2)$$
$$= \sup_{(n_1,n_2)\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}^2)} \mathbf{1}\big(\theta_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1 = n_1', \theta_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2 = n_2',$$
$$p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1 = p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1'', p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2 = p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2''\big)$$
$$\Pi^{\mathcal{N}}(n_1)\Pi^{\mathcal{N}}(n_2) = \sup_{n_1\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{1}\big(\theta_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1 = n_1',$$
$$p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1 = p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1''\big) \sup_{n_2\in\mathbb{C}(\mathbb{R}_+,\mathbb{R})} \mathbf{1}\big(\theta_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2 = n_2',$$
$$p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2 = p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2''\big)\Pi^{\mathcal{N}}(n_1)\Pi^{\mathcal{N}}(n_2)$$
$$= \Pi^{\mathcal{N}}\big(n_1 : p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1 = p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1''\big)\Pi^{\mathcal{N}}(n_1')$$
$$\Pi^{\mathcal{N}}\big(n_2 : p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2 = p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2''\big)\Pi^{\mathcal{N}}(n_2')$$
$$= \Pi^{X,\mathcal{N}_1,\mathcal{N}_2}\big(p_tX = p_t\mathbf{x}'', p_{\int_0^t u_s(X)\,ds}\mathcal{N}_1 = p_{\int_0^t u_s(\mathbf{x}'')\,ds}n_1'',$$
$$p_{\int_0^t v_s(X)\,ds}\mathcal{N}_2 = p_{\int_0^t v_s(\mathbf{x}'')\,ds}n_2''\big)\Pi^{\mathcal{N}}(n_1')\Pi^{\mathcal{N}}(n_2').$$

Equality (2.6.12) follows.

Finally, the fact that $\Pi_x^X(\mathbf{x}) = \sup_{n_1,n_2}\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\mathbf{x},n_1,n_2)$ coincides with the expression for $\Pi^X(\mathbf{x})$ in the statement of the theorem follows by routine calculations. □

Remark 2.6.34. *We refer to* $\big(\mathcal{N}_1\big(\int_0^t u_s(X)\,ds\big), t \in \mathbb{R}_+\big)$ *and* $\big(\mathcal{N}_2\big(\int_0^t v_s(X)\,ds\big), t \in \mathbb{R}_+\big)$ *as Poisson idempotent processes of rates* $u_s(X)$ *and* $v_s(X)$*, respectively.*

2.7 Semimaxingales

In this section we consider idempotent analogues of semimartingales and associated integrals. Let (Ω,Π) be an idempotent probability space with a τ-flow $\mathbf{A}$. Let $(G_t(\lambda;\omega), t \in \mathbb{R}_+, \omega \in \Omega), \lambda \in \mathbb{R}^d$,

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be $\mathbb{R}$-valued $\mathbf{A}$-adapted continuous idempotent processes such that $G_0(\lambda;\omega) = G_t(0;\omega) = 0$. We refer to $G(\lambda) = (G_t(\lambda;\omega), t \in \mathbb{R}_+, \omega \in \Omega), \lambda \in \mathbb{R}^d$, as a cumulant.

Definition 2.7.1. *We say that an $\mathbb{R}^d$-valued $\mathbf{A}$-adapted continuous idempotent process X on (Ω, Π) is an $\mathbf{A}$-semimaxingale with cumulant $G(\lambda)$ if the idempotent process $Y(\lambda) = (Y_t(\lambda;\omega),\ t \in \mathbb{R}_+, \omega \in \Omega)$ defined by*

$$Y_t(\lambda) = \exp(\lambda\cdot(X_t - X_0) - G_t(\lambda)) \tag{2.7.1}$$

is an $\mathbf{A}$-local exponential maxingale for every $\lambda \in \mathbb{R}^d$. If, in addition, $G(\lambda)$ is an increasing function of t for all λ and ω, X is called an $\mathbf{A}$-local maxingale.

Remark 2.7.2. *We occasionally say that X is a semimaxingale on $(\Omega, \mathbf{A}, \Pi)$ rather than that it is an $\mathbf{A}$-semimaxingale.*

Examples of semimaxingales are the idempotent Wiener process and the idempotent Poisson process. Also local maxingales with quadratic characteristics are semimaxingales. Another example is provided by idempotent processes with independent increments. Recall that $\mathbf{A}^X$ denotes the τ-flow generated by an idempotent process X.

Theorem 2.7.3. *Let X be a continuous $\mathbf{A}$-adapted idempotent process with independent increments such that the function $\tilde{G}_t(\lambda) = \ln S\exp(\lambda\cdot(X_t - X_0))$ is finite for all $\lambda \in \mathbb{R}^d$. Then X is an $\mathbf{A}^X$-semimaxingale with cumulant $\tilde{G}(\lambda)$. If, in addition, X_0 is Luzin and $\tilde{G}_t(\lambda)$ is differentiable in λ, then X is Luzin-continuous.*

Proof. We only prove the Luzin-continuity. Since $\tilde{G}_t(\lambda)$ is differentiable in λ, by Lemma 1.11.7 the increments $X_t - X_s$ are Luzin idempotent variables, so by independence of increments, Corollary 1.8.10 and Theorem 1.7.11 X is Luzin. It is Luzin-continuous by Theorem 2.2.13 and the fact that $\tilde{G}_t(\lambda)$ is continuous in t. □

Our primary goal is to show that integrals with respect to semimaxingales also give rise to local exponential maxingales. The methods are analogous to those we used in Section 2.5. In view of applications to large deviation theory we are interested in studying semimaxingales on spaces of trajectories that are also strictly Luzin

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idempotent processes with respect to a deviability. Therefore, both in this section and the next one we assume that Ω is the space $\mathbb{C} = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ and Π is a deviability on $\mathbb{C}$. We equip $\mathbb{C}$ with the flow $\mathbf{C}^\Pi = (\mathcal{C}_s^\Pi,\ s \in \mathbb{R}_+)$ that is the completion of the natural τ-flow $\mathbf{C} = (\mathcal{C}_s,\ s \in \mathbb{R}_+)$ with respect to Π, where the $\mathcal{C}_s$ are the τ-algebras generated by the mappings $\mathbf{x} \to \mathbf{x}_t$, $t \in [0, s]$, for $\mathbf{x} \in \mathbb{C}$. We start with some simple properties.

Definition 2.7.4. *We say that an $\overline{\mathbb{R}}_+$-valued function τ on $\mathbb{C}$ is a strictly Luzin stopping time on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$ if it is a $\mathbf{C}^\Pi$–stopping time and is finite and continuous when restricted to $K_\Pi(a)$ for $a \in (0, 1]$.*

The following result is standard.

Lemma 2.7.5. *Let $(H_t(\mathbf{x}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ be an $\mathbb{R}_+$-valued increasing continuous $\mathbf{C}^\Pi$–adapted strictly Luzin idempotent process. Let, for $c \in \mathbb{R}_+$,*

$$\tau(\mathbf{x}) = \inf\{t \in \mathbb{R}_+ : H_t(\mathbf{x}) + t \geq c\}.$$

Then $\tau(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}$, is a strictly Luzin stopping time on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$.

Proof. The fact that τ is a $\mathbf{C}^\Pi$–stopping time follows by Lemma 2.2.18. Since $H_t(\mathbf{x})$ is increasing, continuous in t and continuous in $\mathbf{x}$ on $K_\Pi(a)$, $a \in (0, 1]$, it is also continuous on $K_\Pi(a)$ as a map from $\mathbb{C}$ to $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ by Polya's theorem, see Liptser and Shiryaev [79, Problem 5.3.2]. Continuity of $\tau(\mathbf{x})$ on $K_\Pi(a)$ follows now by Whitt [135, Theorem 7.2] since $H_t(\mathbf{x}) + t$ is strictly increasing. ☐

Definition 2.7.6. *We say that a semimaxingale (respectively, local maxingale) X with cumulant $G(\lambda)$ is a strictly Luzin semimaxingale (respectively, local maxingale) on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$ if X and $G(\lambda)$ for every $\lambda \in \mathbb{R}^d$ are strictly Luzin idempotent processes. We say that X is a strictly Luzin-continuous semimaxingale (respectively, local maxingale) if X and $G(\lambda)$ for every $\lambda \in \mathbb{R}^d$ are strictly Luzin-continuous idempotent processes.*

Lemma 2.7.7. *If X is either a strictly Luzin local maxingale or a strictly Luzin-continuous semimaxingale on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$, then the local exponential maxingales $Y(\lambda)$ admit localising sequences of strictly Luzin stopping times.*

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Proof. Let X be a strictly Luzin local maxingale with cumulant $G(\lambda)$. Let $\tau_n = \inf\{t \in \mathbb{R}_+ : G_t(2\lambda) + t \geq n\}$, where $n \in \mathbb{N}$. Since $G(2\lambda)$ is a strictly Luzin idempotent process and is increasing, τ_n is a strictly Luzin stopping time by Lemma 2.7.5. The idempotent process $(Y_{t\wedge\tau_n}(\lambda),\, t \in \mathbb{R}_+)$ is uniformly maximable because

$$S_\Pi\big(Y_{t\wedge\tau_n}(\lambda)^2\big) \leq S_\Pi\big(Y_{t\wedge\tau_n}(2\lambda)\exp(G_{t\wedge\tau_n}(2\lambda))\big) \leq e^n,$$

where the first inequality holds since $G_t(\lambda)$ is non-negative and the second follows by the definition of τ_n and the fact that $(Y_{t\wedge\tau_n}(\lambda),\, t \in \mathbb{R}_+)$ is a supermaxingale starting at 1.

If X is a strictly Luzin-continuous semimaxingale, then the above argument applies with $\tau_n = \inf\{t \in \mathbb{R}_+ : \sup_{s\leq t}\big(|G_s(2\lambda)| \vee |G_s(\lambda)|\big) + t \geq n\}$. ☐

In connection with the lemma we introduce the following.

Definition 2.7.8. *A local exponential maxingale M on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$ is called a strictly Luzin-continuous local exponential maxingale if it is a strictly Luzin-continuous idempotent process and admits a localising sequence of strictly Luzin stopping times.*

Remark 2.7.9. *Note that if M is a strictly Luzin-continuous local exponential maxingale and τ is a strictly Luzin stopping time, then $M_{t\wedge\tau}$ is a strictly Luzin idempotent variable.*

In the rest of the section X is the canonical idempotent process on $\mathbb{C}$, i.e., $X_t(\mathbf{x}) = \mathbf{x}_t$, and the following is assumed to hold:

- X is a semimaxingale on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$ with cumulant $G(\lambda)$.

Lemma 2.7.10. *The idempotent process X is a semimaxingale with cumulant $G(\lambda)$ under the deviability $\Pi(\cdot|\mathbf{x}_0 = x)$ for $\Pi \circ \pi_0^{-1}$-almost all $x \in \mathbb{R}^d$.*

We omit a simple proof and only note that $\Pi(\cdot|\mathbf{x}_0 = x)$ is a deviability by Lemma 1.6.12.

We introduce an idempotent measure that is to play an important part in the sequel. Let Λ_0 be the set of all $\mathbb{R}^d$–valued, piecewise constant functions $(\lambda(t), t \in \mathbb{R}_+)$ of the form

$$\lambda(t) = \sum_{i=1}^{k} \lambda_i\, \mathbf{1}(t \in (t_{i-1}, t_i]),$$

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where $0 \le t_0 < t_1 < \ldots < t_k$, $\lambda_i \in \mathbb{R}^d$, $i = 0, \ldots, k$, $k \in \mathbb{N}$. We define for $\mathbf{x} \in \mathbb{C}$ and $x \in \mathbb{R}^d$

$$\mathbf{I}(\mathbf{x}) = \sup_{(\lambda(t)) \in \Lambda_0} \int_0^\infty \big(\lambda(t) \cdot d\mathbf{x}_t - dG_t(\lambda(t); \mathbf{x})\big), \tag{2.7.2}$$

where the integral is understood as a finite sum so that

$$\int_0^\infty (\lambda(t) \cdot d\mathbf{x}_t - dG_t(\lambda(t); \mathbf{x})) = \sum_{i=1}^k \big[\lambda_i \cdot (\mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}}) - (G_{t_i}(\lambda_i; \mathbf{x}) - G_{t_{i-1}}(\lambda_i; \mathbf{x}))\big], \tag{2.7.3}$$

and let

$$\mathbf{\Pi}(\mathbf{x}) = \exp(-\mathbf{I}(\mathbf{x})), \tag{2.7.4}$$

$$\mathbf{I}_x(\mathbf{x}) = \begin{cases} \mathbf{I}(\mathbf{x}), & \text{if } \mathbf{x}_0 = x, \\ \infty, & \text{otherwise,} \end{cases} \tag{2.7.5}$$

$$\mathbf{\Pi}_x(\mathbf{x}) = \exp(-\mathbf{I}_x(\mathbf{x})),\ \mathbf{x} \in \mathbb{C}, \quad \mathbf{\Pi}_x(\Gamma) = \sup_{\mathbf{x} \in \Gamma} \mathbf{\Pi}_x(\mathbf{x}),\ \Gamma \subset \mathbb{C}. \tag{2.7.6}$$

We have our first property of $\mathbf{\Pi}_x$.

Lemma 2.7.11. *For* $\Pi \circ \pi_0^{-1}$*-almost all* $x \in \mathbb{R}^d$, $\Pi(\mathbf{x}|\mathbf{x}_0 = x) \le \mathbf{\Pi}_x(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}$. *In particular,* $\Pi(\mathbf{x}) \le \mathbf{\Pi}(\mathbf{x})$ *and* $\mathbf{\Pi}$ *is an idempotent probability.*

Proof. By Lemma 2.7.10 it is enough to check that $\Pi(\mathbf{x}) \le \mathbf{\Pi}_x(\mathbf{x})$ assuming that $\mathbf{x}_0 = x$ Π-a.e. We follow the argument of the proof of Lemma 2.5.8. Let, for $0 \le s_1 \le t_1 \le \ldots \le s_k \le t_k$ and $\lambda_i \in \mathbb{R}^d$, $i = 1, \ldots, k$,

$$\tilde{Z} = \exp\Big[\sum_{i=1}^k \lambda_i \cdot \big(X_{t_i} - X_{s_i}\big) - \sum_{i=1}^k \big(G_{t_i}(\lambda_i) - G_{s_i}(\lambda_i)\big)\Big].$$

The definition of $\mathbf{\Pi}_x$ and argument of the proof of Lemma 2.5.8 imply that it suffices to show that $S_\Pi \tilde{Z} \le 1$. Let

$$\tau_n = \inf\big\{t \in \mathbb{R}_+ : \max_{i=1,\ldots,k} \big(|G_t(\lambda_i)| \vee |G_t(2\lambda_i)|\big) \ge n\big\}$$

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be a common localising sequence for the $Y(\lambda_i)$, $i = 1, \ldots, k$, and

$$Y_n^i = \exp\bigl(\lambda_i \cdot \bigl(X_{t_i \wedge \tau_n} - X_{s_i \wedge \tau_n}\bigr) - \bigl(G_{t_i \wedge \tau_n}(\lambda_i) - G_{s_i \wedge \tau_n}(\lambda_i)\bigr)\bigr).$$

Then $S_\Pi(Y_n^i | \mathcal{C}_{s_i}^\Pi) = 1$ so that

$$S_\Pi\Bigl(\prod_{i=1}^{k} Y_n^i\Bigr) = S_\Pi\Bigl(\prod_{i=1}^{k-1} Y_n^i S_\Pi(Y_n^k | \mathcal{C}_{s_k}^\Pi)\Bigr)$$
$$= S_\Pi\Bigl(\prod_{i=1}^{k-2} Y_n^i S_\Pi(Y_n^{k-1} | \mathcal{C}_{s_{k-1}}^\Pi)\Bigr) = \ldots = S_\Pi Y_n^1 = 1.$$

Since $\tau_n \to \infty$, it follows that $\tilde{Z} = \lim_{n\to\infty} \prod_{i=1}^k Y_n^i$ so that "the Fatou lemma" (see Theorem 1.4.19) yields the required.

Finally, $\mathbf{\Pi}$ is an idempotent probability since Π is an idempotent probability and $\mathbf{\Pi}(\mathbf{x}) \le 1$ by definition. ☐

Below, we are mostly concerned with the case where the cumulant $G(\lambda) = (G_t(\lambda; \mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}), \lambda \in \mathbb{R}^d$, is absolutely continuous so that it has the form

$$G_t(\lambda; \mathbf{x}) = \int_0^t g_s(\lambda; \mathbf{x})\, ds, \;\; \lambda \in \mathbb{R}^d, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}, \qquad (2.7.7)$$

where $g_s(\lambda; \mathbf{x})$ is Lebesgue integrable in s. The next lemma gives the form of $\mathbf{\Pi}_x$ for absolutely continuous cumulants. It also shows that our usage of the notation $\mathbf{\Pi}$ is consistent with the one in Section 2.6 (see Remark 2.6.20). Let

$$h_s(y; \mathbf{x}) = \sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot y - g_s(\lambda; \mathbf{x})) \qquad (2.7.8)$$

be the convex conjugate, or the Legendre–Fenchel transform, of $g_s(\lambda; \mathbf{x})$. It is non-negative provided $g_s(0; \mathbf{x}) = 0$.

Lemma 2.7.12. *Let an $\mathbb{R}$-valued function $g_s(\lambda; \mathbf{x})$ be Lebesgue measurable in s and continuous in λ, $g_s(0; \mathbf{x}) = 0$, and for $A > 0$, $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$*

$$\int_0^t \sup_{|\lambda| = A} |g_s(\lambda; \mathbf{x})|\, ds < \infty.$$

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If $G(\lambda)$ has the form (2.7.7), then

$$\mathbf{I}(\mathbf{x}) = \begin{cases} \int_0^\infty h_s(\dot{\mathbf{x}}_s;\mathbf{x})\,ds, & \text{if } \mathbf{x} \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases}$$

In particular, X is absolutely continuous under the hypotheses.

Proof. If $\mathbf{x}$ is absolutely continuous, then the desired representation follows by Lemma A.2 in Appendix A with $f(t,\lambda) = \lambda \cdot \dot{\mathbf{x}}_t - g_t(\lambda;\mathbf{x})$, (2.7.3) and (2.7.5).

Let $\mathbf{x}$ not be absolutely continuous on an interval $[0,T]$. Then we can choose $\varepsilon > 0$ such that for every $\delta > 0$ there exist $0 \le t_1 < \ldots < t_{2l} \le T$ satisfying

$$\sum_{i=1}^{l}(t_{2i}-t_{2i-1}) < \delta,\ \sum_{i=1}^{l}|\mathbf{x}_{t_{2i}}-\mathbf{x}_{t_{2i-1}}| > \varepsilon. \tag{2.7.9}$$

For $N > 0$, we take

$$\lambda_N(t) = N\sum_{i=1}^{l}\frac{\mathbf{x}_{t_{2i}} - \mathbf{x}_{t_{2i-1}}}{|\mathbf{x}_{t_{2i}} - \mathbf{x}_{t_{2i-1}}|}\mathbf{1}_{(t_{2i-1},t_{2i}]}(t)$$

(of course we may assume that $|\mathbf{x}_{t_{2i}} - \mathbf{x}_{t_{2i-1}}| > 0$).

Then by (2.7.3), (2.7.5), and (2.7.9)

$$\begin{aligned}\mathbf{I}(\mathbf{x}) &\ge \int_0^\infty [\lambda_N(t)\cdot d\mathbf{x}_t - dG_t(\lambda_N(t);\mathbf{x})] \\ &= N\sum_{i=1}^{l}|\mathbf{x}_{t_{2i}} - \mathbf{x}_{t_{2i-1}}| - \sum_{i=1}^{l}\int_{t_{2i-1}}^{t_{2i}} g_t(\lambda_N(t);\mathbf{x})\,dt \\ &> N\varepsilon - \int_0^T \sup_{|\lambda|=N}|g_t(\lambda;\mathbf{x})|\,\mathbf{1}\Big(t \in \bigcup_{i=1}^{l}(t_{2i-1},t_{2i}]\Big)\,dt.\end{aligned}$$

By (2.7.9) the latter integrand goes to 0 in measure as $\delta \to 0$ so that by Lebesgue's dominated convergence theorem the integral converges to 0 as $\delta \to 0$. Thus, $\mathbf{I}(\mathbf{x}) > N\varepsilon$ for arbitrary N.

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Finally, since by Lemma 2.7.11 $\Pi(X = \mathbf{x}|X_0 = x) \le \mathbf{\Pi}_x(\mathbf{x})$ for $\Pi\circ\pi_0^{-1}$-almost all x, X is absolutely continuous under $\Pi(\cdot|X_0 = x)$ for these x. Since $\Pi(X = \mathbf{x}) = \sup_{x\in\mathbb{R}^d}\Pi(X = \mathbf{x}|X_0 = x)\Pi(X_0 = x)$, it follows that X is absolutely continuous under Π. $\square$

We assume in the rest of the section that $G(\lambda)$ is given by (2.7.7). Let us further assume that

$$G_t(\lambda;\mathbf{x}) = \lambda\cdot B'_t(\mathbf{x}) + \hat{G}_t(\lambda;\mathbf{x}), \tag{2.7.10}$$

where $B' = (B'_t(\mathbf{x}),\, t\in\mathbb{R}_+, \mathbf{x}\in\mathbb{C})$ is an $\mathbb{R}^d$-valued $\mathbf{C}^\Pi$-adapted idempotent process such that $B'_0(\mathbf{x}) = 0$ and $\hat{G}(\lambda) = (\hat{G}_t(\lambda;\mathbf{x}),\, t\in\mathbb{R}_+, \mathbf{x}\in\mathbb{C})$, $\lambda\in\mathbb{R}^d$, are $\mathbb{R}_+$-valued $\mathbf{C}^\Pi$-adapted idempotent processes such that $\hat{G}_0(\lambda;\mathbf{x}) = \hat{G}_t(0;\mathbf{x}) = 0$. Since $G(\lambda)$ is absolutely continuous in t, we assume that both B' and $\hat{G}(\lambda)$ are absolutely continuous so that

$$B'_t(\mathbf{x}) = \int_0^t b_s(\mathbf{x})\,ds, \tag{2.7.11}$$

$$\hat{G}_t(\lambda;\mathbf{x}) = \int_0^t \hat{g}_s(\lambda;\mathbf{x})\,ds, \tag{2.7.12}$$

where $(b_s(\mathbf{x}))$ is $\mathbf{C}^\Pi$-progressively measurable, $\int_0^t|b_s(\mathbf{x})|ds < \infty$, $(\hat{g}_s(\lambda;\mathbf{x}))$ is $\mathbb{R}_+$-valued, $\overline{\mathcal{B}}([0,t])\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{C}_t^\Pi/\mathcal{B}(\mathbb{R}_+)$–measurable as a map from $[0,t]\times\mathbb{R}^d\times\mathbb{C}$ to $\mathbb{R}_+$, $\hat{g}_s(0;\mathbf{x}) = 0$, and $\int_0^t \hat{g}_s(\lambda;\mathbf{x})\,ds < \infty$ for $t\in\mathbb{R}_+$, $\lambda\in\mathbb{R}^d$ and $\mathbf{x}\in\mathbb{C}$. (The product of a σ-algebra and a τ-algebra has been introduced in Definition 1.5.9, the product of two σ-algebras has a standard meaning.) Thus, $g_s(\lambda;\mathbf{x})$ from (2.7.7) has the form

$$g_s(\lambda;\mathbf{x}) = \lambda\cdot b_s(\mathbf{x}) + \hat{g}_s(\lambda;\mathbf{x}). \tag{2.7.13}$$

We now introduce more conditions on $b_s(\mathbf{x})$ and $\hat{g}_s(\lambda;\mathbf{x})$.

(Π_I) The idempotent process $(b_s(\mathbf{x}))$ is strictly Luzin on $(\mathbb{C},\Pi)$ and

$$\int_0^t \sup_{\mathbf{x}\in K_\Pi(a)} |b_s(\mathbf{x})|\,ds < \infty$$

for all $a\in(0,1]$ and $t\in\mathbb{R}_+$.

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(Π_{II}) The function $(\hat{g}_s(\lambda;\mathbf{x}))$ is continuous in $(\lambda,\mathbf{x})$ when restricted to $\mathbb{R}^d \times K_\Pi(a)$ for $a \in (0,1]$, convex in $\lambda \in \mathbb{R}^d$, and

$$\sup_{|\lambda|\le A}\sup_{s\le t}\sup_{\mathbf{x}\in K_\Pi(a)} \hat{g}_s(\lambda;\mathbf{x}) < \infty, \quad \lim_{\lambda\to 0}\sup_{s\le t}\sup_{\mathbf{x}\in K_\Pi(a)} \hat{g}_s(\lambda;\mathbf{x}) = 0$$

for all $a \in (0,1]$, $t \in \mathbb{R}_+$ and $A \in \mathbb{R}_+$.

Let $M_t = X_t - X_0 - B'_t$. Then $M = (M_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ is a $\mathbf{C}^\Pi$–local maxingale with cumulant $\hat{G}(\lambda)$ and the following "canonical decomposition" holds

$$X = X_0 + B' + M. \tag{2.7.14}$$

Under (Π_I) and (Π_{II}) the idempotent processes B' and M are strictly Luzin-continuous. As above, we denote by $\dot{M}$ a $\mathbf{C}^\Pi$-progressively measurable idempotent process that coincides with the Radon-Nikodym derivative of M with respect to Lebesgue measure Π-a.e. We note that for absolutely continuous $\mathbf{x}$

$$\lambda\cdot\dot{\mathbf{x}}_s - g_s(\lambda;\mathbf{x}) = \lambda\cdot\dot{M}_s(\mathbf{x}) - \hat{g}_s(\lambda;\mathbf{x}), \tag{2.7.15}$$

so by (2.7.8), Lemma 2.7.12, (2.7.10), (2.7.14), and Lemma 2.7.11

$$\mathbf{I}(\mathbf{x}) = \int_0^\infty \hat{h}_t(\dot{M}_t(\mathbf{x});\mathbf{x})\, dt < \infty \quad \Pi\text{-a.e.}, \tag{2.7.16}$$

where

$$\hat{h}_t(y;\mathbf{x}) = \sup_{\lambda\in\mathbb{R}^d} (\lambda\cdot y - \hat{g}_t(\lambda;\mathbf{x})),\ y \in \mathbb{R}^d, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}. \tag{2.7.17}$$

Definition 2.7.13. *Let $\hat{\Lambda}_\Pi$ denote the set of all $\mathbb{R}^d$–valued $\mathbf{C}^\Pi$–progressively measurable strictly Luzin idempotent processes $\overline{\lambda} = (\lambda(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ on $(\mathbb{C},\Pi)$ such that for $\alpha \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$*

$$\int_0^t \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})\, ds < \infty \tag{2.7.18}$$

and, moreover,

$$\int_0^t \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A)\, ds \xrightarrow{\Pi} 0 \text{ as } A \to \infty. \tag{2.7.19}$$

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Remark 2.7.14. *If condition* (Π_{II}) *holds, then bounded* $\mathbf{C}^{\Pi}$*-progressively measurable strictly Luzin idempotent processes belong to* $\hat{\Lambda}_{\Pi}$.

Lemma 2.7.15. *Let conditions* (Π_I) *and* (Π_{II}) *hold. Let* $\overline{\lambda} = (\lambda(t,\mathbf{x}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}) \in \hat{\Lambda}_{\Pi}$. *Then* $\int_0^t \lambda(s,\mathbf{x}) \cdot \dot{M}_s\, ds$, *where* $t \in \mathbb{R}_+$, *is well defined and finite* Π*-a.e.*

Proof. Since M is absolutely continuous, the integrand in the statement is well defined Π-a.e. We show that Π-a.e.

$$\int_0^t |\lambda(s,\mathbf{x}) \cdot \dot{M}_s|\, ds < \infty.$$

Since $\hat{h}_s(y;\mathbf{x})$ is the convex conjugate of $\hat{g}_s(\lambda;\mathbf{x})$ and $\hat{g}_s(\lambda;\mathbf{x})$ is nonnegative, by Young's inequality (see, e.g., Krasnosel'skii and Rutickii [75])

$$\begin{aligned}
|\lambda(s,\mathbf{x}) \cdot \dot{M}_s| &= \big[\lambda(s,\mathbf{x}) \text{ sign}\, \big(\lambda(s,\mathbf{x}) \cdot \dot{M}_s\big)\big] \cdot \dot{M}_s \\
&\le \hat{g}_s\big(\lambda(s,\mathbf{x}) \text{ sign}\,(\lambda(s,\mathbf{x}) \cdot \dot{M}_s);\mathbf{x}\big) + \hat{h}_s(\dot{M}_s;\mathbf{x}) \\
&\qquad \le \hat{g}_s(\lambda(s,\mathbf{x});\mathbf{x}) + \hat{g}_s(-\lambda(s,\mathbf{x});\mathbf{x}) + \hat{h}_s(\dot{M}_s;\mathbf{x}).
\end{aligned}$$

The integral from 0 to t of the right-most side is finite Π-a.e. by the definition of $\hat{\Lambda}_{\Pi}$ and (2.7.16). ☐

Given an $\mathbb{R}^d$-valued Lebesgue measurable in s function $\overline{\lambda} = (\lambda(s,\mathbf{x}))$, we introduce an idempotent process $Z(\overline{\lambda}) = (Z_t(\overline{\lambda},\mathbf{x}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ by

$$Z_t(\overline{\lambda},\mathbf{x}) = \exp\Big(\int_0^t \big(\lambda(s,\mathbf{x}) \cdot \dot{\mathbf{x}}_s - g_s(\lambda(s,\mathbf{x});\mathbf{x})\big)\, ds\Big) \tag{2.7.20}$$

if the integral on the right-hand side is well defined and finite, and let $Z_t(\overline{\lambda},\mathbf{x}) = 0$ otherwise. If $\overline{\lambda} \in \hat{\Lambda}_{\Pi}$ and conditions (Π_I) and (Π_{II}) hold, then by Lemma 2.7.15, the definition of $\hat{\Lambda}_{\Pi}$ and (2.7.15) equality (2.7.20) holds for $t \in \mathbb{R}_+$ Π-a.e. The main result of this section is the following theorem.

Theorem 2.7.16. *Let conditions* (Π_I) *and* (Π_{II}) *hold. If* $\overline{\lambda} \in \hat{\Lambda}_{\Pi}$, *then the idempotent process* $Z(\overline{\lambda})$ *is a strictly Luzin-continuous local exponential maxingale on* $(\mathbb{C}, \mathbf{C}^{\Pi}, \Pi)$.

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The proof proceeds through a string of lemmas. Let us first note that by Lemma 2.7.11, Lemma 2.7.12 and condition (Π_{II})

$$S_\Pi Z_\tau(\overline{\lambda}) \leq 1 \tag{2.7.21}$$

for an arbitrary $\mathbb{R}_+$-valued function τ on $\mathbb{C}$. In the lemmas below we assume that conditions (Π_I) and (Π_{II}) hold.

Lemma 2.7.17. *Let $\overline{\lambda} \in \hat{\Lambda}_\Pi$. Then the idempotent process $Z(\overline{\lambda})$ is strictly Luzin-continuous. If $\tau(\mathbf{x}), \mathbf{x} \in \mathbb{C}$, is a strictly Luzin stopping time, then $Z_{t\wedge\tau}(\overline{\lambda})$ is a $\mathcal{C}^\Pi_{t\wedge\tau}$–measurable strictly Luzin variable.*

Proof. The argument is similar to the one we used in the proof of Lemma 2.5.16. We begin the proof of $Z(\overline{\lambda})$ being strictly Luzin-continuous by proving that $Z_t(\overline{\lambda})$ is a strictly Luzin variable. Since by (2.7.21) and "the Chebyshev inequality" $Z_t(\overline{\lambda})$ is a proper idempotent variable, by (2.7.20) it is sufficient to check that the maps $\mathbf{x} \to \int_0^t \lambda(s,\mathbf{x})\cdot\dot{M}_s(\mathbf{x})\,ds$ and $\mathbf{x} \to \int_0^t \hat{g}_s(\lambda(s,\mathbf{x});\mathbf{x})\,ds$ are continuous when restricted to sets $K_\Pi(a)$, $a \in (0,1]$. Let $\mathbf{x}^k \to \hat{\mathbf{x}}$ as $k \to \infty$, where $\mathbf{x}^k, \hat{\mathbf{x}} \in K_\Pi(a)$. We first check the convergence

$$\lim_{k\to\infty} \int_0^t \lambda(s,\mathbf{x}^k)\cdot\dot{M}_s(\mathbf{x}^k)\,ds = \int_0^t \lambda(s,\hat{\mathbf{x}})\cdot\dot{M}_s(\hat{\mathbf{x}})\,ds. \tag{2.7.22}$$

Denoting

$$\lambda^A(t,\mathbf{x}) = \lambda(t,\mathbf{x})i_A(|\lambda(t,\mathbf{x})|), \tag{2.7.23}$$

where

$$i_A(x) = (A+1-x)^+\wedge 1,\ x \in \mathbb{R}_+, \tag{2.7.24}$$

we have that, for $A > 0$,

$$\left|\int_0^t \lambda(s,\mathbf{x}^k)\cdot\dot{M}_s(\mathbf{x}^k)\,ds - \int_0^t \lambda(s,\hat{\mathbf{x}})\cdot\dot{M}_s(\hat{\mathbf{x}})\,ds\right|$$
$$\leq \int_0^t |\lambda(s,\mathbf{x}^k)\cdot\dot{M}_s(\mathbf{x}^k)|\ \mathbf{1}(|\lambda(s,\mathbf{x}^k)| > A)\,ds$$
$$+ \int_0^t |\lambda(s,\hat{\mathbf{x}})\cdot\dot{M}_s(\hat{\mathbf{x}})|\ \mathbf{1}(|\lambda(s,\hat{\mathbf{x}})| > A)\,ds$$

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$$+\int_0^t |(\lambda^A(s,\mathbf{x}^k) - \lambda^A(s,\hat{\mathbf{x}})) \cdot \dot{M}_s(\mathbf{x}^k)|\, ds$$
$$+\left|\int_0^t \lambda^A(s,\hat{\mathbf{x}}) \cdot \dot{M}_s(\mathbf{x}^k)\, ds - \int_0^t \lambda^A(s,\hat{\mathbf{x}}) \cdot \dot{M}_s(\hat{\mathbf{x}})\, ds\right|. \quad (2.7.25)$$

Note that all the terms in (2.7.22) and (2.7.25) are well defined by Lemma 2.7.15 and the fact that $(\lambda^A(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}) \in \hat{\Lambda}_\Pi$. We prove that each term on the right of (2.7.25) tends to 0 as $k \to \infty$ and $A \to \infty$.

Since by Young's inequality for $\alpha > 0$

$$\lambda \cdot y \le \frac{1}{\alpha}\hat{g}_t(\alpha\lambda;\mathbf{x}) + \frac{1}{\alpha}\hat{h}_t(y;\mathbf{x}),$$

we have for $\mathbf{x} \in \mathbb{C}$ and $\alpha > 0$ by (2.7.16)

$$\int_0^t |\lambda(s,\mathbf{x})\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A) \cdot \dot{M}_s(\mathbf{x})|\, ds$$
$$\le \frac{1}{\alpha}\int_0^t \hat{g}_s\big(\alpha\lambda(s,\mathbf{x})\, \text{sign}\,(\lambda(s,\mathbf{x}) \cdot \dot{M}_s(\mathbf{x}));\mathbf{x}\big)\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A)\, ds$$
$$+\frac{1}{\alpha}\int_0^t \hat{h}_s(\dot{M}_s(\mathbf{x});\mathbf{x})\, ds$$
$$\le \frac{1}{\alpha}\int_0^t [\hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x}) + \hat{g}_s(-\alpha\lambda(s,\mathbf{x});\mathbf{x})]\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A)\, ds$$
$$+\frac{1}{\alpha}\mathbf{I}(\mathbf{x}), \quad (2.7.26)$$

so that by (2.7.19) and the inequality $\mathbf{I}(\mathbf{x}) \le -\ln a$ on $K_\Pi(a)$ (which holds since $\Pi(\mathbf{x}) \le \mathbf{\Pi}(\mathbf{x})$)

$$\limsup_{A\to\infty} \sup_{\mathbf{x}\in K_\Pi(a)} \int_0^t |\lambda(s,\mathbf{x}) \cdot \dot{M}_s(\mathbf{x})|\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A)\, ds \le -\frac{\ln a}{\alpha},\ \alpha > 0.$$

Since α is arbitrary, we have thereby proved that the first term on the right of (2.7.25) tends to 0 as $k \to \infty$ and $A \to \infty$, and the second one tends to 0 as $A \to \infty$.

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For the third term, we write analogously to (2.7.26) for $\alpha > 0$

$$\begin{aligned}\int_0^t &|(\lambda^A(s,\mathbf{x}^k) - \lambda^A(s,\hat{\mathbf{x}})) \cdot \dot{M}_s(\mathbf{x}^k)|\, ds \\ &\le \frac{1}{\alpha}\int_0^t \hat{g}_s(\alpha(\lambda^A(s,\mathbf{x}^k) - \lambda^A(s,\hat{\mathbf{x}}));\mathbf{x}^k)\, ds \\ &\qquad + \frac{1}{\alpha}\int_0^t \hat{g}_s(-\alpha(\lambda^A(s,\mathbf{x}^k) - \lambda^A(s,\hat{\mathbf{x}}));\mathbf{x}^k)\, ds + \frac{1}{\alpha}\mathbf{I}(\mathbf{x}^k).\end{aligned} \tag{2.7.27}$$

Since $\overline{\lambda}$ is strictly Luzin, $\lambda(s,\mathbf{x}^k) \to \lambda(s,\hat{\mathbf{x}})$; hence, by (2.7.23) and (2.7.24) $\lambda^A(s,\mathbf{x}^k) - \lambda^A(s,\hat{\mathbf{x}}) \to 0$. Therefore, by condition (Π_{II}) $\hat{g}_s(\alpha(\lambda^A(s,\mathbf{x}^k) - \lambda^A(s,\hat{\mathbf{x}}));\mathbf{x}^k) \to \hat{g}_s(0;\hat{\mathbf{x}}) = 0$ as $k \to \infty$. Thus, by Lebesgue's dominated convergence theorem and condition (Π_{II}) (recall that by (2.7.23) and (2.7.24) $|\lambda^A(s,\mathbf{x}^k)-\lambda^A(s,\hat{\mathbf{x}})| \le 2(A+1)$), the first two terms on the right of (2.7.27) tend to 0 as $k \to \infty$. Since α is arbitrary, the inequality $\mathbf{I}(\mathbf{x}^k) \le -\ln a$ implies that the third term on the right of (2.7.25) tends to 0 as $k \to \infty$.

Thus, we are left to prove that

$$\lim_{k\to\infty}\int_0^t \lambda^A(s,\hat{\mathbf{x}})\cdot\dot{M}_s(\mathbf{x}^k)\, ds = \int_0^t \lambda^A(s,\hat{\mathbf{x}})\cdot\dot{M}_s(\hat{\mathbf{x}})\, ds. \tag{2.7.28}$$

Let $\tilde{\Lambda}$ be the set of bounded $\mathbb{R}^d$–valued Borel functions $(\lambda(t), t \in \mathbb{R}_+)$ such that

$$\lim_{k\to\infty}\int_0^t \lambda(s)\cdot\dot{M}_s(\mathbf{x}^k)\, ds = \int_0^t \lambda(s)\cdot\dot{M}_s(\hat{\mathbf{x}})\, ds.$$

We prove that $\tilde{\Lambda}$ consists, in fact, of all $\mathbb{R}^d$–valued bounded Borel functions, which will imply (2.7.28) since $\lambda^A(s,\hat{\mathbf{x}})$, being Lebesgue-measurable in s, coincides a.e. with some Borel-measurable function. Since $M_s(\mathbf{x})$ is continuous on $K_\Pi(a)$ by (Π_I) and (Π_{II}), the convergence $\mathbf{x}^k \to \hat{\mathbf{x}}$ implies that $\tilde{\Lambda}$ contains all piecewise constant functions $(\lambda(t))$. Now, by a standard monotone class argument, see, e.g.,

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Meyer [88], it is sufficient to prove that $\tilde{\Lambda}$ is closed under bounded pointwise convergence. We prove this by an argument similar to the one we used above: let $\lambda_n(s) \to \lambda(s)$ as $n \to \infty$, where $|\lambda_n(s)| \leq A$ and $|\lambda(s)| \leq A$. Then, as in (2.7.26), we have with the use of Young's inequality for $\alpha > 0$

$$\int_0^t |(\lambda_n(s) - \lambda(s)) \cdot \dot{M}_s(\mathbf{x})|\, ds$$
$$\leq \frac{1}{\alpha} \int_0^t [\hat{g}_s(\alpha(\lambda_n(s) - \lambda(s)); \mathbf{x}) + \hat{g}_s(-\alpha(\lambda_n(s) - \lambda(s)); \mathbf{x})]\, ds + \frac{1}{\alpha}\mathbf{I}(\mathbf{x}),$$

so that condition (Π_{II}) and Lebesgue's dominated convergence theorem yield

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in K_\Pi(a)} \int_0^t |(\lambda_n(s) - \lambda(s)) \cdot \dot{M}_s(\mathbf{x})|\, ds = 0.$$

Hence,

$$\lim_{n\to\infty} \limsup_{k\to\infty} \int_0^t |(\lambda_n(s) - \lambda(s)) \cdot \dot{M}_s(\mathbf{x}^k)|\, ds = 0,$$
$$\lim_{n\to\infty} \int_0^t |(\lambda_n(s) - \lambda(s)) \cdot \dot{M}_s(\hat{\mathbf{x}})|\, ds = 0,$$

which proves the claim. Thus, (2.7.28) and with it (2.7.22) have been proved. Continuity of $\mathbf{x} \to \int_0^t \lambda(s, \mathbf{x}) \cdot \dot{M}_s(\mathbf{x})\, ds$ on $K_\Pi(a)$ has been proved.

In order to prove that

$$\int_0^t \hat{g}_s(\lambda(s, \mathbf{x}^k); \mathbf{x}^k)\, ds \to \int_0^t \hat{g}_s(\lambda(s, \hat{\mathbf{x}}); \hat{\mathbf{x}})\, ds, \tag{2.7.29}$$

we note that since $\lambda(s, \mathbf{x})$ is continuous in $\mathbf{x}$ on $K_\Pi(a)$ by the definition of $\hat{\Lambda}_\Pi$ and $\hat{g}_s(\lambda; \mathbf{x})$ is continuous in $(\lambda, \mathbf{x})$ by condition (Π_{II}),

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$\hat{g}_s(\lambda(s,\mathbf{x}^k);\mathbf{x}^k) \to \hat{g}_s(\lambda(s,\hat{\mathbf{x}});\hat{\mathbf{x}})$ as $k \to \infty$. Therefore, condition (Π_{II}) and Lebesgue's dominated convergence theorem yield for $A > 0$

$$\lim_{k\to\infty}\int_0^t \hat{g}_s(\lambda(s,\mathbf{x}^k);\mathbf{x}^k)i_A(|\lambda(s,\mathbf{x}^k)|)\,ds = \int_0^t \hat{g}_s(\lambda(s,\hat{\mathbf{x}});\hat{\mathbf{x}})i_A(|\lambda(s,\hat{\mathbf{x}})|)\,ds$$

so that

$$\limsup_{k\to\infty}\int_0^t \hat{g}_s(\lambda(s,\mathbf{x}^k);\mathbf{x}^k)\,\mathbf{1}(|\lambda(s,\mathbf{x}^k)| \le A)\,ds \le \int_0^t \hat{g}_s(\lambda(s,\hat{\mathbf{x}});\hat{\mathbf{x}})\,ds.$$

The latter inequality, (2.7.19) and Fatou's lemma imply the convergence (2.7.29).

To complete the proof of $Z(\overline{\lambda})$ being strictly Luzin-continuous, by Theorem 2.2.13 it is sufficient to show that for $T \in \mathbb{R}_+$ and $\eta > 0$

$$\lim_{\delta\to 0}\sup_{\substack{s,t\in[0,T]:\\|s-t|\le\delta}}\Pi\big(|Z_t(\overline{\lambda}) - Z_s(\overline{\lambda})| > \eta\big) = 0.$$

Since for $A > 0$

$$\Pi\big(|Z_t(\overline{\lambda}) - Z_s(\overline{\lambda})| > \eta\big) \le \Pi(Z_s(\overline{\lambda}) > A) \vee \Pi\big(|Z_t(\overline{\lambda})/Z_s(\overline{\lambda}) - 1| > \eta/A\big)$$

and $\Pi(Z_s(\overline{\lambda}) > A) \le 1/A$ by (2.7.21), we deduce that the required would follow by

$$\lim_{\delta\to 0}\Pi\Big(\sup_{\substack{s,t\in[0,T]:\\|s-t|\le\delta}}\Big|\int_s^t \lambda(u,\mathbf{x})\cdot\dot{M}_u(\mathbf{x})\,du\Big| > \eta\Big) = 0,$$

$$\lim_{\delta\to 0}\Pi\Big(\sup_{\substack{s,t\in[0,T]:\\|s-t|\le\delta}}\Big|\int_s^t \hat{g}_u(\lambda(u,\mathbf{x});\mathbf{x})\,du\Big| > \eta\Big) = 0.$$

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The first convergence follows by the inequality

$$\int_s^t |\lambda(u,\mathbf{x})\cdot \dot{M}_u(\mathbf{x})|\,du$$

$$\le \frac{1}{\alpha}\int_0^T [\hat{g}_u(\alpha\lambda(u,\mathbf{x});\mathbf{x})+\hat{g}_u(-\alpha\lambda(u,\mathbf{x});\mathbf{x})]\,\mathbf{1}(|\lambda(u,\mathbf{x})|>A)\,du$$
$$+\frac{1}{\alpha}\int_s^t \sup_{|\lambda|\le\alpha A}\hat{g}_u(\lambda;\mathbf{x})\,du+\frac{1}{\alpha}\mathbf{I}(\mathbf{x}),\quad A>0,\ \alpha>0,$$

derived in analogy with (2.7.26) and condition (Π_{II}). The second convergence follows by condition (Π_{II}) and the inequality

$$\int_s^t \hat{g}_u(\lambda(u,\mathbf{x});\mathbf{x})\,du \le \int_0^T \hat{g}_u(\lambda(u,\mathbf{x});\mathbf{x})\,\mathbf{1}(|\lambda(u,\mathbf{x})|>A)\,du$$
$$+\int_s^t \sup_{|\lambda|\le A}\hat{g}_u(\lambda;\mathbf{x})\,du.$$

Finally, since $\overline{\lambda}$ and $(b_s(\mathbf{x}))$ are $\mathbf{C}^\Pi$-progressively measurable, $\hat{g}_s(\lambda;\mathbf{x})$ is $\overline{\mathcal{B}}([0,t])\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{C}_t^\Pi/\mathcal{B}(\mathbb{R}_+)$–measurable as a map from $[0,t]\times\mathbb{R}^d\times\mathbb{C}$ to $\mathbb{R}_+$, $\dot{M}$ is $\mathbf{C}^\Pi$-progressively measurable, and $\mathbf{C}^\Pi$ is complete, $\mathcal{C}^\Pi_{t\wedge\tau}$–measurability of $Z_{t\wedge\tau}(\overline{\lambda})$ follows by Lemma 2.2.17 and Lemma 2.2.19. The fact that $Z_{t\wedge\tau}(\overline{\lambda})$ is strictly Luzin measurable follows by the first part of the lemma. ☐

We now address the uniform maximability issue. The next lemma is in the theme of Lemmas 2.7.5 and 2.7.7.

Lemma 2.7.18. *Let* $\overline{\lambda}\in\hat{\Lambda}_\Pi$ *and* $(\overline{G}_t,\ t\in\mathbb{R}_+)$, $\overline{G}_0=0$, *be an increasing continuous* $\mathbf{C}^\Pi$*–adapted strictly Luzin idempotent process such that for* Π*-almost all* $\mathbf{x}$

$$\int_0^t \hat{g}_s(2\lambda(s,\mathbf{x});\mathbf{x})\,ds\le\overline{G}_t(\mathbf{x}),\ t\in\mathbb{R}_+.$$

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Let for $N \in \mathbb{N}$

$$\tau_N(\mathbf{x}) = \inf\{t \in \mathbb{R}_+ : \overline{G}_t(\mathbf{x}) + t \geq N\}.$$

Then $\tau_N(\mathbf{x})$ *is a strictly Luzin stopping time, the idempotent process* $\{Z_{t\wedge\tau_N}(\overline{\lambda}), t \in \mathbb{R}_+\}$ *is uniformly maximable, and, moreover,* $S_\Pi\big(Z_{t\wedge\tau_N}(\overline{\lambda})^2\big) \leq e^N$.

Lemma 2.7.19. *Let a function* $\overline{\lambda} = (\lambda(t, \mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}) \in \hat{\Lambda}_\Pi$ *be of the form:*

$$\lambda(t, \mathbf{x}) = \sum_{i=1}^{k} \lambda_i(\mathbf{x}) 1_{(t_{i-1}, t_i]}(t),$$

where $k \in \mathbb{N}, 0 \leq t_0 < t_1 < \ldots < t_k$, *and the* $\lambda_i(\mathbf{x})$ *are* $\mathbb{R}^d$*–valued, bounded and* $\mathcal{C}^\Pi_{t_{i-1}}$*–measurable strictly Luzin variables on* $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$. *Then* $Z(\overline{\lambda})$ *is a strictly Luzin-continuous local exponential maxingale on* $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$.

Proof. Since by Lemma 2.7.17 $Z(\overline{\lambda})$ is a $\mathbb{C}^\Pi$–adapted strictly Luzin-continuous idempotent process, we have to check that there exists an increasing to infinity sequence of strictly Luzin stopping times $\overline{\sigma}_N(\mathbf{x}), N \in \mathbb{N}$, such that the $(Z_{t\wedge\overline{\sigma}_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}), \mathbf{x}),\, t \in \mathbb{R}_+)$ are uniformly maximable exponential maxingales.

Let A be a bound for $\lambda_i(\mathbf{x})$, i.e., $|\lambda_i(\mathbf{x})| \leq A$, $i = 1, \ldots, k$, $\mathbf{x} \in \mathbb{C}$. We introduce for $t \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{C}$ and $N \in \mathbb{N}$

$$\bar{\sigma}_N(\mathbf{x}) = \begin{cases} \inf\Big\{t \in \mathbb{R}_+ : \displaystyle\int_0^t \sup_{|\lambda| \leq 2A} \hat{g}_s(\lambda; \mathbf{x})\, ds + t \geq N\Big\}, & \text{if } \Pi(\mathbf{x}) > 0, \\ N, & \text{if } \Pi(\mathbf{x}) = 0. \end{cases}$$

By condition (Π_{II}) and Lemma 2.7.18 the $\bar{\sigma}_N(\mathbf{x})$ are strictly Luzin stopping times and the idempotent processes $(Z_{t\wedge\bar{\sigma}_N}(\overline{\lambda}), t \in \mathbb{R}_+)$ are uniformly maximable. Let us check that the $(Z_{t\wedge\bar{\sigma}_N}(\overline{\lambda}), t \in \mathbb{R}_+)$, $N \in \mathbb{N}$, are exponential maxingales. Let $0 \leq s < t$. We have to prove that

$$S_\Pi(Z_{t\wedge\bar{\sigma}_N}(\overline{\lambda})|\mathcal{C}^\Pi_s) = Z_{s\wedge\bar{\sigma}_N}(\overline{\lambda}). \tag{2.7.31}$$

(As above relations involving conditional idempotent expectations are understood to hold Π-a.e.) We begin with a proof of

$$S_\Pi(Z_{t\wedge\bar{\sigma}_N}(\overline{\lambda})|\mathcal{C}^\Pi_{t_i\wedge t}) = Z_{t_i\wedge t\wedge\bar{\sigma}_N}(\overline{\lambda}),\ i = 1, \ldots, k. \tag{2.7.32}$$

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We note that by (2.7.1) and (2.7.20) Π-a.e.

$$Z_t(\overline{\lambda}(\mathbf{x}), \mathbf{x}) = \prod_{i=1}^{k} \frac{Y_{t_i \wedge t}(\lambda_i(\mathbf{x}), \mathbf{x})}{Y_{t_{i-1} \wedge t}(\lambda_i(\mathbf{x}), \mathbf{x})}. \tag{2.7.33}$$

Since the idempotent processes $Y(\lambda) = (Y_t(\lambda; \mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$, $\lambda \in \mathbb{R}^d$, are local exponential maxingales, Lemma 2.7.18 and Lemma 2.3.13 imply that the $(Y_{t \wedge \bar{\sigma}_N}(\lambda), t \in \mathbb{R}_+)$, $|\lambda| \leq A$, are uniformly maximable exponential maxingales.

We prove (2.7.32) by inverse induction in i: by (2.7.33) $Z_{t_k \wedge t \wedge \bar{\sigma}_N}(\overline{\lambda}) = Z_{t \wedge \bar{\sigma}_N}(\overline{\lambda})$ so that (2.7.32) holds for $i = k$. Suppose that (2.7.32) holds for some $i \in \{2, \ldots, k\}$. We prove it for $(i-1)$. By properties of conditional idempotent expectations

$$\begin{aligned} S_\Pi(Z_{t \wedge \bar{\sigma}_N}(\overline{\lambda}) | \mathcal{C}^\Pi_{t_{i-1} \wedge t}) &= S_\Pi[S_\Pi(Z_{t \wedge \bar{\sigma}_N}(\overline{\lambda}) | \mathcal{C}^\Pi_{t_i \wedge t}) | \mathcal{C}^\Pi_{t_{i-1} \wedge t}] \\ &= S_\Pi(Z_{t_i \wedge t \wedge \bar{\sigma}_N}(\overline{\lambda}) | \mathcal{C}^\Pi_{t_{i-1} \wedge t}). \end{aligned}$$

By (2.7.33) and properties of conditional idempotent expectations

$$\begin{aligned} S_\Pi(Z_{t_i \wedge t \wedge \bar{\sigma}_N}(\overline{\lambda}) | \mathcal{C}^\Pi_{t_{i-1} \wedge t}) = \prod_{j=1}^{i-1} & \frac{Y_{t_j \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_j(\mathbf{x}), \mathbf{x})}{Y_{t_{j-1} \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_j(\mathbf{x}), \mathbf{x})} \\ & \frac{S_\Pi(Y_{t_i \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_i(\mathbf{x}), \mathbf{x}) | \mathcal{C}^\Pi_{t_{i-1} \wedge t})}{Y_{t_{i-1} \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_i(\mathbf{x}), \mathbf{x})}. \end{aligned} \tag{2.7.34}$$

Let $t \geq t_{i-1}$. Since $\lambda_i(\mathbf{x})$ is $\mathcal{C}^\Pi_{t_{i-1}}$–measurable, by properties of conditional idempotent expectations

$$\begin{aligned} & S_\Pi(Y_{t_i \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_i(\mathbf{x}), \mathbf{x}) | \mathcal{C}^\Pi_{t_{i-1} \wedge t}) \\ &= S_\Pi(Y_{t_i \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_i(\mathbf{x}), \mathbf{x}) | \mathcal{C}^\Pi_{t_{i-1}}) \\ &= S_\Pi(Y_{t_i \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda, \mathbf{x}) | \mathcal{C}^\Pi_{t_{i-1}})|_{\lambda = \lambda_i(\mathbf{x})} \\ &\qquad = Y_{t_{i-1} \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_i(\mathbf{x}), \mathbf{x}), \end{aligned}$$

where in the latter equality we used that $(Y_{t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda, \mathbf{x}), t \in \mathbb{R}_+)$ is an exponential maxingale. So, if $t \geq t_{i-1}$,

$$S_\Pi(Y_{t_i \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_i(\mathbf{x}), \mathbf{x}) | \mathcal{C}^\Pi_{t_{i-1} \wedge t}) = Y_{t_{i-1} \wedge t \wedge \bar{\sigma}_N(\mathbf{x})}(\lambda_i(\mathbf{x}), \mathbf{x}).$$

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This also is true if $t \le t_{i-1}$. Substituting the equality into (2.7.34) obtains

$$S_\Pi(Z_{t_i\wedge t\wedge\bar\sigma_N}(\overline{\lambda})|\mathcal{C}^\Pi_{t_{i-1}\wedge t}) = \prod_{j=1}^{i-1} \frac{Y_{t_j\wedge t\wedge\bar\sigma_N(\mathbf{x})}(\lambda_j(\mathbf{x}),\mathbf{x})}{Y_{t_{j-1}\wedge t\wedge\bar\sigma_N(\mathbf{x})}(\lambda_j(\mathbf{x}),\mathbf{x})} = Z_{t_{i-1}\wedge t\wedge\bar\sigma_N}(\overline{\lambda}).$$

Equality (2.7.32) is proved.

Now, (2.7.31) is obvious if $s \ge t_k$ since in this case $Z_{s\wedge\bar\sigma_N}(\overline{\lambda}) = Z_{t\wedge\bar\sigma_N}(\overline{\lambda}) = Z_{t_k\wedge\bar\sigma_N}(\overline{\lambda})$. Let $s < t_k$ and $i_0 \in \{1,\dots,k\}$ be such that $t_{i_0-1} \le s < t_{i_0}$. By properties of conditional idempotent expectations, (2.7.32) and (2.7.33)

$$\begin{aligned} S_\Pi(Z_{t\wedge\bar\sigma_N}(\overline{\lambda})|\mathcal{C}^\Pi_s) &= S_\Pi[S_\Pi(Z_{t\wedge\bar\sigma_N}(\overline{\lambda})|\mathcal{C}^\Pi_{t_{i_0}\wedge t})|\mathcal{C}^\Pi_s] \\ &= S_\Pi(Z_{t_{i_0}\wedge t\wedge\bar\sigma_N}(\overline{\lambda})|\mathcal{C}^\Pi_s) \\ &= Z_{t_{i_0-1}\wedge s\wedge\bar\sigma_N}(\overline{\lambda})\,\frac{S_\Pi(Y_{t_{i_0}\wedge t\wedge\bar\sigma_N(\mathbf{x})}(\lambda_{i_0}(\mathbf{x}),\mathbf{x})|\mathcal{C}^\Pi_s)}{Y_{t_{i_0-1}\wedge s\wedge\bar\sigma_N(\mathbf{x})}(\lambda_{i_0}(\mathbf{x}),\mathbf{x})}. \end{aligned} \quad (2.7.35)$$

Now, as above, since $\lambda_{i_0}(\mathbf{x})$ is $\mathcal{C}^\Pi_{t_{i_0-1}}$–measurable,

$$\begin{aligned} &S_\Pi(Y_{t_{i_0}\wedge t\wedge\bar\sigma_N(\mathbf{x})}(\lambda_{i_0}(\mathbf{x}),\mathbf{x})|\mathcal{C}^\Pi_s) \\ &= S_\Pi(Y_{t_{i_0}\wedge t\wedge\bar\sigma_N(\mathbf{x})}(\lambda,\mathbf{x})|\mathcal{C}^\Pi_s)|_{\lambda=\lambda_{i_0}(\mathbf{x})} \\ &\qquad = Y_{t_{i_0}\wedge s\wedge\bar\sigma_N(\mathbf{x})}(\lambda_{i_0}(\mathbf{x}),\mathbf{x}). \end{aligned}$$

Substituting this into (2.7.35), we deduce by (2.7.33) that (2.7.31) holds. ☐

Lemma 2.7.19 proves the assertion of Theorem 2.7.16 for piecewise constant and bounded functions $\overline{\lambda}$. To handle general $\overline{\lambda} \in \hat\Lambda_\Pi$, we will use the following approximation result, which extends Lemma 2.5.10.

Lemma 2.7.20. *Let* $\overline{\lambda}^k = (\lambda^k(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}), k \in \mathbb{N}$, *be uniformly bounded functions from* $\hat\Lambda_\Pi$ *such that the* $Z(\overline{\lambda}^k)$ *are strictly Luzin-continuous local exponential maxingales. If* $\overline{\lambda} = (\lambda(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}) \in \hat\Lambda_\Pi$ *is bounded and is a limit of the* $\overline{\lambda}^k$ *in the sense that*

$$\int_0^t \hat g_s(\alpha(\lambda^k(s,\mathbf{x})-\lambda(s,\mathbf{x}));\mathbf{x})ds \xrightarrow{\Pi} 0 \quad \text{as } k\to\infty, \quad \alpha\in\mathbb{R}, t\in\mathbb{R}_+,$$

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then $Z(\overline{\lambda})$ is a strictly Luzin-continuous local exponential maxingale.

Proof. The proof uses the ideas of the proof of Lemma 2.5.10. Let for $N \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\sigma_N(\mathbf{x}) = \inf\bigl\{t \in \mathbb{R}_+ : \int_0^t \hat{g}_s(2\lambda(s,\mathbf{x});\mathbf{x})ds + t \geq N\bigr\}, \tag{2.7.36}$$

$$\sigma_N^k(\mathbf{x}) = \inf\bigl\{t \in \mathbb{R}_+ : \int_0^t \hat{g}_s(2\lambda^k(s,\mathbf{x});\mathbf{x})\,ds + t \geq N+1\bigr\} \wedge \sigma_N(\mathbf{x}). \tag{2.7.37}$$

By Lemma 2.7.18 and condition (Π_{II}) the σ_N and σ_N^k are strictly Luzin stopping times, and $(Z_{t\wedge\sigma_N}(\overline{\lambda}), t \in \mathbb{R}_+)$ and $(Z_{t\wedge\sigma_N^k}(\overline{\lambda}^k), t \in \mathbb{R}_+), k \in \mathbb{N}$, are uniformly maximable idempotent processes. In particular, by Lemma 2.3.13 the $(Z_{t\wedge\sigma_N^k}(\overline{\lambda}^k), t \in \mathbb{R}_+)$, $k \in \mathbb{N}$, are uniformly maximable exponential maxingales. Suppose we have proved that for every $\mathbb{R}_+$-valued bounded and continuous function f on $\mathbb{C}$ and $t \in \mathbb{R}_+$

$$\lim_{k\to\infty} S_\Pi Z_{t\wedge\sigma_N^k(\mathbf{x})}(\overline{\lambda}^k(\mathbf{x}),\mathbf{x})f(\mathbf{x}) = S_\Pi Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}),\mathbf{x})f(\mathbf{x}). \tag{2.7.38}$$

Since the $(Z_{t\wedge\sigma_N^k}(\overline{\lambda}^k), t \in \mathbb{R}_+), k \in \mathbb{N}$, are exponential maxingales, this would imply that $(Z_{t\wedge\sigma_N}(\overline{\lambda}),\ t \in \mathbb{R}_+)$ is an exponential maxingale as well; hence, since $(Z_{t\wedge\sigma_N}(\overline{\lambda}), t \in \mathbb{R}_+)$ is uniformly maximable and $\sigma_N(\mathbf{x}) \to \infty$ as $N \to \infty$, this would prove in view of Lemma 2.7.17 that $Z(\overline{\lambda})$ is a strictly Luzin-continuous local exponential maxingale. Therefore, we prove next (2.7.38).

By Lemma 2.7.18 and (2.7.37) $S_\Pi\bigl(Z_{t\wedge\sigma_N^k(\mathbf{x})}(\overline{\lambda}^k(\mathbf{x}),\mathbf{x})^2\bigr) \leq \exp(N+1)$, so the family $\{Z_{t\wedge\sigma_N^k}(\overline{\lambda}^k),\ k \in \mathbb{N}\}$ is uniformly maximable, and by "the Lebesgue dominated convergence theorem" (see Theorem 1.4.19) (2.7.38) would follow by

$$Z_{t\wedge\sigma_N^k}(\overline{\lambda}^k) \overset{\Pi}{\to} Z_{t\wedge\sigma_N}(\overline{\lambda}) \quad \text{as } k \to \infty. \tag{2.7.39}$$

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As a first step, we prove that, for every $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}_+$,

$$\int_0^t |\hat{g}_s(\alpha\lambda^k(s,\mathbf{x});\mathbf{x}) - \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})|ds \xrightarrow{\Pi} 0 \quad \text{as } k \to \infty. \tag{2.7.40}$$

By convexity of $\hat{g}_s(\lambda;\mathbf{x})$ in λ, for $\varepsilon \in (0, 1/2]$,

$$\hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x}) \le (1-2\varepsilon)\hat{g}_s(\alpha\lambda^k(s,\mathbf{x});\mathbf{x}) + \varepsilon\hat{g}_s(2\alpha\lambda^k(s,\mathbf{x});\mathbf{x}) + \varepsilon\hat{g}_s\big(\frac{\alpha}{\varepsilon}(\lambda(s,\mathbf{x}) - \lambda^k(s,\mathbf{x}));\mathbf{x}\big),$$

hence, since $\hat{g}_s(\lambda;\mathbf{x}) \ge 0$,

$$\hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x}) - \hat{g}_s(\alpha\lambda^k(s,\mathbf{x});\mathbf{x}) \le \varepsilon \sup_{|\lambda|\le 2A|\alpha|} \hat{g}_s(\lambda;\mathbf{x}) + \varepsilon\hat{g}_s\big(\frac{\alpha}{\varepsilon}(\lambda(s,\mathbf{x}) - \lambda^k(s,\mathbf{x}));\mathbf{x}\big),$$

where A is an upper bound for $|\lambda(s,\mathbf{x})|, |\lambda^k(s,\mathbf{x})|, k \in \mathbb{N}, s \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{C}$. Interchanging $\lambda^k(s,\mathbf{x})$ and $\lambda(s,\mathbf{x})$ and integrating, we arrive at the inequality for $\mathbf{x}$ such that $\Pi(\mathbf{x}) > 0$

$$\int_0^t |\hat{g}_s(\alpha\lambda^k(s,\mathbf{x});\mathbf{x}) - \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})|ds$$

$$\le \varepsilon \int_0^t \sup_{|\lambda|\le 2A|\alpha|} \hat{g}_s(\lambda;\mathbf{x})\,ds + \varepsilon \int_0^t \hat{g}_s\big(\frac{\alpha}{\varepsilon}(\lambda(s,\mathbf{x}) - \lambda^k(s,\mathbf{x}));\mathbf{x}\big)ds + \varepsilon \int_0^t \hat{g}_s\big(\frac{\alpha}{\varepsilon}(\lambda^k(s,\mathbf{x}) - \lambda(s,\mathbf{x}));\mathbf{x}\big)ds,$$

where the right-hand side is finite by condition (Π_{II}). By hypotheses, the latter two integrals on the right tend in deviability to 0 as $k \to \infty$, so, for $\eta > 0$,

$$\limsup_{k\to\infty} \Pi(\mathbf{x} : \int_0^t |\hat{g}_s(\alpha\lambda^k(s,\mathbf{x});\mathbf{x}) - \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})|ds > \eta) \le \Pi(\mathbf{x} : \varepsilon \int_0^t \sup_{|\lambda|\le 2A|\alpha|} \hat{g}_s(\lambda;\mathbf{x})ds > \eta/3),$$

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where the right-hand side is not greater than arbitrary $a > 0$ if $\varepsilon > 0$ is chosen small enough to satisfy

$$\varepsilon \sup_{\mathbf{x} \in K_\Pi(a)} \int_0^t \sup_{|\lambda| \le 2A|\alpha|} \hat{g}_s(\lambda; \mathbf{x})\, ds \le \frac{\eta}{3}$$

(use condition (Π_{II})). Limit (2.7.40) is proved. It implies, since by (2.7.36) and (2.7.37)

$$\{\mathbf{x} : \sigma_N(\mathbf{x}) \wedge t \ne \sigma_N^k(\mathbf{x}) \wedge t\} \subset \{\mathbf{x} : \int_0^t |\hat{g}_s(2\lambda^k(s,\mathbf{x}); \mathbf{x}) - \hat{g}_s(2\lambda(s,\mathbf{x}); \mathbf{x})|\, ds > 1\},$$

that $\lim_{k \to \infty} \Pi(\sigma_N(\mathbf{x}) \wedge t \ne \sigma_N^k(\mathbf{x}) \wedge t) = 0$. Hence, by the inequality

$$\begin{aligned}\Pi(|Z_{t \wedge \sigma_N^k(\mathbf{x})}(\overline{\lambda}^k(\mathbf{x}), \mathbf{x}) - Z_{t \wedge \sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}), \mathbf{x})| > \eta) \\ \le \Pi(\sigma_N(\mathbf{x}) \wedge t \ne \sigma_N^k(\mathbf{x}) \wedge t) \\ + \Pi(|Z_{t \wedge \sigma_N(\mathbf{x})}(\overline{\lambda}^k(\mathbf{x}), \mathbf{x}) - Z_{t \wedge \sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}), \mathbf{x})| > \eta),\end{aligned}$$

(2.7.20), and the fact that $Z_{t \wedge \sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}), \mathbf{x})$ is a proper idempotent variable, limit (2.7.39) would follow by

$$\int_0^{t \wedge \sigma_N(\mathbf{x})} \lambda^k(s, \mathbf{x}) \cdot \dot{M}_s(\mathbf{x}) ds \xrightarrow{\Pi} \int_0^{t \wedge \sigma_N(\mathbf{x})} \lambda(s, \mathbf{x}) \cdot \dot{M}_s(\mathbf{x})\, ds \quad \text{as } k \to \infty, \tag{2.7.41}$$

and

$$\int_0^{t \wedge \sigma_N(\mathbf{x})} \hat{g}_s(\lambda^k(s, \mathbf{x}); \mathbf{x}) ds \xrightarrow{\Pi} \int_0^{t \wedge \sigma_N(\mathbf{x})} \hat{g}_s(\lambda(s, \mathbf{x}); \mathbf{x})\, ds \quad \text{as } k \to \infty.$$

The latter limit obviously follows by (2.7.40). Limit (2.7.41) is proved with a tool we have already used: for $\alpha > 0$, by Young's inequality,

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in view of (2.7.17) and (2.7.16),

$$\Big| \int_0^{t\wedge\sigma_N(\mathbf{x})} \lambda^k(s,\mathbf{x})\cdot \dot{M}_s(\mathbf{x})\,ds - \int_0^{t\wedge\sigma_N(\mathbf{x})} \lambda(s,\mathbf{x})\cdot \dot{M}_s(\mathbf{x})\,ds\Big|$$
$$\leq \frac{1}{\alpha}\int_0^t \big(\hat{g}_s(\alpha(\lambda^k(s,\mathbf{x})-\lambda(s,\mathbf{x}));\mathbf{x})$$
$$+ \hat{g}_s(\alpha(\lambda(s,\mathbf{x})-\lambda^k(s,\mathbf{x}));\mathbf{x})\big)\,ds + \frac{1}{\alpha}\mathbf{I}(\mathbf{x}),$$

which implies (2.7.41) in view of the hypotheses. □

The next lemma and its proof are prompted by Theorem 2.5.11.

Lemma 2.7.21. *Let $\overline{\lambda} \in \hat{\Lambda}_\Pi$ and be bounded. Then $Z(\overline{\lambda})$ is a strictly Luzin-continuous local exponential maxingale.*

Proof. Let us first assume that $\overline{\lambda}$ is, in addition, locally uniformly continuous in t uniformly in $\mathbf{x} \in K_\Pi(a)$ for all $a \in (0,1]$, i.e., for all $T > 0$

$$w_{T,a}(\delta) = \sup_{\mathbf{x}\in K_\Pi(a)} \sup_{\substack{s,t\in[0,T]:\\ |s-t|\leq\delta}} |\lambda(t,\mathbf{x})-\lambda(s,\mathbf{x})| \to 0 \quad \text{as } \delta \to 0. \tag{2.7.42}$$

Let for $k \in \mathbb{N}$

$$\lambda^k(t,\mathbf{x}) = \sum_{i=1}^{k^2} \lambda\Big(\frac{i-1}{k},\mathbf{x}\Big)\,\mathbf{1}_{(\frac{i-1}{k},\frac{i}{k}]}(t).$$

Then the $\overline{\lambda}^k = (\lambda^k(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ are bounded and piecewise constant functions from $\hat{\Lambda}_\Pi$, which implies by Lemma 2.7.19 that the $Z(\overline{\lambda}^k), k \in \mathbb{N}$, are strictly Luzin-continuous local exponential maxingales. Also, since $|\lambda^k(s,\mathbf{x}) - \lambda(s,\mathbf{x})| \leq w_{t,a}(1/k)$ for $s \in [0,t]$ and $\mathbf{x} \in K_\Pi(a)$ if $k \geq t$,

$$\sup_{\mathbf{x}\in K_\Pi(a)} \int_0^t \hat{g}_s(\alpha(\lambda^k(s,\mathbf{x})-\lambda(s,\mathbf{x}));\mathbf{x})\,ds$$
$$\leq \sup_{\mathbf{x}\in K_\Pi(a)} \int_0^t \sup_{|\lambda|\leq\alpha w_{t,a}(1/k)} \hat{g}_s(\lambda;\mathbf{x})\,ds.$$

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Since the right-hand side converges to 0 as $k \to \infty$ by (2.7.42) and condition (Π_{II}), we conclude that the $\overline{\lambda}^k$ and $\overline{\lambda}$ meet the conditions of Lemma 2.7.20; hence, $Z(\overline{\lambda})$ is a strictly-Luzin local exponential maxingale.

Let $\overline{\lambda}$ be an arbitrary bounded function from $\hat{\Lambda}_\Pi$. We introduce the Steklov functions

$$\lambda^k(t,\mathbf{x}) = k \int_{t-1/k}^{t} \lambda(s,\mathbf{x})ds, \; t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}, k \in \mathbb{N}, \tag{2.7.43}$$

where $\lambda(s,\mathbf{x}) = 0$ if $s \le 0$. Then the functions $\overline{\lambda}^k = (\lambda^k(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}) \in \hat{\Lambda}_\Pi$ are bounded and $|\lambda^k(t,\mathbf{x}) - \lambda^k(s,\mathbf{x})| \le 2k \sup_{v,\mathbf{x}} |\lambda(v,\mathbf{x})||t-s|$. Hence, by the part just proved the $Z(\overline{\lambda}^k)$ are strictly Luzin-continuous local exponential maxingales. By Lemma 2.7.20 $Z(\overline{\lambda})$ is a strictly Luzin-continuous local exponential maxingale provided

$$\int_0^t \hat{g}_s(\alpha(\lambda^k(s,\mathbf{x}) - \lambda(s,\mathbf{x}));\mathbf{x})ds \xrightarrow{\Pi} 0 \quad \text{as } k \to \infty,$$
$$\alpha \in \mathbb{R}, t \in \mathbb{R}_+. \tag{2.7.44}$$

By convexity of $\hat{g}_s(\lambda;\mathbf{x})$ in λ and (2.7.43)

$$\begin{aligned}
&\int_0^t \hat{g}_s(\alpha(\lambda^k(s,\mathbf{x}) - \lambda(s,\mathbf{x}));\mathbf{x})ds \\
&= \int_0^t \hat{g}_s\Big(\alpha k \int_0^{1/k} (\lambda(s-v,\mathbf{x}) - \lambda(s,\mathbf{x}))dv;\mathbf{x}\Big)ds \\
&\le \int_0^t k \int_0^{1/k} \hat{g}_s(\alpha(\lambda(s-v,\mathbf{x}) - \lambda(s,\mathbf{x}));\mathbf{x})\,dv\,ds \\
&\qquad\qquad \le \sup_{0\le v\le 1/k} \int_0^t \hat{g}_s(\alpha(\lambda(s-v,\mathbf{x}) - \lambda(s,\mathbf{x}));\mathbf{x})\,ds,
\end{aligned}$$

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so we prove (2.7.44) by proving that

$$\int_0^t \hat{g}_s(\alpha(\lambda(s-v,\mathbf{x})-\lambda(s,\mathbf{x}));\mathbf{x})ds \overset{\Pi}{\to} 0 \quad \text{as } v \to 0. \tag{2.7.45}$$

Let $a \in (0,1]$. Firstly, we prove that for every $\hat{\mathbf{x}} \in \mathbb{C}$

$$\lim_{v\to 0} \int_0^t \sup_{\mathbf{x}\in K_\Pi(a)} \hat{g}_s(\alpha(\lambda(s-v,\hat{\mathbf{x}})-\lambda(s,\hat{\mathbf{x}}));\mathbf{x})ds = 0. \tag{2.7.46}$$

The argument is standard. If $(\lambda(s,\hat{\mathbf{x}}),\, s \in \mathbb{R}_+)$ is continuous in s, then $\lambda(s-v,\hat{\mathbf{x}}) - \lambda(s,\hat{\mathbf{x}}) \to 0$ as $v \to 0$ and the required follows by condition (Π_{II}) and boundedness of $(\lambda(s,\hat{\mathbf{x}}),\, s \in \mathbb{R}_+)$. If $(\lambda(s,\hat{\mathbf{x}}),\, s \in \mathbb{R}_+)$ is an arbitrary bounded Lebesgue-measurable function, then, given arbitrary $\varepsilon > 0$, we can choose by Luzin's theorem a bounded continuous function $(\hat{\lambda}(s),\, s \in \mathbb{R}_+)$ such that $\int_0^t \mathbf{1}(\hat{\lambda}(s) \neq \lambda(s,\hat{\mathbf{x}}))\, ds < \varepsilon$, and (2.7.46) then holds for $(\lambda(s,\hat{\mathbf{x}}),\, s \in \mathbb{R}_+)$ since it holds for $(\hat{\lambda}(s),\, s \in \mathbb{R}_+)$ and $(\lambda(s,\hat{\mathbf{x}}),\, s \in \mathbb{R}_+)$ is bounded. Limit (2.7.46) is proved.

We denote

$$\overline{g}_{t,a}(\lambda) = \sup_{s\le t} \sup_{\mathbf{x}\in K_\Pi(a)} \hat{g}_s(\lambda;\mathbf{x}).$$

By continuity of $\lambda(s,\mathbf{y})$ in $\mathbf{y}$ on $K_\Pi(a)$, boundedness of $\lambda(s,\mathbf{y})$ and condition (Π_{II}) we have that

$$\int_0^t \overline{g}_{t,a}(3\alpha(\lambda(s,\mathbf{x}) - \lambda(s,\mathbf{y})))\, ds + \int_0^t \overline{g}_{t,a}(-3\alpha(\lambda(s,\mathbf{x}) - \lambda(s,\mathbf{y})))\, ds \to 0$$

as $\mathbf{y} \to \mathbf{x}, \mathbf{y} \in K_\Pi(a)$. Hence, for $\mathbf{x} \in K_\Pi(a)$ and $\varepsilon > 0$, there exists

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an open subset $U_\varepsilon(\mathbf{x})$ of $K_\Pi(a)$ such that

$$U_\varepsilon(\mathbf{x}) \subset \{\mathbf{y} \in K_\Pi(a) : \int_0^t \overline{g}_{t,a}(3\alpha(\lambda(s,\mathbf{x}) - \lambda(s,\mathbf{y})))\, ds + \int_0^t \overline{g}_{t,a}(-3\alpha(\lambda(s,\mathbf{x}) - \lambda(s,\mathbf{y})))\, ds < \varepsilon\}.$$

By compactness of $K_\Pi(a)$, there exist $\mathbf{x}_1, \dots, \mathbf{x}_k \in K_\Pi(a)$ such that $K_\Pi(a) \subset \cup_{i=1}^k U_\varepsilon(\mathbf{x}_i)$, which means that for every $\mathbf{x} \in K_\Pi(a)$ there exists $i \in \{1, \dots, k\}$ such that

$$\int_0^t \overline{g}_{t,a}(3\alpha(\lambda(s,\mathbf{x}_i) - \lambda(s,\mathbf{x})))\, ds + \int_0^t \overline{g}_{t,a}(-3\alpha(\lambda(s,\mathbf{x}_i) - \lambda(s,\mathbf{x})))\, ds < \varepsilon. \quad (2.7.47)$$

Next, by convexity of $\hat{g}_s(\lambda;\mathbf{x})$ in λ, for $\mathbf{x} \in K_\Pi(a)$,

$$\begin{aligned}
&\int_0^t \hat{g}_s(\alpha(\lambda(s-v,\mathbf{x}) - \lambda(s,\mathbf{x}));\mathbf{x})\, ds \\
&\le \frac{1}{3}\Big(\int_0^t \overline{g}_{t,a}(3\alpha(\lambda(s-v,\mathbf{x}) - \lambda(s-v,\mathbf{x}_i)))\, ds \\
&+ \int_0^t \overline{g}_{t,a}(3\alpha(\lambda(s,\mathbf{x}_i) - \lambda(s,\mathbf{x})))\, ds \\
&\qquad + \int_0^t \hat{g}_s(3\alpha(\lambda(s-v,\mathbf{x}_i) - \lambda(s,\mathbf{x}_i));\mathbf{x})\, ds\Big), \quad (2.7.48)
\end{aligned}$$

where i is chosen so that (2.7.47) holds. By (2.7.46), if v is small enough, then for all $i = 1, \dots, k$

$$\int_0^t \sup_{\mathbf{x} \in K_\Pi(a)} \hat{g}_s(3\alpha(\lambda(s-v,\mathbf{x}_i) - \lambda(s,\mathbf{x}_i));\mathbf{x})\, ds < \varepsilon,$$

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so, by (2.7.47) and (2.7.48)

$$\sup_{\mathbf{x}\in K_\Pi(a)} \int_0^t \hat{g}_s(\alpha(\lambda(s-v,\mathbf{x})-\lambda(s,\mathbf{x}));\mathbf{x})\,ds < \varepsilon.$$

Limit (2.7.45) has been proved. Limit (2.7.44) has been proved. □

Proof of Theorem 2.7.16. Let $\overline{\lambda} \in \hat{\Lambda}_\Pi$ and $\overline{\lambda}_A = (r_A\lambda(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$, where for $\lambda \in \mathbb{R}^d$ and $A > 0$

$$r_A\lambda = \begin{cases} \lambda, & |\lambda| \le A, \\ \dfrac{\lambda}{|\lambda|}A, & |\lambda| > A. \end{cases}$$

Since the map $\lambda \to r_A\lambda$ is continuous, it follows that $\overline{\lambda}_A \in \hat{\Lambda}_\Pi$. Also $\overline{\lambda}_A$ is bounded so that by Lemma 2.7.21 $Z(\overline{\lambda}_A)$ is a strictly Luzin-continuous local exponential maxingale.

Since $\hat{g}_t(\lambda;\mathbf{x})$ is non-negative, convex in λ and $\hat{g}_t(0;\mathbf{x}) = 0$, we have that $\hat{g}_t(r_A\lambda;\mathbf{x})$ is increasing in A, so

$$\hat{g}_t(r_A\lambda;\mathbf{x}) \uparrow \hat{g}_t(\lambda;\mathbf{x}) \quad \text{as } A \to \infty. \tag{2.7.49}$$

Let σ_N be defined by (2.7.36). By (2.7.49) and Lemma 2.7.18 the collections of strictly Luzin variables $\{Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}),\mathbf{x}), t \in \mathbb{R}_+\}$ and $\{Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}_A(\mathbf{x}),\mathbf{x}),\ t \in \mathbb{R}_+,\ A \in \mathbb{R}_+\}$ are uniformly maximable. In particular, $(Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}_A(\mathbf{x}),\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ is a uniformly maximable exponential maxingale, so by Lemma 2.7.17 the theorem is proved if

$$\lim_{A\to\infty} S_\Pi Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}_A(\mathbf{x}),\mathbf{x})f(\mathbf{x}) = S_\Pi Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}),\mathbf{x})f(\mathbf{x})$$

for every bounded, continuous and non-negative f, which by Theorem 1.4.19 is implied by the convergence $Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}_A(\mathbf{x}),\mathbf{x}) \xrightarrow{\Pi} Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}),\mathbf{x})$. Using the fact that $Z_{t\wedge\sigma_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}),\mathbf{x})$ is a proper idempotent variable, we prove the latter convergence by proving that as $A \to \infty$

$$\int_0^t \big(\hat{g}_s(\lambda(s,\mathbf{x});\mathbf{x}) - \hat{g}_s(r_A\lambda(s,\mathbf{x});\mathbf{x})\big)\,ds \xrightarrow{\Pi} 0, \tag{2.7.50a}$$

$$\int_0^t |(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x}))\cdot \dot{M}_s(\mathbf{x})|ds \xrightarrow{\Pi} 0. \tag{2.7.50b}$$

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The first convergence follows by (2.7.49), the fact that the integral in (2.7.50a) is a strictly Luzin variable (use condition (Π_{II}) and (2.7.19)) and Dini's theorem. To prove (2.7.50b) we write as in the proof of Lemma 2.7.17 for $\alpha > 0$

$$\int_0^t |(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x})) \cdot \dot{M}_s(\mathbf{x})| ds$$
$$\leq \frac{1}{\alpha} \int_0^t \big[\hat{g}_s(\alpha(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x}));\mathbf{x})$$
$$+ \hat{g}_s(-\alpha(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x}));\mathbf{x})\big] ds + \frac{1}{\alpha} I(\mathbf{x}). \quad (2.7.51)$$

Next, by the definition of $r_A\lambda$, convexity of $\hat{g}_s(\lambda;\mathbf{x})$ in λ and the fact that $\hat{g}_s(0;\mathbf{x}) = 0$

$$\hat{g}_s(\alpha(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x}));\mathbf{x}) \leq \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A),$$
$$\hat{g}_s(-\alpha(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x}));\mathbf{x}) \leq \hat{g}_s(-\alpha\lambda(s,\mathbf{x});\mathbf{x})\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A).$$

Hence, as $\overline{\lambda} \in \hat{\Lambda}_\Pi$, (2.7.19) implies that

$$\int_0^t \big[\hat{g}_s(\alpha(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x}));\mathbf{x}) + \hat{g}_s(-\alpha(\lambda(s,\mathbf{x}) - r_A\lambda(s,\mathbf{x}));\mathbf{x})\big] ds$$
$$\overset{\Pi}{\to} 0 \quad \text{as } A \to \infty,$$

so, since α is arbitrary, (2.7.51) yields (2.7.50b). ☐

As a byproduct, we can prove that certain integrals with respect to X are semimaxingales.

Definition 2.7.22. *Let Λ_Π be the subset of $\hat{\Lambda}_\Pi$ consisting of functions $\overline{\lambda}$ such that for $t \in \mathbb{R}_+$*

$$\int_0^t |\lambda(s,\mathbf{x}) \cdot b_s(\mathbf{x})|\, ds < \infty,\ \mathbf{x} \in \mathbb{C},$$

and

$$\int_0^t |\lambda(s,\mathbf{x}) \cdot b_s(\mathbf{x})|\, \mathbf{1}(|\lambda(s,\mathbf{x})| > A)\, ds \overset{\Pi}{\to} 0 \quad \text{as } A \to \infty.$$

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For $\overline{\lambda} \in \Lambda_\Pi$, we define the idempotent process $\overline{\lambda}\diamond X = (\overline{\lambda}\diamond X_t,\ t \in \mathbb{R}_+)$ by

$$\overline{\lambda}\diamond X_t = \int_0^t \lambda(s,\mathbf{x})\cdot b_s(\mathbf{x})\,ds + \int_0^t \lambda(s,\mathbf{x})\cdot \dot{M}_s(\mathbf{x})\,ds, \tag{2.7.52}$$

if the integrals are well defined and finite, and $\overline{\lambda}\diamond X_t = \hat{X}_t$ otherwise, where $\hat{X}$ is a continuous idempotent process. By Lemma 2.7.15 and the definition of Λ_Π we have that (2.7.52) holds Π-a.e.

Theorem 2.7.23. *Let* $g_s(\lambda;\mathbf{x})$ *be given by* (2.7.13), *conditions* (Π_I) *and* (Π_{II}) *hold, and* $\overline{\lambda} \in \Lambda_\Pi$. *Then the idempotent process* $\overline{\lambda}\diamond X$ *is a strictly Luzin-continuous semimaxingale on* $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$ *with cumulant* $G^{\overline{\lambda}}(\alpha) = (G_t^{\overline{\lambda}}(\alpha;\mathbf{x}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ *given by*

$$G_t^{\overline{\lambda}}(\alpha;\mathbf{x}) = \int_0^t g_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})\,ds,\ t \in \mathbb{R}_+, \alpha \in \mathbb{R}, \mathbf{x} \in \mathbb{C}.$$

Proof. By Theorem 2.7.16 $(\exp(\alpha\overline{\lambda}\diamond X_t - G_t^{\overline{\lambda}}(\alpha)),\ t \in \mathbb{R}_+)$ is a strictly Luzin-continuous local exponential maxingale. The fact that both $\overline{\lambda}\diamond X$ and $G^{\overline{\lambda}}(\alpha)$ are strictly Luzin-continuous $\mathbf{C}^\Pi$-adapted idempotent processes follows by the proof of Lemma 2.7.17, condition (Π_I), $\mathbf{C}^\Pi$-progressive measurability of $(b_s(\mathbf{x}))$, and the definition of Λ_Π. ☐

In large deviation limit theorems $G(\lambda)$ is often more specific than in (2.7.10) and defined in terms of "characteristics", which we now introduce. Let $\big(c_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}\big)$ be a $\mathbf{C}^\Pi$-progressively measurable idempotent process with values in the space of symmetric, positive semi-definite $d \times d$-matrices such that $\int_0^t \|c_s(\mathbf{x})\|\ ds < \infty$ for $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$; $\big(\nu_s(\Gamma;\mathbf{x}),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d), \mathbf{x} \in \mathbb{C}\big)$ be a transition kernel (for each $\mathbf{x}$) from $(\mathbb{R}_+, \overline{\mathcal{B}}(\mathbb{R}_+))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

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for $t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}$ and $\alpha \in \mathbb{R}_+$

$$\nu_t(\{0\};\mathbf{x}) = 0, \int_{\mathbb{R}^d} |x|^2 \wedge 1\, \nu_t(dx;\mathbf{x}) < \infty,$$

$$\int_0^t \int_{\mathbb{R}^d} |x|^2 \wedge 1\, \nu_s(dx;\mathbf{x})\, ds < \infty, \tag{2.7.53a}$$

$$\int_{\mathbb{R}^d} e^{\alpha|x|}\, \mathbf{1}(|x| > 1)\, \nu_t(dx;\mathbf{x}) < \infty,$$

$$\int_0^t \int_{\mathbb{R}^d} e^{\alpha|x|}\, \mathbf{1}(|x| > 1)\, \nu_s(dx;\mathbf{x})\, ds < \infty, \tag{2.7.53b}$$

and the functions $(\int_{\mathbb{R}^d} f(x)\, \nu_s(dx;\mathbf{x}),\ s \in \mathbb{R}_+)$ are $\mathbf{C}^{\Pi}$-progressively measurable for Borel functions f such that the integrals are well defined; $\big(\hat{\nu}_s(\Gamma;\mathbf{x}),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d), \mathbf{x} \in \mathbb{C}\big)$ be a transition kernel (for each $\mathbf{x}$) from $(\mathbb{R}_+, \overline{\mathcal{B}}(\mathbb{R}_+))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that for $s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$\hat{\nu}_s(\Gamma;\mathbf{x}) \le \nu_s(\Gamma;\mathbf{x}),\ \hat{\nu}_s(\mathbb{R}^d;\mathbf{x}) \le 1, \tag{2.7.54}$$

and the functions $(\int_{\mathbb{R}^d} f(x)\, \hat{\nu}_s(dx;\mathbf{x}),\ s \in \mathbb{R}_+)$ are $\mathbf{C}^{\Pi}$-progressively measurable for Borel functions f such that the integrals are well defined.

We say that the semimaxingale X has local characteristics $(b, c, \nu, \hat{\nu})$, where $(b_s(\mathbf{x}))$ is as above, if the associated cumulant is given by (2.7.7), where

$$g_s(\lambda;\mathbf{x}) = \lambda \cdot b_s(\mathbf{x}) + \frac{1}{2}\lambda \cdot c_s(\mathbf{x})\lambda + \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda \cdot x)\nu_s(dx;\mathbf{x})$$
$$+ \Big(\ln\big(1 + \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1)\hat{\nu}_s(dx;\mathbf{x})\big) - \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1)\hat{\nu}_s(dx;\mathbf{x})\Big). \tag{2.7.55}$$

Remark 2.7.24. *The right-hand side is well defined since* $\int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1)\hat{\nu}_s(dx;\mathbf{x}) > -1$ *by the fact that* $\hat{\nu}_s(\mathbb{R}^d;\mathbf{x}) \le 1$*. It is also not difficult to check that* $\int_0^t \sup_{|\lambda|\le A} |g_s(\lambda;\mathbf{x})| ds < \infty$ *for* $t \in \mathbb{R}_+$ *and* $A \in \mathbb{R}_+$.

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Let

$$C_t(\mathbf{x}) = \int_0^t c_s(\mathbf{x})\, ds. \tag{2.7.56}$$

We call the idempotent process $B' = (B'_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ (defined by (2.7.11)) the first characteristic of X "without truncation", $C = (C_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ the second characteristic, $\nu_s(dx; \mathbf{x})$ the density of the measure of jumps, and $\hat{\nu}_s(dx; \mathbf{x})$ the density of the discontinuous measure of jumps. The quadruplet $(B', C, \nu, \hat{\nu})$ is referred to as the characteristics of the semimaxingale X "without truncation". In large deviation limit theorems we will also need characteristics "associated with limiters".

Definition 2.7.25. *A Borel function $h : \mathbb{R}^d \to \mathbb{R}^d$ is said to be a limiter if it is bounded and $h(x) = x$ in a neighbourhood of the origin.*

For Borel functions f, for which the integrals below are well defined, we introduce the notation

$$f(x) * \nu_t(\mathbf{x}) = \int_0^t \int_{\mathbb{R}^d} f(x)\, \nu_s(dx; \mathbf{x}) ds, \;\; f(x) \bullet \hat{\nu}_s(\mathbf{x}) = \int_{\mathbb{R}^d} f(x) \hat{\nu}_s(dx; \mathbf{x}).$$

The first characteristic of X associated with a limiter h is an idempotent process $B = (B_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ defined by

$$B_t(\mathbf{x}) = B'_t(\mathbf{x}) + (h(x) - x) * \nu_t\, (\mathbf{x}). \tag{2.7.57}$$

The modified second characteristic associated with h is an idempotent process $\tilde{C} = (\tilde{C}_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ such that the $\tilde{C}_t(\mathbf{x})$ are symmetric positive semi-definite $d \times d$-matrices, specified by the equalities

$$\lambda \cdot \tilde{C}_t(\mathbf{x}) \lambda = \lambda \cdot C_t(\mathbf{x}) \lambda + (\lambda \cdot h(x))^2 * \nu_t\, (\mathbf{x}) - \int_0^t (\lambda \cdot h(x) \bullet \hat{\nu}_s(\mathbf{x}))^2 ds, \; \lambda \in \mathbb{R}^d. \tag{2.7.58}$$

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Analogously, the modified second characteristic "without truncation" $\tilde{C}' = (\tilde{C}'_t(\mathbf{x}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ is specified by the equalities

$$\lambda \cdot \tilde{C}'_t(\mathbf{x})\lambda = \lambda \cdot C_t(\mathbf{x})\lambda + (\lambda \cdot x)^2 * \nu_t\,(\mathbf{x}) - \int_0^t (\lambda \cdot x \bullet \hat{\nu}_s(\mathbf{x}))^2 ds. \tag{2.7.59}$$

Note that (2.7.57) implies that if $\bar{B} = (\bar{B}_t(\mathbf{x}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ is the first characteristic associated with a limiter $\bar{h}(x)$, then

$$\bar{B}_t(\mathbf{x}) = B_t(\mathbf{x}) + (\bar{h}(x) - h(x)) * \nu_t\,(\mathbf{x}). \tag{2.7.60}$$

Obviously, $B_t(\mathbf{x}), C_t(\mathbf{x}), \tilde{C}_t(\mathbf{x}), \tilde{C}'_t(\mathbf{x})$, and $f(x) * \nu_t(\mathbf{x})$ (when well defined) are continuous in t and $\mathcal{C}_t^{\Pi}$–measurable in $\mathbf{x}$. We note that the cumulant assumes the form

$$G_t(\lambda; \mathbf{x}) = \lambda \cdot B_t(\mathbf{x}) + \frac{1}{2}\lambda \cdot C_t(\mathbf{x})\lambda + (e^{\lambda \cdot x} - 1 - \lambda \cdot h(x)) * \nu_t\,(\mathbf{x})$$
$$+ \int_0^t \Big(\ln\big(1 + (e^{\lambda \cdot x} - 1) \bullet \hat{\nu}_s(\mathbf{x})\big) - (e^{\lambda \cdot x} - 1) \bullet \hat{\nu}_s(\mathbf{x})\Big) ds. \tag{2.7.61}$$

Remark 2.7.26. *The definition of the characteristics of a semimaxingale is motivated by the definition of the characteristics of a semimartingale and the fact that semimaxingales are "large deviation limits" of semimartingales as is shown in part II. Note that the expression* (2.7.61) *for the cumulant is analogous to the logarithm of the stochastic exponential of a semimartingale, Liptser and Shiryaev [79], Jacod and Shiryaev [67]. Thus,* B_t *is "the drift term",* C_t *is "the diffusion term",* $\nu_s\,ds$ *is "the predictable measure of jumps", and* $\hat{\nu}_s\,ds$ *is "the discontinuous part of the predictable measure of jumps". As we will see in part II, this analogy is not only in form, but it also helps us to formulate conditions under which large deviation limit theorems for semimartingales can be proved. The terminology "truncated" and "nontruncated" characteristics is also inherited from semimartingales. The analogy with semimartingales would be more complete if we required in Definition 2.7.1 that* $G_t(\lambda; \omega)$ *be, in addition, of locally bounded variation in* t*. In fact, this property holds if a semimaxingale admits characteristics as is the case in all our examples of semimaxingales. However, since a number of*

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properties of semimaxingales do not depend on $G(\lambda)$ being of locally bounded variation, we have decided not to include this requirement in the definition.

We now state conditions on the characteristics that imply conditions (Π_I) and (Π_{II}).

Lemma 2.7.27. *Let the canonical idempotent process X be a semimaxingale on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$ with local characteristics $(b, c, \nu, \hat{\nu})$, where b and c are strictly Luzin idempotent processes, and $\int_{\mathbb{R}^d}(\exp(\lambda \cdot x) - 1 - \lambda \cdot x)\, \nu_s(dx; \mathbf{x})$ and $\int_{\mathbb{R}^d} \exp(\lambda \cdot x)\, \hat{\nu}_s(dx; \mathbf{x})$ are continuous in $(\lambda, \mathbf{x})$ when restricted to $\mathbb{R}^d \times K_\Pi(a)$ for $a \in (0,1]$ and $s \in \mathbb{R}_+$. If for all $a \in (0,1]$, $A \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$*

$$\int_0^t \sup_{\mathbf{x} \in K_\Pi(a)} |b_s(\mathbf{x})|\, ds < \infty, \quad \sup_{s \le t} \sup_{\mathbf{x} \in K_\Pi(a)} \|c_s(\mathbf{x})\| < \infty,$$

$$\sup_{s \le t} \sup_{\mathbf{x} \in K_\Pi(a)} \sup_{|\lambda| \le A} \int_{\mathbb{R}^d} (e^{\lambda \cdot x} - 1 - \lambda \cdot x) \nu_s(dx; \mathbf{x}) < \infty,$$

$$\lim_{\lambda \to 0} \sup_{s \le t} \sup_{\mathbf{x} \in K_\Pi(a)} \int_{\mathbb{R}^d} (e^{\lambda \cdot x} - 1 - \lambda \cdot x) \nu_s(dx; \mathbf{x}) = 0,$$

$$\sup_{s \le t} \sup_{\mathbf{x} \in K_\Pi(a)} \sup_{|\lambda| \le A} \int_{\mathbb{R}^d} e^{\lambda \cdot x} \hat{\nu}_s(dx; \mathbf{x})\, ds < \infty,$$

$$\lim_{\lambda \to 0} \sup_{s \le t} \sup_{\mathbf{x} \in K_\Pi(a)} \left| \int_{\mathbb{R}^d} \big(e^{\lambda \cdot x} - 1\big) \hat{\nu}_s(dx; \mathbf{x}) \right| = 0,$$

then the associated cumulant satisfies conditions (Π_I) and (Π_{II}).

2.8 Maxingale problems

In this section we are concerned with identifying deviabilities for which the canonical idempotent process on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ is a semimaxingale with a given cumulant. As in the preceding section we denote $\mathbb{C} = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$, $\mathcal{C}_t = \mathcal{C}_t(\mathbb{R}_+, \mathbb{R}^d)$, and $\mathbf{C} = (\mathcal{C}_t,\, t \in \mathbb{R}_+)$. We also assume as given a cumulant $G(\lambda)$ on $\mathbb{C}$. Let $x \in \mathbb{R}^d$.

Definition 2.8.1. *We say that a deviability Π on $\mathbb{C}$ is a solution to the maxingale problem (x, G) if the canonical process X is a semimaxingale with cumulant $G(\lambda)$ on $(\mathbb{C}, \mathbf{C}, \Pi)$ such that $X_0 = x$ Π-a.e.*

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Examples are provided by the Wiener and Poisson idempotent processes: by Theorem 2.4.2 the Wiener idempotent probability on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ solves the maxingale problem $(0, G)$ with $G_t(\lambda; \mathbf{x}) = \lambda^2 t/2$ and by Theorem 2.4.16 the Poisson idempotent probability on $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ solves the maxingale problem $(0, G)$ with $G_t(\lambda; \mathbf{x}) = (\exp \lambda - 1)t$. In view of applications to large deviation theory, we are interested in finding conditions when a maxingale problem has a unique solution. Existence issues are of less importance to us. Besides, large deviation convergence theorems of part II also imply that under their hypotheses the associated maxingale problems have solutions.

Definition 2.8.2. *We say that uniqueness holds for the maxingale problem (x, G) if it has at most one solution.*

Our candidate for a solution of (x, G) is the idempotent measure $\mathbf{\Pi}_x$ defined in (2.7.6). We begin with the case where the cumulant does not depend on $\mathbf{x}$.

Lemma 2.8.3. *Let $G_t(\lambda; \mathbf{x})$ not depend on $\mathbf{x} \in \mathbb{C}$, be differentiable in λ for all $t \in \mathbb{R}_+$ and the differences $G_t(\lambda) - G_s(\lambda)$ be convex in λ for all $0 \le s < t$. Then the idempotent measures $\mathbf{\Pi}_{t_1,\ldots,t_k}$ on $(\mathbb{R}^d)^k$, where $0 \le t_1 < t_2 < \ldots < t_k$, specified by the densities*

$$\mathbf{\Pi}_{t_1,\ldots,t_k}(x_1, \ldots, x_k) = \prod_{i=1}^{k} \inf_{\lambda \in \mathbb{R}^d} e^{-\lambda \cdot (x_i - x_{i-1})} e^{G_{t_i}(\lambda) - G_{t_{i-1}}(\lambda)},$$

where $t_0 = 0$ and $x_0 = x$, form a projective system of deviabilities, which has $\mathbf{\Pi}_x$ as the projective limit.

Proof. By differentiability of $G_t(\lambda)$, Lemma 1.11.4 and Lemma 1.11.7 the $\mathbf{\Pi}_{t_1,\ldots,t_k}$ are deviabilities. In order to prove that they form a projective system it suffices to check that, given $t_{i-1} < t_i < t_{i+1}$, x_{i-1} and x_{i+1}, we have

$$\inf_{x_i \in \mathbb{R}^d} \sup_{\substack{\lambda_1 \in \mathbb{R}^d, \\ \lambda_2 \in \mathbb{R}^d}} \big(\lambda_1 \cdot (x_i - x_{i-1}) + \lambda_2 \cdot (x_{i+1} - x_i) - (G_{t_i}(\lambda_1) - G_{t_{i-1}}(\lambda_1)) - (G_{t_{i+1}}(\lambda_2) - G_{t_i}(\lambda_2))\big)$$
$$= \sup_{\lambda \in \mathbb{R}^d} \big(\lambda \cdot (x_{i+1} - x_{i-1}) - (G_{t_{i+1}}(\lambda) - G_{t_{i-1}}(\lambda))\big).$$

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The latter equality follows by a minimax argument, see, e.g., Aubin and Ekeland [6]. By Theorem 2.2.4 the projective limit of the $\mathbf{\Pi}_{t_1,\ldots,t_k}$ coincides with $\mathbf{\Pi}_x$. □

Remark 2.8.4. *As a consequence, $\mathbf{\Pi}_x$ is a deviability under the hypotheses.*

The following existence and uniqueness result, which is a converse to Theorem 2.7.3, shows that the distributions of certain idempotent processes with independent increments are uniquely specified by the associated cumulants. We denote $G^*_t(\lambda) = \sup_{0\le s\le t}|G_s(\lambda)|$.

Theorem 2.8.5. *Let $G_t(\lambda;\mathbf{x})$ not depend on $\mathbf{x} \in \mathbb{C}$. Let, in addition, $G_t(\lambda)$ be differentiable in λ for all $t \in \mathbb{R}_+$ and the differences $G_t(\lambda) - G_s(\lambda)$ be convex in λ for all $0 \le s < t$. Then $\mathbf{\Pi}_x$ is a unique solution to problem (x, G). The canonical idempotent process X has independent increments and starts at x under $\mathbf{\Pi}_x$.*

Proof. We define $\mathbf{\Pi}_{t_1,\ldots,t_k}$ as in Lemma 2.8.3. The construction of the $\mathbf{\Pi}_{t_1,\ldots,t_k}$ and the fact that they induce deviability $\mathbf{\Pi}_x$ on $\mathbb{C}$ imply that the canonical idempotent process X has independent increments and starts at x under $\mathbf{\Pi}_x$, and $S_{\mathbf{\Pi}_x} \exp(\lambda\cdot(X_t - X_s)) = \exp(G_t(\lambda) - G_s(\lambda))$ so that X is a semimaxingale with cumulant $G_t(\lambda)$.

We prove uniqueness. Let Π be a solution of (x, G) and X be the canonical idempotent process on $\mathbb{C}$. We note that for every $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}^d$ the family $\{Y_{t\wedge n}(\lambda), t \in \mathbb{R}_+\}$ is Π–uniformly maximable. (Recall that $Y(\lambda)$ is defined by (2.7.1).) To see this, we write by (2.7.1)

$$S_\Pi\big(Y_{t\wedge n}(\lambda)^2\big) = \sup_{\mathbf{x}\in\mathbb{C}} Y_{t\wedge n}(2\lambda;\mathbf{x}) \exp(-2G_{t\wedge n}(\lambda)) \exp(G_{t\wedge n}(2\lambda))\Pi(\mathbf{x})$$
$$\le \big(S_\Pi Y_{t\wedge n}(2\lambda)\big) \exp(2G^*_n(\lambda) + G^*_n(2\lambda)).$$

Since $S_\Pi Y_{t\wedge n}(2\lambda) \le S_\Pi Y_0(2\lambda) = 1$ by Lemma 2.3.13, the uniform maximability is proved.

Then by “the Doob stopping theorem” (Theorem 2.3.8) the sequence $\{n, n \in \mathbb{N}\}$ is a localising sequence for $Y(\lambda)$ and, in particular, $Y(\lambda)$ is a $\mathbf{C}$–exponential maxingale under Π. Therefore, for $0 \le s < t$,

$$S_\Pi\big(\exp(\lambda\cdot(X_t - X_s))|\mathcal{C}_s\big) = \exp(G_t(\lambda) - G_s(\lambda)), \qquad (2.8.1)$$

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so by Lemma 1.11.9 X has independent increments under Π. Let $\Pi^X_{t_0,t_1,\ldots,t_k}$, $0 = t_0 < t_1 < t_1 < \ldots < t_k$, denote finite-dimensional idempotent distributions of X so that $\Pi^X_{t_0,t_1,\ldots,t_k}(x_0, x_1, \ldots, x_k) = \Pi(X_{t_0} = x_0, X_{t_1} = x_1, \ldots, X_{t_k} = x_k)$. By independence of increments of X we have that $\Pi^X_{t_0,t_1,\ldots,t_k}(x_0, x_1, \ldots, x_k) = \Pi^X_0(x_0) \prod_{i=1}^k \Pi^X_{t_{i-1},t_i}(x_{i-1}, x_i)$. Also by (2.8.1) and Lemma 1.11.5

$$\Pi^X_{t_{i-1},t_i}(x_{i-1}, x_i) = \inf_{\lambda \in \mathbb{R}^d} e^{-\lambda\cdot(x_i - x_{i-1})} \exp(G_{t_i}(\lambda) - G_{t_{i-1}}(\lambda)).$$

Thus, $\Pi^X_{t_0,t_1,\ldots,t_k} = \mathbf{\Pi}_{t_0,t_1,\ldots,t_k}$ Since by Theorem 2.2.2 and Corollary 1.7.12 $\Pi^X(\mathbf{x}) = \inf_{t_1,\ldots,t_k} \Pi^X_{t_0,t_1,\ldots,t_k}(\mathbf{x}_{t_0}, \mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_k})$, the required follows by Lemma 2.8.3. $\square$

Remark 2.8.6. *Note also that if X is an idempotent process with independent increments and cumulant $G(\lambda)$ on an idempotent probability space (Ω, Π), then $G_t(\lambda) - G_s(\lambda) = \ln S_\Pi \exp\big(\lambda \cdot (X_t - X_s)\big)$ so that $G_t(\lambda) - G_s(\lambda)$ is convex for $s < t$.*

If $G_t(\lambda)$ is given in terms of characteristics by (2.7.7) and (2.7.55), we have the following consequence.

Corollary 2.8.7. *Let $G_t(\lambda)$ have the form (2.7.7) and (2.7.55), where b_s, c_s, ν_s, and $\hat{\nu}_s$ do not depend on $\mathbf{x}$. If, in addition,*

$$(L_0) \qquad 1 + \inf_{s \le t}\big(e^{-\alpha|x|} - 1\big) \bullet \hat{\nu}_s > 0,\ t \in \mathbb{R}_+,\ \alpha \in \mathbb{R}_+,$$

then $\mathbf{\Pi}_x$ is a unique deviability on $\mathbb{C}$ such that X is a semimaxingale starting at x with independent increments and local characteristics $(b, c, \nu, \hat{\nu})$ under $\mathbf{\Pi}_x$.

Proof. By Theorem 2.8.5 we only need to prove that $G_t(\lambda)$ is differentiable in λ, which follows by condition (L_0). $\square$

The next lemma gives sufficient conditions for condition (L_0) to hold.

Lemma 2.8.8. *Condition (L_0) holds if at least one of the following conditions holds*

1. $\lim_{a\to\infty} \inf_{s \le t} \hat{\nu}_s(|x| \le a) > 0$, $t \in \mathbb{R}_+$,

2. $\sup_{s \le t} \hat{\nu}_s(\mathbb{R}^d) < 1$, $t \in \mathbb{R}_+$,

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3. for every $t \in \mathbb{R}_+$ there exists $\epsilon > 0$ such that $\sup_{s \le t} e^{\epsilon|x|} \bullet \hat{\nu}_s < \infty$.

Proof. It is straightforward to see that conditions 1 and 2 imply (L_0). Let condition 3 hold. We have for $\alpha \in \mathbb{R}_+$, $s \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$

$$\begin{aligned}
&1 + (e^{-\alpha|x|} - 1) \bullet \hat{\nu}_s \ge 1 + \left(e^{-\alpha|x|}\,\mathbf{1}(\epsilon|x| < b) - 1\right) \bullet \hat{\nu}_s \\
&\ge 1 + \left(e^{-\alpha b/\epsilon}(1 - e^{-b}e^{\epsilon|x|}) - 1\right) \bullet \hat{\nu}_s \\
&\qquad = 1 + (e^{-\alpha b/\epsilon} - 1)\hat{\nu}_s(\mathbb{R}^d) - e^{-\alpha b/\epsilon}e^{-b}e^{\epsilon|x|} \bullet \hat{\nu}_s. \quad (2.8.2)
\end{aligned}$$

Since $\hat{\nu}_s(\mathbb{R}^d) \in [0,1]$, it follows that $1 + (e^{-\alpha b/\epsilon} - 1)\hat{\nu}_s(\mathbb{R}^d) \ge e^{-\alpha b/\epsilon}$, and, hence picking b such that $e^b \ge 2\sup_{s\le t} e^{\epsilon|x|} \bullet \hat{\nu}_s$, which is possible by hypotheses, we conclude that the left-most side of (2.8.2) is bounded from below by $e^{-\alpha b/\epsilon}/2$ for $s \le t$. □

We now consider the case where the cumulant depends on $\mathbf{x}$. The following is an existence result for a diffusion maxingale problem. It is a consequence of Theorem 2.5.19 and Theorem 2.6.24. We denote $\mathbf{x}_t^* = \sup_{0\le s\le t}|\mathbf{x}_s|$.

Theorem 2.8.9. *Let the maxingale problem (x, G) be specified by the cumulant*

$$G_t(\lambda; \mathbf{x}) = \int_0^t \lambda \cdot b_s(\mathbf{x})\, ds + \frac{1}{2}\int_0^t \lambda \cdot c_s(\mathbf{x})\lambda\, ds,$$

where functions $(b_s(\mathbf{x}))$ and $(c_s(\mathbf{x}))$, assuming values in $\mathbb{R}^d$ and the space of symmetric positive semi-definite $d \times d$-matrices, respectively, are $\mathbf{C}$-progressively measurable and continuous in $\mathbf{x}$, and $\int_0^t |b_s(\mathbf{x})|\, ds < \infty$ and $\int_0^t \|c_s(\mathbf{x})\|\, ds < \infty$. Let also $b_t(\mathbf{x})$ and $c_t(\mathbf{x})$ satisfy the linear-growth conditions

$$|b_t(\mathbf{x})| \le l_t(1+\mathbf{x}_t^*), \quad \|c_t(\mathbf{x})\| \le l_t(1+\mathbf{x}_t^{*2}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C},$$

where l_t is Lebesgue measurable and $\int_0^t l_s\, ds < \infty$, $t \in \mathbb{R}_+$. Let X be a Luzin solution to the idempotent Ito differential equation

$$\dot{X}_t = b_t(X) + c_t(X)^{1/2}\dot{W}_t,\ X_0 = x,$$

where W is an $\mathbb{R}^d$-valued idempotent Wiener process. Then the idempotent distribution of X solves problem (x, G). In particular, $\mathbf{\Pi}_x$ is

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a solution to the maxingale problem (x, G)*. It assumes the form*

$$\mathbf{\Pi}_x(\mathbf{x}) = \exp\Bigl(-\frac{1}{2}\int_0^\infty (\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\cdot c_t(\mathbf{x})^{\oplus}(\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\,dt\Bigr)$$

if $\mathbf{x}_0 = x$*,* $\mathbf{x}$ *is absolutely continuous and* $\dot{\mathbf{x}}_t - b_t(\mathbf{x})$ *is in the range of* $c_t(\mathbf{x})$ *a.e., and* $\mathbf{\Pi}_x(\mathbf{x}) = 0$ *otherwise.*

The next result concerns existence for maxingale problems of the Poisson type.

Theorem 2.8.10. *Let* $(u_s(\mathbf{x}),\, s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}))$ *and* $(v_s(\mathbf{x}),\, s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}))$ *be* $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$*-progressively measurable* $\mathbb{R}_+$*-valued functions, which are continuous in* $\mathbf{x}$ *and satisfy the linear-growth condition* $u_s(\mathbf{x}) + v_s(\mathbf{x}) \le l_s(1 + \mathbf{x}_s^*)$*, where* l_s *is locally integrable. Let*

$$G_t(\lambda; \mathbf{x}) = (e^{\lambda} - 1)\int_0^t u_s(\mathbf{x})\,ds + (e^{-\lambda} - 1)\int_0^t v_s(\mathbf{x})\,ds,\ \lambda \in \mathbb{R}.$$

Let $(X, \mathcal{N}_1, \mathcal{N}_2) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^3)$ *be the Luzin solution to the idempotent equation*

$$X_t = x + \mathcal{N}_1\Bigl(\int_0^t u_s(X)\,ds\Bigr) - \mathcal{N}_2\Bigl(\int_0^t v_s(X)\,ds\Bigr),\ x \in \mathbb{R},$$

where $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are independent Poisson idempotent processes, that was constructed in the proof of Theorem 2.6.33. Then the idempotent distribution of* X *is a solution to problem* (x, G) *so that* $\mathbf{\Pi}_x$ *is a solution to* (x, G) *and assumes the form*

$$\mathbf{\Pi}_x(\mathbf{x}) = \exp\Bigl(-\int_0^\infty \sup_{\lambda \in \mathbb{R}}\bigl(\lambda\dot{\mathbf{x}}_t - (e^{\lambda} - 1)u_t(\mathbf{x}) - (e^{-\lambda} - 1)v_t(\mathbf{x})\bigr)\,dt\Bigr)$$

if $\mathbf{x}$ *is absolutely continuous and* $\mathbf{x}_0 = x$*, and* $\mathbf{\Pi}_x(\mathbf{x}) = 0$ *otherwise.*

Proof. Let $\mathcal{A}_t$ denote the τ-algebra on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^3)$ with atoms $p_t\mathbf{x}$, $p_{\int_0^t u_s(\mathbf{x})\,ds} n_1$ and $p_{\int_0^t v_s(\mathbf{x})\,ds} n_2$, where $(\mathbf{x}, n_1, n_2) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^3)$, $\mathbf{A} = (\mathcal{A}_t,\, t \in \mathbb{R}_+)$ and idempotent probability $\Pi^{X, \mathcal{N}_1, \mathcal{N}_2}$ on $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^3)$ be

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defined as in the proof of Theorem 2.6.33. It is sufficient to check that $Z = (Z_t(\mathbf{x}),\, t \in \mathbb{R}_+)$ defined by

$$Z_t(\mathbf{x}) = \exp\Bigl(\lambda(\mathbf{x}_{t\wedge\tau_N(\mathbf{x})} - x) - (e^\lambda - 1)\int\limits_0^{t\wedge\tau_N(\mathbf{x})} u_s(\mathbf{x})\,ds - (e^{-\lambda} - 1)\int\limits_0^{t\wedge\tau_N(\mathbf{x})} v_s(\mathbf{x})\,ds\Bigr),$$

where $\tau_N(\mathbf{x}) = \inf\{t \in \mathbb{R}_+ : \mathbf{x}_t^* \ge N\}$ and $\lambda \in \mathbb{R}$, is an $\mathbf{A}$-exponential maxingale on $\bigl(\mathbb{C}(\mathbb{R}_+, \mathbb{R}^3), \Pi^{X,\mathcal{N}_1,\mathcal{N}_2}\bigr)$. Let $(\mathbf{x}', n_1', n_2') \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^3)$ be such that $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}(\mathbf{x}', n_1', n_2') > 0$. Then for $s \le t$ by the definition of $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}$

$$\begin{aligned} S(Z_t|\mathcal{A}_{s\wedge\tau_N})(\mathbf{x}', n_1', n_2') &= \sup_{(\mathbf{x},n_1,n_2)\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}^3)} \\ &\exp\Bigl(\bigl(\lambda n_1(\int\limits_0^{t\wedge\tau_N(\mathbf{x})} u_s(\mathbf{x})\,ds) - (e^\lambda - 1)\int\limits_0^{t\wedge\tau_N(\mathbf{x})} u_s(\mathbf{x})\,ds\bigr) \\ &+ \bigl(-\lambda n_2(\int\limits_0^{t\wedge\tau_N(\mathbf{x})} v_s(\mathbf{x})\,ds) - (e^{-\lambda} - 1)\int\limits_0^{t\wedge\tau_N(\mathbf{x})} v_s(\mathbf{x})\,ds\bigr)\Bigr) \\ &\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}\bigl((\mathbf{x}, n_1, n_2)|\mathcal{A}_{s\wedge\tau_N}\bigr)(\mathbf{x}', n_1', n_2'). \end{aligned} \quad (2.8.3)$$

Since

$$\begin{aligned} &\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}\bigl((\mathbf{x}, n_1, n_2)|\mathcal{A}_{s\wedge\tau_N}\bigr)(\mathbf{x}', n_1', n_2') \\ &\quad\le \Pi^{\mathcal{N}}(\theta_{\int_0^{s\wedge\tau_N(\mathbf{x}')} u_r(\mathbf{x}')\,dr}\, n_1)\Pi^{\mathcal{N}}(\theta_{\int_0^{s\wedge\tau_N(\mathbf{x}')} v_r(\mathbf{x}')\,dr}\, n_2), \end{aligned}$$

it follows that the right-hand side of (2.8.3) is not greater than $Z_{s\wedge\tau_N(\mathbf{x}')}(\mathbf{x}')$. For the reverse inequality, let $\hat{\mathbf{x}}$ be defined on $[0, s \wedge \tau_N(\mathbf{x}')]$ by $\hat{\mathbf{x}}(q) = \mathbf{x}'(q)$, on $[s \wedge \tau_N(\mathbf{x}'), t \wedge \tau_N(\hat{\mathbf{x}})]$ as a solution of the equation

$$\hat{\mathbf{x}}_q = \mathbf{x}'_{s\wedge\tau_N(\mathbf{x}')} + e^\lambda \int\limits_{s\wedge\tau_N(\mathbf{x}')}^{q} u_r(\hat{\mathbf{x}})\,dr - e^{-\lambda}\int\limits_{s\wedge\tau_N(\mathbf{x}')}^{q} v_r(\hat{\mathbf{x}})\,dr,$$

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and on $[t \wedge \tau_N(\hat{\mathbf{x}}), \infty)$ by

$$\hat{\mathbf{x}}_q = \hat{\mathbf{x}}_{t\wedge\tau_N(\hat{\mathbf{x}})} + \int\limits_{t\wedge\tau_N(\hat{\mathbf{x}})}^{q} u_r(\hat{\mathbf{x}})\,dr - \int\limits_{t\wedge\tau_N(\hat{\mathbf{x}})}^{q} v_r(\hat{\mathbf{x}})\,dr.$$

The latter two equations have solutions by a standard argument. We define $\hat{n}_1$ on $\big[0, \int_0^{s\wedge\tau_N(\mathbf{x}')} u_r(\mathbf{x}')\,dr\big]$ by $\hat{n}_1(q) = n_1'(q)$, on $\big[\int_0^{s\wedge\tau_N(\mathbf{x}')} u_r(\mathbf{x}')\,dr, \int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} u_r(\hat{\mathbf{x}})\,dr\big]$ by $\hat{n}_1(q) = n_1'\big(\int_0^{s\wedge\tau_N(\mathbf{x}')} u_r(\mathbf{x}')\,dr\big) + e^{\lambda}\big(q - \int_0^{s\wedge\tau_N(\mathbf{x}')} u_r(\mathbf{x}')\,dr\big)$ and on $\big[\int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} u_r(\hat{\mathbf{x}})\,dr, \infty)$ by $\hat{n}_1(q) = \hat{n}_1\big(\int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} u_r(\hat{\mathbf{x}})\,dr\big) + \big(q - \int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} u_r(\hat{\mathbf{x}})\,dr\big)$. Similarly, $\hat{n}_2(q) = n_2'(q)$ on $\big[0, \int_0^{s\wedge\tau_N(\mathbf{x}')} v_r(\mathbf{x}')\,dr\big]$, $\hat{n}_2(q) = n_2'\big(\int_0^{s\wedge\tau_N(\mathbf{x}')} v_r(\mathbf{x}')\,dr\big) + e^{-\lambda}\big(q - \int_0^{s\wedge\tau_N(\mathbf{x}')} v_r(\mathbf{x}')\,dr\big)$ on $\big[\int_0^{s\wedge\tau_N(\mathbf{x}')} v_r(\mathbf{x}')\,dr, \int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} v_r(\hat{\mathbf{x}})\,dr\big]$, and $\hat{n}_2(q) = \hat{n}_2\big(\int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} v_r(\hat{\mathbf{x}})\,dr\big) + \big(q - \int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} v_r(\hat{\mathbf{x}})\,dr\big)$ on $\big[\int_0^{t\wedge\tau_N(\hat{\mathbf{x}})} v_r(\hat{\mathbf{x}})\,dr, \infty)$. Then $(\hat{\mathbf{x}}, \hat{n}_1, \hat{n}_2)$ satisfies equation (2.6.11), so $\Pi^{X,\mathcal{N}_1,\mathcal{N}_2}\big((\hat{\mathbf{x}}, \hat{n}_1, \hat{n}_2)\big) = \Pi^{\mathcal{N}}(\hat{n}_1)\Pi^{\mathcal{N}}(\hat{n}_2)$. Since

$$\exp\Big(\lambda\hat{n}_1\big(\int\limits_0^{t\wedge\tau_N(\hat{\mathbf{x}})} u_r(\hat{\mathbf{x}})\,dr\big) - (e^{\lambda}-1)\int\limits_0^{t\wedge\tau_N(\hat{\mathbf{x}})} u_s(\hat{\mathbf{x}})\,ds\Big)\Pi^{\mathcal{N}}(\hat{n}_1)$$
$$= \exp\Big(\lambda n_1'\big(\int\limits_0^{s\wedge\tau_N(\mathbf{x}')} u_r(\mathbf{x}')\,dr\big)$$
$$- (e^{\lambda}-1)\int\limits_0^{s\wedge\tau_N(\mathbf{x}')} u_r(\mathbf{x}')\,dr\Big)\Pi^{\mathcal{N}}\big([n_1']_{\mathcal{A}_{s\wedge\tau_N(\mathbf{x}')}}\big)$$

and

$$\exp\Big(-\lambda\hat{n}_2\big(\int\limits_0^{t\wedge\tau_N(\hat{\mathbf{x}})} v_r(\hat{\mathbf{x}})\,dr\big) - (e^{-\lambda}-1)\int\limits_0^{t\wedge\tau_N(\hat{\mathbf{x}})} v_s(\hat{\mathbf{x}})\,ds\Big)\Pi^{\mathcal{N}}(\hat{n}_2)$$

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$$= \exp\Big(-\lambda n_2'\big(\int_0^{s\wedge\tau_N(\mathbf{x}')} v_r(\mathbf{x}')\,dr\big) - (e^{-\lambda}-1)\int_0^{s\wedge\tau_N(\mathbf{x}')} v_r(\mathbf{x}')\,dr\Big)\Pi^{\mathcal{N}}\big([n_2']_{\mathcal{A}_{s\wedge\tau_N(\mathbf{x}')}}\big),$$

we conclude that the expression in the supremum on the right-hand side of (2.8.3) evaluated at $(\hat{\mathbf{x}},\hat{n}_1,\hat{n}_2)$ equals $Z_{s\wedge\tau_N(\mathbf{x}')}(\mathbf{x}')$.

Thus, $S(Z_{t\wedge\tau_N}|\mathcal{A}_{s\wedge\tau_N}) = Z_{s\wedge\tau_N}$. Therefore,

$$\begin{aligned} S(Z_{t\wedge\tau_N}|\mathcal{A}_s) &= \big(S(Z_{t\wedge\tau_N}|\mathcal{A}_{s\wedge\tau_N})\,\mathbf{1}(s\le\tau_N)\big)\vee\big(Z_{\tau_N}\,\mathbf{1}(s>\tau_N)\big)\\ &= \big(Z_{s\wedge\tau_N}\,\mathbf{1}(s\le\tau_N)\big)\vee\big(Z_{\tau_N}\,\mathbf{1}(s>\tau_N)\big) = Z_{s\wedge\tau_N}. \end{aligned}$$

□

We now study the uniqueness issue. We introduce for future use for $t\in\mathbb{R}_+$

$$\mathbf{I}_t(\mathbf{x}) = \sup_{(\lambda(s))\in\Lambda_0}\Big(\int_0^t \lambda(s)\cdot d\mathbf{x}_s - dG_s(\lambda(s);\mathbf{x})\Big), \quad (2.8.4)$$

$$\mathbf{I}_{x,t}(\mathbf{x}) = \begin{cases}\mathbf{I}_t(\mathbf{x}), & \text{if } \mathbf{x}_0 = x,\\ \infty, & \text{otherwise,}\end{cases} \quad (2.8.5)$$

$$\mathbf{\Pi}_{x,t}(\mathbf{x}) = \exp(-\mathbf{I}_{x,t}(\mathbf{x})),\ \mathbf{\Pi}_{x,t}(\Gamma) = \sup_{\mathbf{x}\in\mathbb{C}}\mathbf{\Pi}_{x,t}(\mathbf{x}),\ \Gamma\subset\mathbb{C}. \quad (2.8.6)$$

As we have seen, $\mathbf{\Pi}_x$ is a natural candidate for a solution to (x,G). We first show that it is a tight τ-smooth idempotent measure on $\mathbb{C}$ under fairly general assumptions.

Definition 2.8.11. *We say that $G(\lambda)$ satisfies the linear-growth condition if there exist $\mathbb{R}_+$-valued, increasing and continuous in t functions $F^l(\lambda) = (F^l_t(\lambda), t\in\mathbb{R}_+)$, $\lambda\in\mathbb{R}^d$, such that $F^l_0(\lambda) = F^l_t(0) = 0$ and for some increasing function $k_t\in\mathbb{R}_+$ we have for all $0\le s<t$, $\mathbf{x}\in\mathbb{C}$ and $\lambda\in\mathbb{R}^d$*

$$G_t(\lambda;\mathbf{x}) - G_s(\lambda;\mathbf{x}) \le F^l_t(\lambda(1+k_t\mathbf{x}^*_t)) - F^l_s(\lambda(1+k_t\mathbf{x}^*_t)).$$

Lemma 2.8.12. *Let $G(\lambda)$ satisfy the linear-growth condition. Then $\mathbf{\Pi}_x$ is a tight τ-smooth idempotent measure on $\mathbb{C}$.*

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Proof. With no loss of generality we assume that $x = 0$ and check that $\mathbf{I}_0(\mathbf{x}) = -\ln \mathbf{\Pi}_0(\mathbf{x})$ defined by (2.7.2), (2.7.3), and (2.7.5), where $x = 0$, is a tight rate function on $\mathbb{C}$ in the sense of Remark 1.7.17, i.e., the sets $L_{\mathbf{I}_0}(a) = \{\mathbf{x} \in \mathbb{C} : \mathbf{I}_0(\mathbf{x}) \le a\}$ are compact for all $a \in \mathbb{R}_+$. Let $\mathbf{x} \in L_{\mathbf{I}_0}(a)$. Then $\mathbf{x}_0 = 0$ by (2.7.5). By (2.7.2), (2.7.3), (2.7.5), and the linear-growth condition we have for $0 \le s < t$, denoting by e_i, $i = 1, \dots, 2d$, the d-vector, whose $\lfloor (i+1)/2 \rfloor$th entry is 1 if i is odd, -1 if i is even, and the rest of the entries are equal to 0,

$$\frac{|\mathbf{x}_t - \mathbf{x}_s|}{1 + k_t \mathbf{x}_t^*} \le d \max_{i=1,\dots,2d} \frac{e_i \cdot (\mathbf{x}_t - \mathbf{x}_s)}{1 + k_t \mathbf{x}_t^*}$$
$$\le d \max_{i=1,\dots,2d} \Big(G_t\Big(\frac{e_i}{1 + k_t \mathbf{x}_t^*}\Big) - G_s\Big(\frac{e_i}{1 + k_t \mathbf{x}_t^*}\Big) + a \Big)$$
$$\le d \max_{i=1,\dots,2d} \big(F_t^l(e_i) - F_s^l(e_i) \big) + da. \quad (2.8.7)$$

Since $F_t^l(\lambda)$ is increasing in t, $\mathbf{x}$ has bounded variation over bounded intervals; therefore, since k_t is increasing, for $T > 0$

$$\int_0^T \frac{d\,\mathrm{Var}_t \mathbf{x}}{1 + k_T \mathbf{x}_t^*} \le d \sum_{i=1}^{2d} F_T^l(e_i) + da$$

(recall that $F_0^l(\lambda) = 0$), which implies that

$$\int_0^T \frac{d\mathbf{x}_t^*}{1 + c_T \mathbf{x}_t^*} \le d \sum_{i=1}^{2d} F_T^l(e_i) + da, \quad (2.8.8)$$

where $c_T = k_T \vee 1$. By (2.8.8) and the fact that $\mathbf{x}_0 = 0$ we deduce that

$$\ln(1 + c_T \mathbf{x}_T^*) \le c_T d \Big(\sum_{i=1}^{2d} F_T^l(e_i) + a\Big). \quad (2.8.9)$$

Hence,

$$\sup_{\mathbf{x} \in L_{\mathbf{I}_0}(a)} \mathbf{x}_T^* < \infty. \quad (2.8.10)$$

In analogy with (2.8.7) we can write for $T > 0$ and $b > 0$

$$b\,\frac{|\mathbf{x}_t - \mathbf{x}_s|}{1 + k_t \mathbf{x}_t^*} \le d \max_{i=1,\dots,2d} \frac{b\, e_i \cdot (\mathbf{x}_t - \mathbf{x}_s)}{1 + k_t \mathbf{x}_t^*}$$
$$\le d \max_{i=1,\dots,2d} \big(F_t^l(be_i) - F_s^l(be_i) \big) + da.$$

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Therefore, for $\delta > 0$

$$\sup_{\mathbf{x}\in L_{\mathbf{I}_0}(a)} \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} |\mathbf{x}_t - \mathbf{x}_s|$$
$$\le \frac{d}{b}\big[\max_{i=1,\dots,2d} \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} \big|F_t^l(be_i)-F_s^l(be_i)\big|+a\big](1+k_T \sup_{\mathbf{x}\in L_{\mathbf{I}_0}(a)} \mathbf{x}_T^*),$$

so by continuity of $F_t^l(\lambda)$ in t

$$\limsup_{\delta\to 0} \sup_{\mathbf{x}\in L_{\mathbf{I}_0}(a)} \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} |\mathbf{x}_t-\mathbf{x}_s| \le \frac{da}{b}\left(1 + k_T \sup_{\mathbf{x}\in L_{\mathbf{I}_0}(a)} \mathbf{x}_T^*\right).$$

Letting $b \to \infty$ and using (2.8.10), we conclude that the left-hand side of the latter inequality is 0. An application of Arzelà–Ascoli's theorem ends the proof. □

Remark 2.8.13. *The above proof shows that if $\mathbf{I}_{x,T}(\mathbf{x}) \le a$, then the bound* (2.8.9) *holds, which implies that the sets $\cup_{s\le t}\{\mathbf{x}_s^* : \mathbf{\Pi}_{x,s}(\mathbf{x}) \ge \epsilon\}$ are bounded under the hypotheses for $t \in \mathbb{R}_+$ and $\epsilon \in (0,1]$.*

By Lemma 2.7.11 if Π solves the maxingale problem (x, G), then

$$\Pi(\mathbf{x}) \le \mathbf{\Pi}_x(\mathbf{x}),\ \mathbf{x} \in \mathbb{C}. \tag{2.8.11}$$

Our goal is to establish conditions when we actually have equality above. Clearly, we need only to be concerned with the case $\mathbf{\Pi}_x(\mathbf{x}) > 0$. We assume in the sequel that the cumulant is given by (2.7.7) and (2.7.13), where $(b_s(\mathbf{x}))$ is $\mathbf{C}$-progressively measurable, $\int_0^t|b_s(\mathbf{x})|ds < \infty$, $(\hat{g}_s(\lambda;\mathbf{x}))$ is non-negative, continuous in λ and $\overline{\mathcal{B}}([0,t])\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{C}_t/\mathcal{B}(\mathbb{R}_+)$–measurable as a map from $[0,t]\times\mathbb{R}^d\times\mathbb{C}$ to $\mathbb{R}_+$, $\hat{g}_s(0;\mathbf{x}) = 0$, and $\int_0^t \sup_{|\lambda|=A} \hat{g}_s(\lambda;\mathbf{x})\,ds < \infty$ for $t \in \mathbb{R}_+$, $A \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$. By Lemma 2.7.12 if $\mathbf{\Pi}_x(\mathbf{x}) > 0$, then $\mathbf{x}$ is absolutely continuous. We note that by (2.7.2), (2.7.5), (2.7.6), (2.8.4), (2.8.5), and (2.8.6)

$$\mathbf{\Pi}_{x,t}(\mathbf{x}) \downarrow \mathbf{\Pi}_x(\mathbf{x}),\ \mathbf{x} \in \mathbb{C},\quad \text{as } t \to \infty. \tag{2.8.12}$$

The argument of the proof of Lemma 2.7.12 also shows that

$$\mathbf{I}_t(\mathbf{x}) = \begin{cases} \int_0^t h_s(\dot{\mathbf{x}}_s;\mathbf{x})\,ds & \text{if } \mathbf{x} \text{ is absolutely continuous on } [0,t], \\ \infty & \text{otherwise.} \end{cases} \tag{2.8.13}$$

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For the following theorem we recall that $Z(\overline{\lambda}) = (Z_t(\overline{\lambda}, \mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ is given by (2.7.20) if the integral on the right-hand side is well defined and finite, and $Z_t(\overline{\lambda}, \mathbf{x}) = 0$ otherwise.

Theorem 2.8.14. *Let deviability* Π *solve the maxingale problem* (x, G) *and* $\hat{\mathbf{x}} \in \mathbb{C}$. *Let there exist an* $\mathbb{R}^d$*–valued function* $\hat{\lambda} = (\hat{\lambda}(s, \mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ *with the following properties.*

a) $Z(\hat{\lambda})$ *is a strictly Luzin-continuous local exponential maxingale on* $(\mathbb{C}, \mathbf{C}^{\Pi}, \Pi)$ *and admits a localising sequence of strictly Luzin* $\mathbf{C}$*-stopping times;*

b) if $\tilde{\mathbf{x}}$ *is such that* $\mathbf{\Pi}_x(\tilde{\mathbf{x}}) > 0$ *and a.e. in* $s \in [0, t]$

$$\hat{\lambda}(s, \tilde{\mathbf{x}}) \cdot \dot{\tilde{\mathbf{x}}}_s - g_s(\hat{\lambda}(s, \tilde{\mathbf{x}}); \tilde{\mathbf{x}}) = \sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot \dot{\tilde{\mathbf{x}}}_s - g_s(\lambda; \tilde{\mathbf{x}})), \tag{2.8.14}$$

then $\tilde{\mathbf{x}}_s = \hat{\mathbf{x}}_s$, $s \in [0, t]$, *where* $t \in \mathbb{R}_+$.

Then $\Pi(p_s^{-1} \circ p_s \hat{\mathbf{x}}) = \mathbf{\Pi}_{x,s}(\hat{\mathbf{x}})$, $s \in \mathbb{R}_+$, *and* $\Pi(\hat{\mathbf{x}}) = \mathbf{\Pi}_x(\hat{\mathbf{x}})$.

Proof. Let $\{\tau_N,\ N \in \mathbb{N}\}$ be a localising sequence of strictly Luzin $\mathbf{C}$-stopping times for $Z(\hat{\lambda})$ so that the $(Z_{t \wedge \tau_N}(\hat{\lambda}),\ t \in \mathbb{R}_+)$ are strictly Luzin uniformly maximable exponential maxingales under Π. Since $S_\Pi Z_{t \wedge \tau_N}(\hat{\lambda}) = 1$ and $\lim_{A \to \infty} S_\Pi Z_{t \wedge \tau_N}(\hat{\lambda})\, \mathbf{1}(Z_{t \wedge \tau_N}(\hat{\lambda}) > A) = 0$, for A large enough $S_\Pi Z_{t \wedge \tau_N}(\hat{\lambda})\, \mathbf{1}(Z_{t \wedge \tau_N}(\hat{\lambda}) \le A) = 1$ so that the inequality

$$S_\Pi Z_{t \wedge \tau_N}(\hat{\lambda})\, \mathbf{1}(Z_{t \wedge \tau_N}(\hat{\lambda}) \le A) \le \Big[\sup_{\mathbf{x} \in K_\Pi(a)} Z_{t \wedge \tau_N(\mathbf{x})}(\hat{\lambda}(\mathbf{x}), \mathbf{x}) \Pi(\mathbf{x}) \Big] \vee (aA),$$

where $a \in (0, 1]$, implies that for a small enough $\sup_{\mathbf{x} \in K_\Pi(a)} Z_{t \wedge \tau_N(\mathbf{x})}(\hat{\lambda}(\mathbf{x}), \mathbf{x}) \Pi(\mathbf{x}) \ge 1$. On the other hand, by the definitions of $Z(\hat{\lambda})$ and $\mathbf{\Pi}_x$, and (2.8.11)

$$Z_{t \wedge \tau_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}), \mathbf{x}) \Pi(\mathbf{x}) \le Z_{t \wedge \tau_N(\mathbf{x})}(\overline{\lambda}(\mathbf{x}), \mathbf{x}) \mathbf{\Pi}_x(\mathbf{x}) \le 1. \tag{2.8.15}$$

Thus, if $a > 0$ is small enough, then

$$\sup_{\mathbf{x} \in K_\Pi(a)} Z_{t \wedge \tau_N(\mathbf{x})}(\hat{\lambda}(\mathbf{x}), \mathbf{x}) \Pi(\mathbf{x}) = 1. \tag{2.8.16}$$

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Being a strictly Luzin idempotent variable on $(\mathbb{C}, \Pi)$, $Z_{t\wedge\tau_N(\mathbf{x})}(\hat{\lambda}(\mathbf{x}), \mathbf{x})$ is continuous in $\mathbf{x}$ when restricted to $K_\Pi(a)$. Also $\Pi(\mathbf{x})$, being a deviability density, is upper semi-continuous. Therefore, the product $Z_{t\wedge\tau_N(\mathbf{x})}(\hat{\lambda}(\mathbf{x}), \mathbf{x})\Pi(\mathbf{x})$ is upper semi-continuous when restricted to $K_\Pi(a)$. As the latter set is compact, the supremum in (2.8.16) is attained, so for some $\mathbf{x}^N \in \mathbb{C}$

$$Z_{t\wedge\tau_N(\mathbf{x}^N)}(\hat{\lambda}(\mathbf{x}^N), \mathbf{x}^N)\Pi(\mathbf{x}^N) = 1 \tag{2.8.17}$$

(we suppress in $\mathbf{x}^N$ dependence on t). Then by (2.8.15)

$$Z_{t\wedge\tau_N(\mathbf{x}^N)}(\hat{\lambda}(\mathbf{x}^N), \mathbf{x}^N)\mathbf{\Pi}_x(\mathbf{x}^N) = 1. \tag{2.8.18}$$

In particular, $\mathbf{\Pi}_x(\mathbf{x}^N) > 0$ so that by (2.7.6) and Lemma 2.7.12 $\mathbf{x}^N$ is absolutely continuous and $\mathbf{x}_0^N = x$. Thus, by the definitions of Z and $\mathbf{\Pi}_x$ as well as Lemma 2.7.12 we have that almost everywhere on $[0, t\wedge\tau_N(\mathbf{x}^N)]$

$$\hat{\lambda}(s, \mathbf{x}^N)\cdot\dot{\mathbf{x}}_s^N - g_s(\hat{\lambda}(s, \mathbf{x}^N); \mathbf{x}^N) = \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot\dot{\mathbf{x}}_s^N - g_s(\lambda; \mathbf{x}^N)) \tag{2.8.19}$$

and

$$\int_{t\wedge\tau_N(\mathbf{x}^N)}^{\infty} \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot\dot{\mathbf{x}}_s^N - g_s(\lambda; \mathbf{x}^N))\,ds = 0. \tag{2.8.20}$$

Hence, by the requirements on $\hat{\lambda}$ we conclude that $\mathbf{x}_s^N = \hat{\mathbf{x}}_s$, $0 \le s \le t\wedge\tau_N(\mathbf{x}^N)$. Since $t\wedge\tau_N(\mathbf{x})$ is a $\mathbf{C}$–stopping time, by Lemma 2.2.21 $t\wedge\tau_N(\mathbf{x}^N) = t\wedge\tau_N(\hat{\mathbf{x}})$. Therefore, by (2.7.5), (2.7.6), Lemma 2.7.12, (2.8.5), (2.8.6), (2.8.20), and (2.8.13)

$$\mathbf{\Pi}_{x,t\wedge\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}}) = \mathbf{\Pi}_{x,t\wedge\tau_N(\mathbf{x}^N)}(\mathbf{x}^N) = \mathbf{\Pi}_x(\mathbf{x}^N). \tag{2.8.21}$$

Thus, by (2.8.17), (2.8.18) and (2.8.21), $\Pi(\mathbf{x}^N) = \mathbf{\Pi}_x(\mathbf{x}^N) = \mathbf{\Pi}_{x,t\wedge\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}})$, which implies, since $\mathbf{x}^N \in p_{t\wedge\tau_N(\hat{\mathbf{x}})}^{-1} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\,\hat{\mathbf{x}}$, that

$$\Pi(p_{t\wedge\tau_N(\hat{\mathbf{x}})}^{-1}\circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\hat{\mathbf{x}}) \ge \Pi(\mathbf{x}^N) = \mathbf{\Pi}_{x,t\wedge\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}}). \tag{2.8.22}$$

On the other hand, by (2.8.12) $\mathbf{\Pi}_{x,t}(\mathbf{x}) \ge \mathbf{\Pi}_x(\mathbf{x})$ for $\mathbf{x} \in \mathbb{C}$ and

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$t \in \mathbb{R}_+$, so by (2.8.11), (2.8.4), (2.8.5), (2.8.6), and Lemma 2.2.21

$$\Pi(p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\,\hat{\mathbf{x}}) = \sup\{\Pi(\mathbf{x}); \mathbf{x} \in p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\,\hat{\mathbf{x}}\}$$
$$\leq \sup\{\mathbf{\Pi}_x(\mathbf{x}); \mathbf{x} \in p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\,\hat{\mathbf{x}}\}$$
$$\leq \sup\{\mathbf{\Pi}_{x,t\wedge\tau_N(\mathbf{x})}(\mathbf{x}); \mathbf{x} \in p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\,\hat{\mathbf{x}}\}$$
$$= \mathbf{\Pi}_{x,t\wedge\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}}).$$

Comparing this with (2.8.22) yields

$$\Pi(p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\hat{\mathbf{x}}) = \mathbf{\Pi}_{x,t\wedge\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}}). \tag{2.8.23}$$

Since $\tau_N(\hat{\mathbf{x}}) \to \infty$, it follows that $\cap_{N=1}^{\infty} p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\hat{\mathbf{x}} = p_t^{-1} \circ p_t\hat{\mathbf{x}}$; so since the sets $p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\,\hat{\mathbf{x}}$ are closed, by the τ-smoothness property of deviability

$$\Pi(p_t^{-1} \circ p_t\hat{\mathbf{x}}) = \lim_{N\to\infty} \Pi(p^{-1}_{t\wedge\tau_N(\hat{\mathbf{x}})} \circ p_{t\wedge\tau_N(\hat{\mathbf{x}})}\hat{\mathbf{x}}). \tag{2.8.24}$$

Also the convergence $\tau_N(\hat{\mathbf{x}}) \to \infty$, (2.8.4), (2.8.5), and (2.8.6) yield

$$\mathbf{\Pi}_{x,t}(\hat{\mathbf{x}}) = \lim_{N\to\infty} \mathbf{\Pi}_{x,t\wedge\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}}). \tag{2.8.25}$$

Putting together (2.8.23), (2.8.24) and (2.8.25) results in the equality $\Pi(p_t^{-1} \circ p_t\hat{\mathbf{x}}) = \mathbf{\Pi}_{x,t}(\hat{\mathbf{x}})$. The final assertion follows by taking in both sides of the latter equality the limit as $t \to \infty$, using τ-smoothness of Π and (2.8.12). □

Remark 2.8.15. *In the sequel we routinely omit indications that certain relations hold almost everywhere with respect to Lebesgue measure when this is understood.*

Conditions for $Z(\overline{\lambda})$ to be a strictly Luzin-continuous exponential maxingale on $(\mathbb{C}, \mathbf{C}^{\Pi}, \Pi)$ are given in Theorem 2.7.16. However, the deviability Π is not known to us, so it would be difficult to verify the hypotheses of the theorem. Therefore, we introduce somewhat cruder conditions, which have the advantage of not involving Π. The following conditions replace conditions (Π_I) and (Π_{II}).

I The function $b_s(\mathbf{x})$ is continuous in $\mathbf{x}$ and

$$\int_0^t \sup_{\mathbf{x}\in K} |b_s(\mathbf{x})|\, ds < \infty$$

for all compacts $K \subset \mathbb{C}$ and $t \in \mathbb{R}_+$.

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II The function $\hat{g}_s(\lambda;\mathbf{x})$ is convex in $\lambda \in \mathbb{R}^d$, continuous in $(\lambda, \mathbf{x}) \in \mathbb{R}^d \times \mathbb{C}$, and

$$\sup_{|\lambda|\le A}\sup_{s\le t}\sup_{\mathbf{x}\in K}\hat{g}_s(\lambda;\mathbf{x}) < \infty, \qquad \lim_{\lambda\to 0}\sup_{s\le t}\sup_{\mathbf{x}\in K}\hat{g}_s(\lambda;\mathbf{x}) = 0$$

for all compacts $K \subset \mathbb{C}$, $t \in \mathbb{R}_+$ and $A \in \mathbb{R}_+$.

If $g_s(\lambda;\mathbf{x})$ has the form (2.7.55), Lemma 2.7.27 provides sufficient conditions for conditions I and II to hold in terms of characteristics.

Lemma 2.8.16. *Let $g_s(\lambda;\mathbf{x})$ be given by* (2.7.55), *where $\big(c_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}\big)$, $\big(\nu_s(\Gamma;\mathbf{x}),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d), \mathbf{x} \in \mathbb{C}\big)$, and $\big(\hat{\nu}_s(\Gamma;\mathbf{x}),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d), \mathbf{x} \in \mathbb{C}\big)$ are as defined in Section 2.7 with $\mathbf{C}^{\Pi}$-progressive measurability replaced by $\mathbf{C}$-progressive measurability. Let $b_s(\mathbf{x})$ and $c_s(\mathbf{x})$ be continuous in $\mathbf{x}$, and $\int_{\mathbb{R}^d}(\exp(\lambda\cdot x) - 1 - \lambda\cdot x)\,\nu_s(dx;\mathbf{x})$ and $\int_{\mathbb{R}^d}\exp(\lambda\cdot x)\,\hat{\nu}_s(dx;\mathbf{x})$ be continuous in $(\lambda,\mathbf{x})$. If for all compacts $K \subset \mathbb{C}$, $A \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$*

$$\int_0^t \sup_{\mathbf{x}\in K}|b_s(\mathbf{x})|\,ds < \infty, \quad \sup_{s\le t}\sup_{\mathbf{x}\in K}\|c_s(\mathbf{x})\| < \infty,$$

$$\sup_{s\le t}\sup_{\mathbf{x}\in K}\int_{\mathbb{R}^d}(e^{A|x|} - 1 - A|x|)\,\nu_s(dx;\mathbf{x}) < \infty,$$

$$\lim_{\lambda\to 0}\sup_{s\le t}\sup_{\mathbf{x}\in K}\int_{\mathbb{R}^d}(e^{\lambda\cdot x} - 1 - \lambda\cdot x)\,\nu_s(dx;\mathbf{x}) = 0,$$

$$\sup_{s\le t}\sup_{\mathbf{x}\in K}\int_{\mathbb{R}^d} e^{A|x|}\,\hat{\nu}_s(dx;\mathbf{x})\,ds < \infty,$$

$$\lim_{\lambda\to 0}\sup_{s\le t}\sup_{\mathbf{x}\in K}\left|\int_{\mathbb{R}^d}\big(e^{\lambda\cdot x} - 1\big)\,\hat{\nu}_s(dx;\mathbf{x})\right| = 0,$$

then conditions I and II are satisfied.

We now define the class of integrands.

Definition 2.8.17. *Let $\hat{\Lambda}$ denote the set of all $\mathbb{R}^d$–valued $\mathbf{C}$–progressively measurable idempotent processes $\overline{\lambda} = (\lambda(t,\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ such that the $\lambda(t,\mathbf{x})$ are continuous in $\mathbf{x}$, for $\alpha \in \mathbb{R}$,*

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$t \in \mathbb{R}_+$ *and* $\mathbf{x} \in \mathbb{C}$

$$\int_0^t \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})\,ds < \infty$$

and for every compact $K \subset \mathbb{C}$

$$\sup_{\mathbf{x}\in K} \int_0^t \hat{g}_s(\alpha\lambda(s,\mathbf{x});\mathbf{x})\,\mathbf{1}(|\lambda(s,\mathbf{x})| > A)\,ds \to 0 \text{ as } A \to \infty.$$

Obviously, $\hat{\Lambda} \subset \hat{\Lambda}_\Pi$ for every deviability Π on $\mathbb{C}$.

Remark 2.8.18. *If* $\mathbf{\Pi}_x$ *is a deviability, then both in conditions I and II and the definition of* $\hat{\Lambda}$ *we could consider only compacts* $K_{\mathbf{\Pi}_x}(a)$, *where* $a \in (0,1]$ *and* $K_{\mathbf{\Pi}_x}(a) = \{\mathbf{x} : \mathbf{\Pi}_x(\mathbf{x}) \geq a\}$, *and require that* $\mathbf{x}$ *be such that* $\mathbf{\Pi}_x(\mathbf{x}) > 0$.

The following consequence of Theorem 2.7.16 allows us to check that $Z(\overline{\lambda})$ satisfies condition a) of Theorem 2.8.14.

Theorem 2.8.19. *Let conditions I and II hold. If* $\overline{\lambda} \in \hat{\Lambda}$ *and* Π *solves the maxingale problem* (x, G), *then the idempotent process* $Z(\overline{\lambda})$ *is a strictly Luzin-continuous local exponential maxingale on* $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$, *which admits a localising sequence of strictly Luzin* $\mathbf{C}$-*stopping times.*

Proof. By Theorem 2.7.16 $Z(\overline{\lambda})$ is a strictly Luzin-continuous local exponential maxingale on $(\mathbb{C}, \mathbf{C}^\Pi, \Pi)$. It is straightforward to check that the sequence $\{\tau_N,\, N \in \mathbb{N}\}$ defined by

$$\tau_N(\mathbf{x}) = \inf\{t \in \mathbb{R}_+ : \int_0^t \hat{g}_s(2\lambda(s,\mathbf{x});\mathbf{x})\,ds + t \geq N\}$$

is a localising sequence of strictly Luzin-continuous $\mathbf{C}$-stopping times. ☐

We now consider the issue of choosing the function $\hat{\lambda}$ to satisfy condition b) of Theorem 2.8.14. There are two ways of doing this as is shown by the following lemma. We denote by $\nabla g_s(\lambda;\mathbf{x})$ the gradient of $g_s(\lambda;\mathbf{x})$ with respect to λ if it is well defined.

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Lemma 2.8.20. *Let $g_s(\lambda;\mathbf{x})$ be differentiable in λ. Let $\hat{\mathbf{x}} \in \mathbb{C}$ be absolutely continuous.*

1. *If a function $\hat{\lambda} = (\hat{\lambda}(s,\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ is such that*

$$\dot{\hat{\mathbf{x}}}_s = \nabla g_s(\hat{\lambda}(s,\mathbf{x});\mathbf{x})$$

for $\mathbf{\Pi}_x$-almost all $\mathbf{x}$ and almost all $s \in \mathbb{R}_+$, then condition b) of Theorem 2.8.14 is satisfied.

2. *If a function $\hat{\lambda} = (\hat{\lambda}(s),\ s \in \mathbb{R}_+)$ is such that the equation*

$$\dot{\mathbf{x}}_s = \nabla g_s(\hat{\lambda}(s);\mathbf{x}),\ \mathbf{x}_0 = x,$$

has the only solution $\mathbf{x} = \hat{\mathbf{x}}$, then condition b) of Theorem 2.8.14 is satisfied.

Proof. By the requirements on $\tilde{\mathbf{x}}$ in part b) of Theorem 2.8.14 and the necessary condition for attaining supremum $\dot{\tilde{\mathbf{x}}}_s = \nabla g_s(\hat{\lambda}(s,\tilde{\mathbf{x}});\tilde{\mathbf{x}})$. Hence, if $\hat{\lambda}$ is from part 1 of the lemma, then $\dot{\hat{\mathbf{x}}}_s = \dot{\tilde{\mathbf{x}}}_s$ so that since $\hat{\mathbf{x}}_0 = \tilde{\mathbf{x}}_0 = x$ we conclude that $\hat{\mathbf{x}} = \tilde{\mathbf{x}}$. If $\hat{\lambda}$ is from part 2 of the lemma, then $\hat{\mathbf{x}} = \tilde{\mathbf{x}}$ by definition. □

We are able to obtain somewhat general uniqueness results only for the case where $\hat{\lambda}$ is chosen as in part 1 of Lemma 2.8.20, so we concentrate on that case. However, we give examples that show an application of the approach in part 2. We next state a uniqueness result for a "diffusion" maxingale problems.

Theorem 2.8.21. *Let the canonical process X be a semimaxingale under Π with local characteristics $(b, c, 0, 0)$ starting at x. Let the following conditions hold:*

1. *the functions $b_s(\mathbf{x})$ and $c_s(\mathbf{x})$ are continuous in $\mathbf{x} \in \mathbb{C}$,*

2. *for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$*

$$\int_0^t \sup_{\mathbf{x}\in K} |b_s(\mathbf{x})|^2 ds < \infty, \quad \sup_{s\le t}\sup_{\mathbf{x}\in K} \|c_s(\mathbf{x})\| < \infty,$$

3. *for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$*

$$\inf_{s\le t}\inf_{\mathbf{x}\in K}\inf_{\substack{\lambda\in\mathbb{R}^d:\\ |\lambda|=1}} \lambda\cdot c_s(\mathbf{x})\lambda > 0.$$

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Then $\Pi = \mathbf{\Pi}_x$.

Proof. We first note that $\hat{g}_t(\lambda; \mathbf{x}) = \lambda \cdot c_t(\mathbf{x})\lambda/2$. Let $\hat{\mathbf{x}} \in \mathbb{C}$ be such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$. We prove that $\Pi(\hat{\mathbf{x}}) = \mathbf{\Pi}_x(\hat{\mathbf{x}})$. We apply Theorem 2.8.14. By Lemma 2.8.16 conditions I and II are satisfied. Since $\hat{\mathbf{x}}$ is absolutely continuous and $c_t(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathbb{C}$, we can define

$$\hat{\lambda}(t, \mathbf{x}) = c_t(\mathbf{x})^{-1} \cdot (\dot{\hat{\mathbf{x}}}_t - b_t(\mathbf{x})). \tag{2.8.26}$$

Since $\dot{\hat{\mathbf{x}}}_t = \nabla g_t(\hat{\lambda}(t, \mathbf{x}); \mathbf{x})$ for all $\mathbf{x}$, by Lemma 2.8.20 $\hat{\lambda}$ satisfies the condition of part b) of Theorem 2.8.14, so by part a) of Theorem 2.8.14 and Theorem 2.8.19 it suffices to check that $\hat{\lambda} \in \hat{\Lambda}$. Firstly, we note that for $\mathbf{x} \in K$, where K is compact,

$$\int_0^t \sup_{\mathbf{x} \in K} |\dot{\hat{\mathbf{x}}}_s - b_s(\mathbf{x})|^2 ds < \infty. \tag{2.8.27}$$

Indeed, we have

$$\int_0^t \sup_{\mathbf{x} \in K} |\dot{\hat{\mathbf{x}}}_s - b_s(\mathbf{x})|^2 ds \leq 2 \int_0^t |\dot{\hat{\mathbf{x}}}_s - b_s(\hat{\mathbf{x}})|^2 ds + 2 \int_0^t \sup_{\mathbf{x} \in K} |b_s(\hat{\mathbf{x}}) - b_s(\mathbf{x})|^2 ds. \tag{2.8.28}$$

Since by Lemma 2.7.12

$$\mathbf{I}(\hat{\mathbf{x}}) = \frac{1}{2} \int_0^\infty (\dot{\hat{\mathbf{x}}}_s - b_s(\hat{\mathbf{x}})) \cdot c_s(\hat{\mathbf{x}})^{-1} (\dot{\hat{\mathbf{x}}}_s - b_s(\hat{\mathbf{x}}))\, ds \geq \frac{1}{2} \big(\sup_{s \leq t} \|c_s(\hat{\mathbf{x}})\|\big)^{-1} \int_0^t |\dot{\hat{\mathbf{x}}}_s - b_s(\hat{\mathbf{x}})|^2\, ds,$$

we conclude that the first term on the right of (2.8.28) is finite. The second one is finite by hypotheses. Inequality (2.8.27) is proved.

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Next, we have by (2.8.26) for $\mathbf{x} \in K_{\Pi}(a)$ and $\alpha \in \mathbb{R}_+$, denoting $c = \inf_{s \le t} \inf_{\mathbf{x} \in K} \inf_{\substack{\lambda \in \mathbb{R}^d: \\ |\lambda|=1}} \lambda \cdot c_s(\mathbf{x})\lambda$, that

$$\int_0^t \hat{g}_s(\alpha\hat{\lambda}(s,\mathbf{x});\mathbf{x})\,\mathbf{1}(|\hat{\lambda}(s,\mathbf{x})| > A)ds$$
$$= \frac{\alpha^2}{2}\int_0^t (\dot{\mathbf{x}}_s - b_s(\mathbf{x}))\cdot c_s(\mathbf{x})^{-1}(\dot{\mathbf{x}}_s - b_s(\mathbf{x}))\,\mathbf{1}(|\hat{\lambda}(s,\mathbf{x})| > A)ds$$
$$\le \frac{\alpha^2}{2}c^{-1}\int_0^t |\dot{\mathbf{x}}_s - b_s(\mathbf{x})|^2\,\mathbf{1}(|\hat{\lambda}(s,\mathbf{x})| > A)ds$$

so that by (2.8.27) and absolute continuity of the Lebesgue integral the required limit

$$\lim_{A\to\infty}\sup_{\mathbf{x}\in K}\int_0^t \hat{g}_s(\alpha\hat{\lambda}(s,\mathbf{x});\mathbf{x})\,\mathbf{1}(|\hat{\lambda}(s,\mathbf{x})| > A)ds = 0$$

would follow by

$$\lim_{A\to\infty}\sup_{\mathbf{x}\in K}\int_0^t \mathbf{1}(|\hat{\lambda}(s,\mathbf{x})| > A)ds = 0.$$

The latter limit follows since by (2.8.26), (2.8.27) and hypotheses

$$\sup_{\mathbf{x}\in K}\int_0^t |\hat{\lambda}(s,\mathbf{x})|^2 ds \le c^{-2}\int_0^t \sup_{\mathbf{x}\in K}|\dot{\mathbf{x}}_s - b_s(\mathbf{x})|^2 ds < \infty.$$

□

As a consequence of this result, Theorem 2.6.24 and Theorem 2.6.30, we have the following existence and uniqueness result for idempotent Ito equations.

Theorem 2.8.22. *Let* $(b_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ *and* $(\sigma_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ *be respective* $\mathbb{R}^d$*-valued and* $\mathbb{R}^{d\times d}$*-valued* $\mathbf{C}$*-progressively measurable idempotent processes. Let the following conditions hold:*

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1. *$b_s(\mathbf{x})$ and $\sigma_s(\mathbf{x})$ are continuous in $\mathbf{x} \in \mathbb{C}$,*
2. *linear growth: for every $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$*

$$\int_0^t \frac{|b_s(\mathbf{x})|^2}{1+\mathbf{x}_s^{*2}}\,ds + \sup_{s\le t} \frac{\|\sigma_s(\mathbf{x})\|^2}{1+\mathbf{x}_s^{*2}} < \infty,$$

3. *for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$*

$$\inf_{s\le t}\inf_{\mathbf{x}\in K}\inf_{\substack{\lambda\in\mathbb{R}^d:\\ |\lambda|=1}} \lambda\cdot\sigma_s(\mathbf{x})\sigma_s(\mathbf{x})^T\lambda > 0.$$

Then the equation

$$\dot{X}_t = b_t(X) + \sigma_t(X)\dot{W}_t,\ X_0 = x,$$

has a unique Luzin solution. The idempotent distribution of X is given by

$$\Pi^X(\mathbf{x}) = \exp\Bigl(-\frac{1}{2}\int_0^\infty (\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\cdot\bigl(\sigma_t(\mathbf{x})\sigma_t(\mathbf{x})^T\bigr)^{-1}(\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\,dt\Bigr)$$

if $\mathbf{x}$ is absolutely continuous and $\mathbf{x}_0 = x$, and $\Pi^X(\mathbf{x}) = 0$ otherwise.

Proof. By Theorem 2.6.24 the equation has a Luzin solution X. By Theorem 2.6.30 and Theorem 2.8.21 uniqueness holds. The form of Π^X is given in Lemma 2.6.19. □

Remark 2.8.23. *One can weaken the conditions of Theorem 2.8.22 by requiring that the non-degeneracy condition 3 in the hypotheses hold for compacts $K_{\mathbf{\Pi}_x}(a)$, where $a \in (0,1]$, and $\mathbf{x}$ in conditions 1 and 2 is such that $\mathbf{\Pi}_x(\mathbf{x}) > 0$.*

If the function $g_s(\lambda;\mathbf{x})$ is more general than in Theorem 2.8.21, we cannot apply Theorem 2.8.14 to all $\mathbf{x} \in \mathbb{C}$ such that $\mathbf{\Pi}_x(\mathbf{x}) > 0$, so we have to introduce additional regularity conditions, e.g., require that the derivative $\dot{\mathbf{x}}_s$ be locally bounded. As for “nonregular” $\mathbf{x}$, a way to establish for them the equality $\Pi(\mathbf{x}) = \mathbf{\Pi}_x(\mathbf{x})$ is to see to it that each such $\mathbf{x}$ can properly be approximated by “regular” $\mathbf{x}$. Below we assume without further mentioning that Π is a solution to the maxingale problem (x, G).

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Definition 2.8.24. *Let $D \subset \mathbb{C}$. We define the $\mathbf{\Pi}_x$–closure of D as the set of all $\mathbf{x} \in \mathbb{C}$ such that $\mathbf{\Pi}_x(\mathbf{x}) > 0$ for which there exists a sequence $\mathbf{x}^k \in D$ such that $\mathbf{x}^k \to \mathbf{x}$ and $\mathbf{\Pi}_{x,t}(\mathbf{x}^k) \to \mathbf{\Pi}_{x,t}(\mathbf{x})$ as $k \to \infty$ for all $t \in \mathbb{R}_+$. We say that D is $\mathbf{\Pi}_x$-dense in $\mathbb{C}$ if its $\mathbf{\Pi}_x$–closure coincides with the set $\{\mathbf{x} \in \mathbb{C} : \mathbf{\Pi}_x(\mathbf{x}) > 0\}$.*

Remark 2.8.25. *If conditions I and II hold, then by (2.8.4), (2.8.5) and (2.8.6) $\mathbf{\Pi}_{x,t}(\mathbf{x})$ is an upper semi-continuous function of $\mathbf{x}$. In this case in the above definition of the $\mathbf{\Pi}_x$–closure of D it is sufficient to require that $\mathbf{x}^k \in D$ be such that $\mathbf{x}^k \to \mathbf{x}$ and $\liminf_{k\to\infty} \mathbf{\Pi}_{x,t}(\mathbf{x}^k) \ge \mathbf{\Pi}_{x,t}(\mathbf{x})$, $t \in \mathbb{R}_+$.*

Lemma 2.8.26. *If $\Pi(p_t^{-1} \circ p_t\, \mathbf{x}) = \mathbf{\Pi}_{x,t}(\mathbf{x})$, $t \in \mathbb{R}_+$, for all $\mathbf{x} \in D \subset \mathbb{C}$, then $\Pi(p_t^{-1} \circ p_t\, \mathbf{x}) = \mathbf{\Pi}_{x,t}(\mathbf{x})$, $t \in \mathbb{R}_+$, for all $\mathbf{x}$ from the $\mathbf{\Pi}_x$–closure of D. If, in addition, the set D is $\mathbf{\Pi}_x$–dense in $\mathbb{C}$, then uniqueness holds for the maxingale problem (x, G) with $\Pi = \mathbf{\Pi}_x$.*

Proof. Let $\mathbf{x}$ belong to the $\mathbf{\Pi}_x$–closure of D. Let $\mathbf{x}^k \in D$ be such that $\mathbf{x}^k \to \mathbf{x}$ and $\mathbf{\Pi}_{x,t}(\mathbf{x}^k) \to \mathbf{\Pi}_{x,t}(\mathbf{x})$, $t \in \mathbb{R}_+$. Since $p_t\mathbf{x}^k \to p_t\mathbf{x}$, it follows that, for arbitrary $\varepsilon > 0$, $p_t^{-1} \circ p_t\mathbf{x}^k \subset p_t^{-1}\big(B_\varepsilon(p_t\mathbf{x})\big)$ for all k large enough, where $B_\varepsilon(p_t\mathbf{x})$ is the closed ε-ball about $p_t\mathbf{x}$. Since by the τ-smoothness property of deviability $\lim_{\varepsilon\to 0} \Pi\big(p_t^{-1}\big(B_\varepsilon(p_t\mathbf{x})\big)\big) = \Pi(p_t^{-1} \circ p_t\, \mathbf{x})$, we conclude that $\limsup_{k\to\infty} \Pi(p_t^{-1} \circ p_t\, \mathbf{x}^k) \le \Pi(p_t^{-1} \circ p_t\mathbf{x})$. Since also $\Pi(p_t^{-1} \circ p_t\, \mathbf{x}^k) = \mathbf{\Pi}_{x,t}(\mathbf{x}^k) \to \mathbf{\Pi}_{x,t}(\mathbf{x})$ as $k \to \infty$, we have that $\mathbf{\Pi}_{x,t}(\mathbf{x}) \le \Pi(p_t^{-1} \circ p_t\, \mathbf{x})$. On the other hand, $\mathbf{\Pi}_{x,t}(\mathbf{x}) \ge \mathbf{\Pi}_x(p_t^{-1} \circ p_t\, \mathbf{x}) \ge \Pi(p_t^{-1} \circ p_t\, \mathbf{x})$ by the definitions of $\mathbf{\Pi}_{x,t}$ and $\mathbf{\Pi}_x$, and (2.8.11), so $\mathbf{\Pi}_{x,t}(\mathbf{x}) = \Pi(p_t^{-1} \circ p_t\, \mathbf{x})$, $t \in \mathbb{R}_+$.

Finally, if D is $\mathbf{\Pi}_x$-dense in $\mathbb{C}$, then by the part just proved, the τ-smoothness property of deviability and (2.8.12) $\Pi(\mathbf{x}) = \lim_{t\to\infty} \Pi(p_t^{-1} \circ p_t\, \mathbf{x}) = \lim_{t\to\infty} \mathbf{\Pi}_{x,t}(\mathbf{x}) = \mathbf{\Pi}_x(\mathbf{x})$ when $\mathbf{\Pi}_x(\mathbf{x}) > 0$. If $\mathbf{\Pi}_x(\mathbf{x}) = 0$, then $\Pi(\mathbf{x}) = 0 = \mathbf{\Pi}_x(\mathbf{x})$ by (2.8.11). ☐

Theorem 2.8.27. *Let $g_s(\lambda; \mathbf{x})$ meet conditions I and II and be differentiable in λ. Let there exist a family $\{G_m,\, m \in \mathbb{N}\}$ of subsets of $\mathbb{R}^d$ and an $\mathbb{R}^d$-valued function $(\Lambda(s, \mathbf{x}, y),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}, y \in \mathbb{R}^d)$, which is $\overline{\mathcal{B}}[0,t] \otimes \mathcal{C}_t \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$ –measurable when restricted to $[0,t] \times \mathbb{C} \times \mathbb{R}^d$ for $t \in \mathbb{R}_+$, continuous in $\mathbf{x}$, bounded on the sets $[0,t] \times K \times G_m$, where $t \in \mathbb{R}_+$, $K \subset \mathbb{C}$ and is compact, and $m \in \mathbb{N}$, and such that*

$$y = \nabla g_s(\Lambda(s, \mathbf{x}, y); \mathbf{x}) \tag{2.8.29}$$

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for $y \in \cup_{m=1}^{\infty} G_m$*, (almost all)* $s \in \mathbb{R}_+$ *and* $\mathbf{\Pi}_x$*–almost all* $\mathbf{x}$*.*

Let

$$D = \bigcup_{m=1}^{\infty} \{\mathbf{x} \in \mathbb{C} : \mathbf{x} \textit{ is absolutely continuous, } \mathbf{x}_0 = x, \textit{ and } \dot{\mathbf{x}}_s \in G_m,\ s \in \mathbb{R}_+\}. \quad (2.8.30)$$

Then $\Pi(p_t^{-1} \circ p_t \mathbf{x}) = \mathbf{\Pi}_{x,t}(\mathbf{x})$*,* $t \in \mathbb{R}_+$*, whenever* $\mathbf{x}$ *is in the* $\mathbf{\Pi}_x$*–closure of* D*. If the set* D *is, in addition,* $\mathbf{\Pi}_x$*–dense in* $\mathbb{C}$*, then uniqueness holds for the maxingale problem* (x, G) *with* $\Pi = \mathbf{\Pi}_x$*.*

Proof. For $\hat{\mathbf{x}} \in D$ we define $\hat{\lambda}(s, \mathbf{x}) = \Lambda(s, \mathbf{x}, \dot{\hat{\mathbf{x}}}_s)$. The function $\hat{\lambda}(s, \mathbf{x})$ is $\mathbf{C}$-progressively measurable, bounded on the sets $[0, T] \times K$ for $T \in \mathbb{R}_+$ and continuous in $\mathbf{x}$. Therefore $(\hat{\lambda}(s, \mathbf{x}), s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}) \in \hat{\Lambda}$ so that by Theorem 2.8.19 $Z(\hat{\lambda})$ is a strictly Luzin-continuous local exponential maxingale under Π and admits a localising sequence of $\mathbf{C}$-stopping times. Since also by (2.8.29) $\dot{\hat{\mathbf{x}}}_s = \nabla g_s(\hat{\lambda}(s, \mathbf{x}); \mathbf{x})$ for (almost all) $s \in \mathbb{R}_+$ and $\mathbf{\Pi}_x$-almost all $\mathbf{x}$, Theorem 2.8.14 and Lemma 2.8.20 imply that $\Pi(p_t^{-1} \circ p_t\, \hat{\mathbf{x}}) = \mathbf{\Pi}_{x,t}(\hat{\mathbf{x}})$, $t \in \mathbb{R}_+$, and an application of Lemma 2.8.26 ends the proof. □

We give an application.

Theorem 2.8.28. *Let* $d = 1$*. Let the canonical process* X *on* $\mathbb{C}$ *be a semimaxingale starting at* $x \in \mathbb{R}_+$ *under* Π *with local characteristics* $(bq, 0, \nu, 0)$*, where* $\nu_s(\Gamma; \mathbf{x}) = \mathbf{1}(q_s(\mathbf{x}) \in \Gamma) b_s(\mathbf{x})$*, and* $b_s(\mathbf{x})$ *and* $q_s(\mathbf{x})$ *are* $\mathbb{R}_+$*–valued functions, which are* $\mathbf{C}$*-progressively measurable in* $(s, \mathbf{x})$ *and continuous in* $\mathbf{x} \in \mathbb{C}$*. Let for every* $t \in \mathbb{R}_+$ *and compact* $K \subset \mathbb{C}$

$$\inf_{s \le t} \inf_{\mathbf{x} \in K} b_s(\mathbf{x}) > 0, \quad \sup_{s \le t} \sup_{\mathbf{x} \in K} b_s(\mathbf{x}) < \infty,$$
$$\inf_{s \le t} \inf_{\mathbf{x} \in K} q_s(\mathbf{x}) > 0, \quad \sup_{s \le t} \sup_{\mathbf{x} \in K} q_s(\mathbf{x}) < \infty.$$

Then $\Pi = \mathbf{\Pi}_x$*.*

Proof. We apply Theorem 2.8.27. Conditions I and II are met by Lemma 2.8.16. The function $g_s(\lambda; \mathbf{x}) = \big(e^{\lambda q_s(\mathbf{x})} - 1\big) b_s(\mathbf{x})$ is differentiable in λ. We take in the hypotheses of Theorem 2.8.27 $G_m = [1/m, m]$ and

$$\Lambda(s, \mathbf{x}, y) = \begin{cases} \dfrac{1}{q_s(\mathbf{x})} \ln \dfrac{y}{b_s(\mathbf{x}) q_s(\mathbf{x})}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

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We check that D is $\mathbf{\Pi}_x$–dense in $\mathbb{C}$. Let $\mathbf{x} \in \mathbb{C}$ be such that $\mathbf{\Pi}_x(\mathbf{x}) > 0$. By Lemma 2.7.12 $\mathbf{x}$ is absolutely continuous; also

$$\mathbf{I}(\mathbf{x}) = \int_0^\infty \sup_{\lambda \in \mathbb{R}} (\lambda \dot{\mathbf{x}}_t - (e^{\lambda q_t(\mathbf{x})} - 1) b_t(\mathbf{x})) dt$$

so that $\dot{\mathbf{x}}_t \geq 0$ a.e. We define $\mathbf{x}^k$ by $\mathbf{x}_0^k = x$ and $\dot{\mathbf{x}}_s^k = (\dot{\mathbf{x}}_s \mathbf{1}(\dot{\mathbf{x}}_s \leq k)) \vee \frac{1}{k}$. Convergence $\mathbf{x}^k \to \mathbf{x}$ is obvious. We prove that for $t \in \mathbb{R}_+$

$$\lim_{k \to \infty} \int_0^t \sup_{\lambda \in \mathbb{R}} (\lambda \dot{\mathbf{x}}_s^k - (e^{\lambda q_s(\mathbf{x}^k)} - 1) b_s(\mathbf{x}^k)) ds = \int_0^t \sup_{\lambda \in \mathbb{R}} (\lambda \dot{\mathbf{x}}_s - (e^{\lambda q_s(\mathbf{x})} - 1) b_s(\mathbf{x})) ds. \quad (2.8.31)$$

We have

$$\begin{aligned}
&\int_0^t \sup_{\lambda \in \mathbb{R}} (\lambda \dot{\mathbf{x}}_s^k - (e^{\lambda q_s(\mathbf{x}^k)} - 1) b_s(\mathbf{x}^k))\, ds \\
&= \int_0^t \Big(\frac{\dot{\mathbf{x}}_s}{q_s(\mathbf{x}^k)} \ln \frac{\dot{\mathbf{x}}_s}{b_s(\mathbf{x}^k) q_s(\mathbf{x}^k)} - \frac{\dot{\mathbf{x}}_s}{q_s(\mathbf{x}^k)} + b_s(\mathbf{x}^k) \Big) \\
&\mathbf{1}\big(\dot{\mathbf{x}}_s \leq k, \dot{\mathbf{x}}_s \geq \frac{1}{k}\big) ds \\
&+ \int_0^t \Big(\frac{1}{k q_s(\mathbf{x}^k)} \ln \frac{1}{k b_s(\mathbf{x}^k) q_s(\mathbf{x}^k)} - \frac{1}{k q_s(\mathbf{x}^k)} + b_s(\mathbf{x}^k) \Big) \\
&\qquad \Big(\mathbf{1}\big(\dot{\mathbf{x}}_s < \frac{1}{k}\big) + \mathbf{1}(\dot{\mathbf{x}}_s > k) \Big) ds.
\end{aligned}$$

Since $\int_0^t \dot{\mathbf{x}}_s \ln \dot{\mathbf{x}}_s \mathbf{1}(\dot{\mathbf{x}}_s > 1) ds < \infty$ by the fact that $\mathbf{I}(\mathbf{x}) < \infty$ and hypotheses, $b_s(\mathbf{x}^k) \to b_s(\mathbf{x})$ and $q_s(\mathbf{x}^k) \to q_s(\mathbf{x})$ as $k \to \infty$, Lebesgue's dominated convergence theorem implies that the right-hand side con-

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verges to

$$\int_0^t \left(\frac{\dot{\mathbf{x}}_s}{q_s(\mathbf{x})} \ln \frac{\dot{\mathbf{x}}_s}{b_s(\mathbf{x})q_s(\mathbf{x})} - \frac{\dot{\mathbf{x}}_s}{q_s(\mathbf{x})} + b_s(\mathbf{x}) \right) \mathbf{1}(\dot{\mathbf{x}}_s > 0)\, ds + \int_0^t b_s(\mathbf{x})\, \mathbf{1}(\dot{\mathbf{x}}_s = 0)\, ds$$

ending the proof of (2.8.31). □

We now consider a version of the above result, which will be used in an application to the analysis of a many-server queue in part II. This result also shows the use of the other method of choosing the function $\hat{\lambda}$.

Theorem 2.8.29. *Let $d = 1$. Let deviability Π on $\mathbb{C}$ be such that the canonical process X is a semimaxingale on $(\mathbb{C}, \Pi)$ starting at $x \in \mathbb{R}_+$ with local characteristics $(b, 0, \nu, 0)$, where*

$$\nu_s(\Gamma; \mathbf{x}) = \mathbf{1}(1 \in \Gamma) v_s(\mathbf{x}) + \mathbf{1}(-1 \in \Gamma) u_s(\mathbf{x})(\mathbf{x}_s \wedge m_s(\mathbf{x})),$$
$$b_s(\mathbf{x}) = v_s(\mathbf{x}) - u_s(\mathbf{x})(\mathbf{x}_s \wedge m_s(\mathbf{x})),$$

and $v_s(\mathbf{x})$, $u_s(\mathbf{x})$ and $m_s(\mathbf{x})$ are $\mathbb{R}_+$–valued functions, which are $\mathbf{C}$-progressively measurable in $(s, \mathbf{x})$ and locally Lipshitz-continuous in $\mathbf{x} \in \mathbb{C}$. Let also for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$

$$\inf_{s \le t} \inf_{\mathbf{x} \in K} v_s(\mathbf{x}) > 0, \quad \sup_{s \le t} \sup_{\mathbf{x} \in K} v_s(\mathbf{x}) < \infty,$$
$$\inf_{s \le t} \inf_{\mathbf{x} \in K} u_s(\mathbf{x}) > 0, \quad \sup_{s \le t} \sup_{\mathbf{x} \in K} u_s(\mathbf{x}) < \infty,$$
$$\inf_{s \le t} \inf_{\mathbf{x} \in K} m_s(\mathbf{x}) > 0.$$

Then $\Pi = \mathbf{\Pi}_x$.

Proof. We first consider the case where $x > 0$. Let $\hat{\mathbf{x}}$ be such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$, $\sup_{s \in [0,t]} |\dot{\hat{\mathbf{x}}}_s| < \infty$ and $\inf_{s \in [0,t]} \hat{\mathbf{x}}_s > 0$ for $t \in \mathbb{R}_+$. We define a function $\hat{\lambda}(s)$ by the equality

$$\dot{\hat{\mathbf{x}}}_s = e^{\hat{\lambda}(s)} v_s(\hat{\mathbf{x}}) - e^{-\hat{\lambda}(s)} u_s(\hat{\mathbf{x}}) \big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big). \qquad (2.8.32)$$

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The function $\hat{\lambda}(s)$ is well defined, satisfies the conditions of part 2 of Lemma 2.8.20 and, being locally bounded and Lebesgue measurable, is an element of $\hat{\Lambda}$; also by Lemma 2.8.16 conditions I and II are met; so $Z(\hat{\lambda})$ is a strictly Luzin exponential maxingale by Theorem 2.8.19 and admits a localising sequence of strictly Luzin **C**-stopping times. Therefore, by Theorem 2.8.14 $\Pi(\hat{\mathbf{x}}) = \mathbf{\Pi}_x(\hat{\mathbf{x}})$ and $\Pi(p_t^{-1} \circ p_t\hat{\mathbf{x}}) = \mathbf{\Pi}_{x,t}(\hat{\mathbf{x}})$, $t \in \mathbb{R}_+$.

For more general functions $\hat{\mathbf{x}}$ such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$ we apply Lemma 2.8.26. Let us consider the instance where $\inf_{s\in[0,t]} \hat{\mathbf{x}}_s > 0$ for $t \in \mathbb{R}_+$. We define

$$\hat{\mathbf{x}}_s^k = x + \int_0^s \dot{\hat{\mathbf{x}}}_p \, \mathbf{1}\big(|\dot{\hat{\mathbf{x}}}_p| \le k\big)\, dp. \tag{2.8.33}$$

Then $\sup_{s\in[0,t]}|\hat{\mathbf{x}}_s^k - \hat{\mathbf{x}}_s| \to 0$ as $k \to \infty$; in particular, $\liminf_{k\to\infty} \inf_{s\in[0,t]} \hat{\mathbf{x}}_s^k > 0$ so by the part just proved $\Pi(p_t^{-1} \circ p_t\hat{\mathbf{x}}^k) = \mathbf{\Pi}_{x,t}(\hat{\mathbf{x}}^k)$, $t \in \mathbb{R}_+$. Since for absolutely continuous $\mathbf{x}$

$$\mathbf{I}_t(\mathbf{x}) = \int_0^t \sup_{\lambda\in\mathbb{R}} \big(\lambda\dot{\mathbf{x}}_s - (e^{\lambda} - 1)v_s(\mathbf{x}) - (e^{-\lambda} - 1)u_s(\mathbf{x})\big(\mathbf{x}_s \wedge m_s(\mathbf{x})\big)\big)\, ds,$$

to prove that $\Pi(\hat{\mathbf{x}}) = \mathbf{\Pi}_x(\hat{\mathbf{x}})$ it is sufficient to show by Lemma 2.8.26 and Remark 2.8.25 that for all $t \in \mathbb{R}_+$

$$\begin{aligned} &\limsup_{k\to\infty} \int_0^t \sup_{\lambda\in\mathbb{R}} \big(\lambda\dot{\hat{\mathbf{x}}}_s^k - (e^{\lambda} - 1)v_s(\hat{\mathbf{x}}^k) \\ &- (e^{-\lambda} - 1)u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\big)\big)\, ds \\ &\le \int_0^t \sup_{\lambda\in\mathbb{R}} \big(\lambda\dot{\hat{\mathbf{x}}}_s - (e^{\lambda} - 1)v_s(\hat{\mathbf{x}}) - (e^{-\lambda} - 1)u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)\big)\, ds. \end{aligned} \tag{2.8.34}$$

We have

$$\int_0^t \sup_{\lambda\in\mathbb{R}} \big(\lambda\dot{\hat{\mathbf{x}}}_s^k - (e^{\lambda} - 1)v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1)u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\big)\big)\, ds$$

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$$\leq \int_0^t \sup_{\lambda\in\mathbb{R}} \bigl(\lambda \dot{\hat{\mathbf{x}}}_s - (e^{\lambda}-1)v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda}-1)u_s(\hat{\mathbf{x}}^k)\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)\,ds$$
$$+ \int_0^t \sup_{\lambda\in\mathbb{R}} \bigl(-(e^{\lambda}-1)v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda}-1)u_s(\hat{\mathbf{x}}^k)\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)$$
$$\mathbf{1}\bigl(|\dot{\hat{\mathbf{x}}}_s| > k\bigr)\,ds. \quad (2.8.35)$$

The second integral on the right converges to 0 as $k \to \infty$. We work with the first integral. Let $C_1 > 0$ and $C_2 > 0$ be respective upper and lower bounds over large values of k and $s \in [0,t]$ for the $\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr)$, $C_3 > 0$ and $C_4 > 0$ – upper and lower bounds for the $v_s(\hat{\mathbf{x}}^k)$, and $C_5 > 0$ and $C_6 > 0$ – upper and lower bounds for the $u_s(\hat{\mathbf{x}}^k)$. Let $\lambda^k(s)$ be the points where the supremums in the first integral on the right-hand side of (2.8.35) are attained so that

$$\dot{\hat{\mathbf{x}}}_s = e^{\lambda^k(s)}v_s(\hat{\mathbf{x}}^k) - e^{-\lambda^k(s)}u_s(\hat{\mathbf{x}}^k)\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr). \quad (2.8.36)$$

Since for $\lambda^k(s)$ positive $e^{\lambda^k(s)}v_s(\hat{\mathbf{x}}^k) - e^{-\lambda^k(s)}u_s(\hat{\mathbf{x}}^k)\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr) \geq C_4 e^{\lambda^k(s)} - C_5C_1$ and for $\lambda^k(s)$ negative $e^{\lambda^k(s)}v_s(\hat{\mathbf{x}}^k) - e^{-\lambda^k(s)}u_s(\hat{\mathbf{x}}^k)\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr) \leq C_3 - e^{-\lambda^k(s)}C_6C_2$, we conclude that

$$e^{\lambda^k(s)} \leq \frac{|\dot{\hat{\mathbf{x}}}_s| + C_5C_1}{C_4} \vee 1, \quad e^{-\lambda^k(s)} \leq \frac{|\dot{\hat{\mathbf{x}}}_s| + C_3}{C_6C_2} \vee 1, \quad s \in [0,t]. \quad (2.8.37)$$

We thus write for the first integral on the right of (2.8.35)

$$\int_0^t \sup_{\lambda\in\mathbb{R}} \bigl(\lambda \dot{\hat{\mathbf{x}}}_s - (e^{\lambda}-1)v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda}-1)u_s(\hat{\mathbf{x}}^k)\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)\,ds$$
$$= \int_0^t \bigl(\lambda^k(s)\dot{\hat{\mathbf{x}}}_s - (e^{\lambda^k(s)}-1)v_s(\hat{\mathbf{x}}^k)$$
$$- (e^{-\lambda^k(s)}-1)u_s(\hat{\mathbf{x}}^k)\bigl(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)\,ds$$

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$$\leq \int_0^t \big(\lambda^k(s)\dot{\hat{\mathbf{x}}}_s - (e^{\lambda^k(s)}-1)v_s(\hat{\mathbf{x}}) - (e^{-\lambda^k(s)}-1)u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)\big)\,ds$$

$$+ \int_0^t |e^{\lambda^k(s)} - 1||v_s(\hat{\mathbf{x}}^k) - v_s(\hat{\mathbf{x}})|\,ds$$

$$+ \int_0^t |e^{-\lambda^k(s)} - 1||u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\big) - u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)|\,ds$$

$$\leq \int_0^t \sup_{\lambda\in\mathbb{R}}\big(\lambda\dot{\hat{\mathbf{x}}}_s - (e^{\lambda}-1)v_s(\hat{\mathbf{x}}) - (e^{-\lambda}-1)u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)\big)\,ds$$

$$+ \int_0^t |e^{\lambda^k(s)} - 1||v_s(\hat{\mathbf{x}}^k) - v_s(\hat{\mathbf{x}})|\,ds$$

$$+ \int_0^t |e^{-\lambda^k(s)} - 1||u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\big) - u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)|\,ds.$$

By (2.8.37) and the facts that $\hat{\mathbf{x}}$ is absolutely continuous, $u_s(\mathbf{x})$, $v_s(\mathbf{x})$ and $m_s(\mathbf{x})$ are continuous in $\mathbf{x}$ and bounded, and the $\hat{\mathbf{x}}^k$ converge uniformly on $[0,t]$ to $\hat{\mathbf{x}}$ as $k \to \infty$, the latter two integrals converge to 0 as $k \to \infty$, so (2.8.34) follows.

Let us now assume that $\hat{\mathbf{x}}$ is an arbitrary function such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$. Clearly, $\hat{\mathbf{x}}$ is $\mathbb{R}_+$-valued, absolutely continuous, and $\hat{\mathbf{x}}_0 = x > 0$. Let $\hat{\mathbf{x}}^k_s = x + \int_0^s \dot{\hat{\mathbf{x}}}_u\,\mathbf{1}(\hat{\mathbf{x}}_u \geq 1/k)\,du$. Since $\dot{\hat{\mathbf{x}}}_s = 0$ on the set $\{\hat{\mathbf{x}}_s = 0\}$ (a.e.), we have that $\hat{\mathbf{x}}^k \to \hat{\mathbf{x}}$ uniformly on bounded intervals. Since $\hat{\mathbf{x}}^k_s = \hat{\mathbf{x}}_s \vee (1/k)$ for k large, by the part proved $\Pi(p_t^{-1} \circ p_t\hat{\mathbf{x}}^k) = \mathbf{\Pi}_{x,t}(\hat{\mathbf{x}}^k)$, $t \in \mathbb{R}_+$, so by Lemma 2.8.26 in order to prove that $\Pi(\hat{\mathbf{x}}) = \mathbf{\Pi}_x(\hat{\mathbf{x}})$ it is sufficient to show that (2.8.34) holds. We have

$$\int_0^t \sup_{\lambda\in\mathbb{R}}\big(\lambda\dot{\hat{\mathbf{x}}}^k_s - (e^{\lambda}-1)v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda}-1)u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\big)\big)\,ds$$

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$$= \int_0^t \sup_{\lambda \in \mathbb{R}} \big(\lambda \dot{\hat{\mathbf{x}}}_s - (e^{\lambda} - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}}^k)\big)\big)$$
$$\mathbf{1}(\hat{\mathbf{x}}_s \geq 1/k)\, ds$$
$$+ \int_0^t \sup_{\lambda \in \mathbb{R}} \big(-(e^{\lambda} - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\big)\big)$$
$$\mathbf{1}(\hat{\mathbf{x}}_s < 1/k)\, ds. \quad (2.8.38)$$

Routine calculations show that the second integral on the right-hand side of (2.8.38) converges to $\int_0^t v_s(\hat{\mathbf{x}})\, \mathbf{1}(\hat{\mathbf{x}}_s = 0)\, ds$ as $k \to \infty$. Defining $\lambda^k(s)$ on the set $\{s : \hat{\mathbf{x}}_s \geq 1/k\}$ by

$$\dot{\hat{\mathbf{x}}}_s = e^{\lambda^k(s)} v_s(\hat{\mathbf{x}}^k) - e^{-\lambda^k(s)} u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}}^k)\big),$$

we have estimates analogous to (2.8.37) with the right inequality replaced by

$$\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}}^k)\big) e^{-\lambda^k(s)} \leq \frac{|\dot{\hat{\mathbf{x}}}_s| + C_3}{C_6} \vee \big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}}^k)\big)$$

so that

$$\int_0^t \sup_{\lambda \in \mathbb{R}} \big(\lambda \dot{\hat{\mathbf{x}}}_s - (e^{\lambda} - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}}^k)\big)\big)$$
$$\mathbf{1}(\hat{\mathbf{x}}_s \geq 1/k)\, ds$$
$$\leq \int_0^t \sup_{\lambda \in \mathbb{R}} \big(\lambda \dot{\hat{\mathbf{x}}}_s - (e^{\lambda} - 1) v_s(\hat{\mathbf{x}}) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)\big)$$
$$\mathbf{1}(\hat{\mathbf{x}}_s > 0)\, ds$$
$$+ \int_0^t |e^{\lambda^k(s)} - 1| |v_s(\hat{\mathbf{x}}^k) - v_s(\hat{\mathbf{x}})|\, ds$$
$$+ \int_0^t \big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}}^k)\big) |e^{-\lambda^k(s)} - 1| |u_s(\hat{\mathbf{x}}^k) - u_s(\hat{\mathbf{x}})|\, ds$$
$$+ \int_0^t |e^{-\lambda^k(s)} - 1| |m_s(\hat{\mathbf{x}}^k) - m_s(\hat{\mathbf{x}})| u_s(\hat{\mathbf{x}})\, ds.$$

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The required convergence follows by the fact that the last three integrals on the right-hand side converge to 0 by the same argument as above.

We now consider the case $x = 0$. Let $\hat{\mathbf{x}}$ be such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$, $\sup_{s\in[0,t]}|\dot{\hat{\mathbf{x}}}_s| < \infty$, $\inf_{s\in[0,\epsilon]}\dot{\hat{\mathbf{x}}}_s > 0$, and $\inf_{s\in[\epsilon,t]}\hat{\mathbf{x}}_s > 0$ for $t \in \mathbb{R}_+$ and some $\epsilon > 0$. We again define $\hat{\lambda}(s)$ by (2.8.32). It is evidently bounded on $[\epsilon, t]$. Next, since by (2.8.32) $\dot{\hat{\mathbf{x}}}_s \le e^{\hat{\lambda}(s)}v_s(\hat{\mathbf{x}})$, we have that $e^{\hat{\lambda}(s)} \ge \dot{\hat{\mathbf{x}}}_s/C_3'$; hence, $\dot{\hat{\mathbf{x}}}_s = e^{\hat{\lambda}(s)}v_s(\hat{\mathbf{x}}) - e^{-\hat{\lambda}(s)}u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big) \ge e^{\hat{\lambda}(s)}C_4' - C_3'C_5'\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)/\dot{\hat{\mathbf{x}}}_s$, where C_3', C_4' and C_5' have a similar meaning as above. We thus conclude that there exist A_1, A_2 and A_3, which depend on $\hat{\mathbf{x}}_s, s \in [0,\epsilon]$, but do not explicitly depend on $\dot{\hat{\mathbf{x}}}_s$, such that for $s \in [0, \epsilon]$

$$A_1\dot{\hat{\mathbf{x}}}_s \le e^{\hat{\lambda}(s)} \le A_2\dot{\hat{\mathbf{x}}}_s + \frac{A_3}{\dot{\hat{\mathbf{x}}}_s}. \tag{2.8.39}$$

Since $\dot{\hat{\mathbf{x}}}_s$ is bounded both from below and above on $[0, \epsilon]$, so is $\hat{\lambda}(s)$. Hence, by Theorem 2.8.14 $\Pi(\hat{\mathbf{x}}) = \mathbf{\Pi}_x(\hat{\mathbf{x}})$ and $\Pi(p_t^{-1} \circ p_t\hat{\mathbf{x}}) = \mathbf{\Pi}_{x,t}(\hat{\mathbf{x}})$, $t \in \mathbb{R}_+$.

Next, let $\hat{\mathbf{x}}$ be such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$, $\inf_{s\in[0,\epsilon]}\dot{\hat{\mathbf{x}}}_s > 0$, and $\inf_{s\in[\epsilon,t]}\hat{\mathbf{x}}_s > 0$ for $t \in \mathbb{R}_+$ and some $\epsilon > 0$. We define $\hat{\mathbf{x}}^k$ in analogy with the case $x > 0$ by (2.8.33). Then following the same line of reasoning we need bounds on $\lambda^k(s)$ defined by (2.8.36) in order to prove that

$$\int_\epsilon^t |e^{\lambda^k(s)} - 1||v_s(\hat{\mathbf{x}}^k) - v_s(\hat{\mathbf{x}})|\,ds \to 0, \tag{2.8.40}$$

$$\int_\epsilon^t |e^{-\lambda^k(s)} - 1||u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\big) - u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)|\,ds \to 0, \tag{2.8.41}$$

$$\int_0^\epsilon |e^{\lambda^k(s)} - 1||v_s(\hat{\mathbf{x}}^k) - v_s(\hat{\mathbf{x}})|\,ds \to 0, \tag{2.8.42}$$

$$\int_0^\epsilon |e^{-\lambda^k(s)} - 1||u_s(\hat{\mathbf{x}}^k)\big(\hat{\mathbf{x}}_s^k \wedge m_s(\hat{\mathbf{x}}^k)\big) - u_s(\hat{\mathbf{x}})\big(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}})\big)|\,ds \to 0. \tag{2.8.43}$$

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On the interval $[\epsilon, t]$ bounds (2.8.37) apply yielding convergences (2.8.40) and (2.8.41). Bounds on $[0, \epsilon]$ that imply limits (2.8.42) and (2.8.43) are given by (2.8.39) with suitable A_1, A_2 and A_3.

Next, let $\hat{\mathbf{x}}$ be such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$ and not identically equal to 0. Let $\epsilon_k = \inf\{s \in \mathbb{R}_+ : \hat{\mathbf{x}}_s = 1/k\}$. We define $\hat{\mathbf{x}}^k$ by $\hat{\mathbf{x}}^k_s = s\hat{\mathbf{x}}_{\epsilon_k}/\epsilon_k$ for $s \in [0, \epsilon_k]$ and $\hat{\mathbf{x}}^k_s = \hat{\mathbf{x}}_s \vee (1/k)$ for $s \geq \epsilon_k$. Then $\Pi(p_t^{-1} \circ p_t \hat{\mathbf{x}}^k) = \mathbf{\Pi}_{x,t}(\hat{\mathbf{x}}^k)$, $t \in \mathbb{R}_+$, so we need to prove that (2.8.34) holds. We have

$$\int_0^t \sup_{\lambda \in \mathbb{R}} \bigl(\lambda \dot{\hat{\mathbf{x}}}^k_s - (e^\lambda - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}}^k) \bigl(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)\, ds$$
$$\leq \int_0^{\epsilon_k} \sup_{\lambda \in \mathbb{R}} \bigl(\lambda \dot{\hat{\mathbf{x}}}^k_s - (e^\lambda - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}}^k) \bigl(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)\, ds$$
$$+ \int_0^t \sup_{\lambda \in \mathbb{R}} \bigl(\lambda \dot{\hat{\mathbf{x}}}_s - (e^\lambda - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}}^k) \bigl(\hat{\mathbf{x}}_s \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)$$
$$\mathbf{1}(\hat{\mathbf{x}}_s \geq 1/k)\, ds$$
$$+ \int_0^t \sup_{\lambda \in \mathbb{R}} \bigl(-(e^\lambda - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda} - 1) u_s(\hat{\mathbf{x}}^k) \bigl(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\bigr)\bigr)$$
$$\mathbf{1}(\hat{\mathbf{x}}_s < 1/k)\, ds. \quad (2.8.44)$$

By an argument similar to the one used for deriving an asymptotic bound for the right-hand side of (2.8.38) the limit superior as $k \to \infty$ of the sum of the latter two integrals is not greater than $\mathbf{I}_t(\hat{\mathbf{x}})$. Let $\lambda^k(s)$ be the points, where the supremums in the first integral on the right of (2.8.44) are attained. Then the integrand takes the form

$$\lambda^k(s) \dot{\hat{\mathbf{x}}}^k_s - (e^{\lambda^k(s)} - 1) v_s(\hat{\mathbf{x}}^k) - (e^{-\lambda^k(s)} - 1) u_s(\hat{\mathbf{x}}^k) \bigl(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\bigr)$$
$$\leq \lambda^k(s) \dot{\hat{\mathbf{x}}}^k_s + v_s(\hat{\mathbf{x}}^k) + u_s(\hat{\mathbf{x}}^k) \bigl(\hat{\mathbf{x}}^k_s \wedge m_s(\hat{\mathbf{x}}^k)\bigr).$$

Since the derivatives $\dot{\hat{\mathbf{x}}}^k_s$ equal $\hat{\mathbf{x}}_{\epsilon_k}/\epsilon_k$ and $\epsilon_k \to 0$ as $k \to \infty$, the estimates (2.8.39) applied to $\hat{\mathbf{x}}^k_s$ show that the first integral on the right of (2.8.44) converges to 0 as $k \to \infty$ provided $\lim_{k\to\infty} \hat{\mathbf{x}}_{\epsilon_k} \bigl|\ln(\hat{\mathbf{x}}_{\epsilon_k}/\epsilon_k)\bigr| = 0$. The latter limit follows since $\hat{\mathbf{x}}_{\epsilon_k} \bigl|\ln(\hat{\mathbf{x}}_{\epsilon_k}/\epsilon_k)\bigr| \leq \int_0^{\epsilon_k} |\dot{\hat{\mathbf{x}}}_s \ln|\dot{\hat{\mathbf{x}}}_s||\, ds$ and $\int_0^t |\dot{\hat{\mathbf{x}}}_s \ln|\dot{\hat{\mathbf{x}}}_s||\, ds < \infty$ since $\mathbf{I}_t(\hat{\mathbf{x}}) < \infty$.

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Finally, if $\hat{\mathbf{x}}_s = 0$ for all $s \in \mathbb{R}_+$, then we let $\hat{\mathbf{x}}^k_s = s/k$ for $s \in [0, 1/k]$ and $\hat{\mathbf{x}}^k_s = 1/k$ for $s \geq 1/k$. □

Remark 2.8.30. *This theorem illustrates the general feature that if the function $\hat{\lambda}$ is chosen as in part 2 of Lemma 2.8.20, then one needs to impose Lipshitz continuity conditions on the coefficients.*

Our purpose now is to state uniqueness results for more general functions $g_s(\lambda; \mathbf{x})$. In the next lemma, given a closed convex set $F \subset \mathbb{R}^m$, we denote as $\mathrm{proj}_F \lambda$ the projection of $\lambda \in \mathbb{R}^m$ onto F; for a closed convex cone $N \in \mathbb{R}^d$, we denote as $\mathrm{aff}\, N$ the affine hull of N; $N^\perp = \{\lambda \in \mathbb{R}^d : \lambda \cdot y \leq 0 \text{ for all } y \in N\}$ denotes the polar cone of N. As above, $\mathrm{ri}\, N$ denotes the relative interior of N.

Lemma 2.8.31. *Let conditions I and II hold, and $g_s(\lambda; \mathbf{x})$ be differentiable in λ. Let there exist a closed convex cone $N \subset \mathbb{R}^d$ such that $g_s(\lambda; \mathbf{x})$ is strictly convex in $\lambda \in \mathrm{aff}\, N$, $g_s(\lambda; \mathbf{x}) \geq g_s(\mathrm{proj}_{\mathrm{aff}\, N} \lambda; \mathbf{x})$, $\lambda \in \mathbb{R}^d$, $s \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{C}$, and the following holds:*

1. *for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$*

$$\lim_{\substack{\lambda \in \mathrm{aff}\, N: \\ |\mathrm{proj}_N \lambda| \to \infty}} \inf_{s \leq t} \inf_{\mathbf{x} \in K} \frac{g_s(\lambda; \mathbf{x})}{|\mathrm{proj}_N \lambda|} = \infty;$$

2. *for every $t \in \mathbb{R}_+$, compact $K \subset \mathbb{C}$ and $A \in \mathbb{R}_+$*

$$\inf_{\substack{\lambda \in \mathrm{aff}\, N: \\ |\mathrm{proj}_N \lambda| \leq A}} \inf_{s \leq t} \inf_{\mathbf{x} \in K} g_s(\lambda; \mathbf{x}) > -\infty;$$

3. *for every $t \in \mathbb{R}_+$, $A \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{C}$, and sequence $\mathbf{x}^k \to \mathbf{x}$*

$$\lim_{k \to \infty} \int_0^t \sup_{\substack{\lambda \in \mathrm{aff}\, N: \\ |\mathrm{proj}_N \lambda| \leq A}} |g_s(\lambda; \mathbf{x}^k) - g_s(\lambda; \mathbf{x})|\, ds = 0;$$

4. *if $N^\perp \neq 0$, then for every $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$*

$$\limsup_{\substack{\lambda \in N^\perp: \\ |\lambda| \to \infty}} \sup_{s \leq t} \frac{g_s(\lambda; \mathbf{x})}{|\lambda|} \leq 0.$$

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Then $\Pi(p_t^{-1} \circ p_t \mathbf{x}) = \mathbf{\Pi}_{x,t}(\mathbf{x})$, $t \in \mathbb{R}_+$, *for every* $\mathbf{x}$ *such that* $\mathbf{x}_0 = x$, $\dot{\mathbf{x}}_s \in N$ *(a.e.) and* $\sup_{s\in\mathbb{R}_+} |\dot{\mathbf{x}}_s| < \infty$. *Also, if* $\mathbf{\Pi}_x(\mathbf{x}) > 0$, *then* $\dot{\mathbf{x}}_s \in N$ *(a.e.)*

Proof. We apply Theorem 2.8.27. Denote as N' the polar cone of N relative to $\operatorname{aff} N$, i.e., $N' = N^\perp \cap \operatorname{aff} N$ and let for $m \in \mathbb{N}$

$$\begin{aligned} N_m &= \{y \in \operatorname{aff} N : \lambda \cdot y \le -\frac{1}{m}|\lambda||y| \text{ for all } \lambda \in N'\}, \\ G_m &= \{y \in N_m : |y| \le m\}. \end{aligned}$$

We next define $\Lambda(s, \mathbf{x}, y)$ in the statement of Theorem 2.8.27. If $y \notin \operatorname{ri} N$, we set $\Lambda(s, \mathbf{x}, y) = 0$. Let $y \in \operatorname{ri} N$ (the latter set is nonempty since N is convex, von Leichtweiss [131]). Since $\operatorname{ri} N = \cup_{m=1}^\infty G_m$, we have that $y \in G_m$ for some m.

We now prove that for every compact $K \subset \mathbb{C}$ and $t \in \mathbb{R}_+$ there exists $C_0 > 0$ such that the inequality $\lambda \cdot y - g_s(\lambda; \mathbf{x}) \le 0$ holds for $\mathbf{x} \in K$, $s \le t$ and $\lambda \in \operatorname{aff} N$ such that $|\lambda| > C_0$. If $|y| = 0$, then $\operatorname{aff} N = N$, so $\lambda = \operatorname{proj}_N \lambda$ and in view of condition 1 $\lambda \cdot y - g_s(\lambda; \mathbf{x})$ is negative for $\mathbf{x} \in K$ and $s \le t$ if $|\lambda|$ is large enough.

Let us assume now that $|y| > 0$. We first consider the case when $\lambda \in \operatorname{aff} N$ is such that $\lambda \cdot y \le -\dfrac{1}{2m}|\lambda||y|$. Then for $s \le t$

$$\lambda \cdot y - g_s(\lambda; \mathbf{x}) \le -\frac{1}{2m}|\lambda||y| - \inf_{\lambda \in \operatorname{aff} N} \inf_{u \le t} g_u(\lambda; \mathbf{x})$$

which by conditions 1 and 2 of the lemma is negative for all $\mathbf{x} \in K$ and $s \le t$ if $|\lambda|$ is large enough. Now, let $\lambda \in \operatorname{aff} N$ be such that $\lambda \cdot y \ge -\dfrac{1}{2m}|\lambda||y|$. Then

$$\frac{|\operatorname{proj}_N \lambda|}{|\lambda|} \ge \frac{1}{2(m+1)}. \tag{2.8.45}$$

To see this, write $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1 = \operatorname{proj}_{N'} \lambda$ and $\lambda_2 = \operatorname{proj}_N \lambda$. Since $y \in G_m \subset N_m$ and $\lambda_1 \in N'$, it follows that $\lambda_1 \cdot y \le -\dfrac{1}{m}|\lambda_1||y|$; on the other hand, $\lambda \cdot y \ge -\dfrac{1}{2m}|\lambda||y|$, so

$$|\lambda_2||y| \ge \lambda_2 \cdot y = \lambda \cdot y - \lambda_1 \cdot y \ge \frac{|y|}{m}\big(|\lambda_1| - \frac{|\lambda|}{2}\big) \ge \frac{|y|}{m}\big(\frac{|\lambda|}{2} - |\lambda_2|\big).$$

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Hence, $|\lambda_2|\big(1+\dfrac{1}{m}\big) \ge \dfrac{|\lambda|}{2m}$ (recall that $|y|>0$) which is equivalent to (2.8.45) by the definition of λ_2.

Inequality (2.8.45) and condition 1 imply that, given $L>0$, we have, if $|\lambda|$ is large enough, that for $\mathbf{x}\in K$ and $s\le t$

$$\lambda\cdot y - g_s(\lambda;\mathbf{x}) \le |\lambda||y| - \frac{L}{2(m+1)}|\lambda|,$$

which is negative if L has been chosen large enough.

Thus, in all the cases we have that $\lambda\cdot y - g_s(\lambda;\mathbf{x}) \le 0$ for all $\mathbf{x}\in K$ and $s\le t$ if $\lambda\in$ aff N is such that $|\lambda|$ is large enough. The claim is proved.

We thus conclude, since $\lambda\cdot y = \mathrm{proj}_{\mathrm{aff}\, N}\lambda\cdot y$ and by hypotheses $g_s(\lambda;\mathbf{x}) \ge g_s(\mathrm{proj}_{\mathrm{aff}\, N}\lambda;\mathbf{x})$, that for $\mathbf{x}\in K$ and $s\le t$

$$\begin{aligned}\sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot y - g_s(\lambda;\mathbf{x})) &= \sup_{\lambda\in\,\mathrm{aff}\, N}(\lambda\cdot y - g_s(\lambda;\mathbf{x}))\\ &= \sup_{\substack{\lambda\in\,\mathrm{aff}\, N:\\ |\lambda|\le C_0}}(\lambda\cdot y - g_s(\lambda;\mathbf{x})),\ s\le t. \quad (2.8.46)\end{aligned}$$

Since the function $\lambda\cdot y - g_s(\lambda;\mathbf{x})$ is strictly concave in $\lambda\in\mathrm{aff}\, N$ by the hypotheses, it attains supremum on $\{\lambda\in\mathrm{aff}\, N:\ |\lambda|\le C_0\}$ at a unique point, which we take as $\Lambda(s,\mathbf{x},y)$.

Thus, we have defined $\Lambda(s,\mathbf{x},y)$ for all $y\in\mathbb{R}^d$. We check that it satisfies the conditions of Theorem 2.8.27. By definition

$$\Lambda(s,\mathbf{x},y)\cdot y - g_s(\Lambda(s,\mathbf{x},y);\mathbf{x}) = \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot y - g_s(\lambda;\mathbf{x}))$$

for $s\in\mathbb{R}_+$, $y\in\cup_{m=1}^\infty G_m$, $\mathbf{x}\in\mathbb{C}$, which implies (2.8.29). Equality (2.8.46) shows that $\Lambda(s,\mathbf{x},y)$ is bounded on the sets $[0,t]\times K\times G_m$, where $t\in\mathbb{R}_+$, K is compact, and $m\in\mathbb{N}$. Also it is $\overline{\mathcal{B}}[0,t]\otimes\mathcal{C}_t\otimes\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$ –measurable when restricted to $[0,t]\times\mathbb{C}\times\mathbb{R}^d$ for $t\in\mathbb{R}_+$ since $g_s(\lambda;\mathbf{x})$ is $\mathbf{C}$-progressively measurable in $(s,\mathbf{x})$ and continuous in λ, and for $\Gamma\in\mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}&\{(s,\mathbf{x},y):\Lambda(s,\mathbf{x},y)\in\Gamma\}\\ &=\{(s,\mathbf{x},y):\ \sup_{\lambda\in\,\Gamma\,\cap\,\mathrm{aff}\, N}(\lambda\cdot y - g_s(\lambda;\mathbf{x})) = \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot y - g_s(\lambda;\mathbf{x}))\},\end{aligned}$$

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if $\Gamma \not\ni 0$, and

$$\{(s,\mathbf{x},y):\Lambda(s,\mathbf{x},y)\in\Gamma\}$$
$$=\{(s,\mathbf{x},y):\sup_{\lambda\in\Gamma\cap\operatorname{aff}N}(\lambda\cdot y-g_s(\lambda;\mathbf{x}))=\sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot y-g_s(\lambda;\mathbf{x}))\}$$
$$\bigcup\{(s,\mathbf{x},y)\,:\,y\notin\operatorname{ri}N\},$$

if $\Gamma\ni 0$, where $\sup_\emptyset=\infty$. (Note that by continuity of $g_s(\lambda;\mathbf{x})$ in λ the supremums may be taken over the rationals.)

Finally, $\Lambda(s,\mathbf{x},y)$ is continuous in $\mathbf{x}$. Indeed, let $\mathbf{x}^k\to\mathbf{x}$ as $k\to\infty$. If $y\notin\operatorname{ri}N$, then $\Lambda(s,\mathbf{x}^k,y)=\Lambda(s,\mathbf{x},y)=0$. Let $y\in\operatorname{ri}N$. Hence, $y\in G_m$ for some m and, since the set $\{\mathbf{x}^k,k\in\mathbb{N}\}$ is relatively compact, the set $\{\Lambda(s,\mathbf{x}^k,y),k\in\mathbb{N}\}$ is bounded; so there exist $\lambda'\in\operatorname{aff}N$ and subsequence k' such that $\Lambda(s,\mathbf{x}^{k'},y)\to\lambda'$ as $k'\to\infty$. Since $\Lambda(s,\mathbf{x}^k,y)\cdot y-g_s(\Lambda(s,\mathbf{x}^k,y);\mathbf{x}^k)\geq\lambda\cdot y-g_s(\lambda;\mathbf{x}^k)$ for $\lambda\in\mathbb{R}^d$ and $g_s(\lambda;\mathbf{x})$ is continuous in $(\lambda,\mathbf{x})$, we conclude that $\lambda'\cdot y-g_s(\lambda';\mathbf{x})\geq\lambda\cdot y-g_s(\lambda;\mathbf{x})$ so that $\lambda'\cdot y-g_s(\lambda';\mathbf{x})=\sup_{\lambda\in\operatorname{aff}N}(\lambda\cdot y-g_s(\lambda;\mathbf{x}))$. Since the point where the supremum is attained is unique, $\lambda'=\Lambda(s,\mathbf{x},y)$ proving that $\Lambda(s,\mathbf{x}^k,y)\to\Lambda(s,\mathbf{x},y)$ as $k\to\infty$. Thus, existence of $\Lambda(s,\mathbf{x},y)$ in the statement of Theorem 2.8.27 is proved.

Let $\hat{\mathbf{x}}$ be such that

$$\hat{\mathbf{x}}_0=x,\ \dot{\hat{\mathbf{x}}}_s\in N,\ \ A=\sup_{s\in\mathbb{R}_+}|\dot{\hat{\mathbf{x}}}_s|<\infty. \tag{2.8.47}$$

We prove that $\Pi(p_t^{-1}\circ p_t\hat{\mathbf{x}})=\mathbf{\Pi}_{x,t}(\hat{\mathbf{x}})$, $t\in\mathbb{R}_+$. Let D be defined as in Theorem 2.8.27. According to Theorem 2.8.27, the required equality will follow if we find a sequence $\mathbf{x}^k\in D$, which converges to $\hat{\mathbf{x}}$ as $k\to\infty$, and is such that

$$\lim_{k\to\infty}\mathbf{\Pi}_{x,t}(\mathbf{x}^k)=\mathbf{\Pi}_{x,t}(\hat{\mathbf{x}}),\ t\in\mathbb{R}_+. \tag{2.8.48}$$

Since $\operatorname{ri}N$ is nonempty, there exist $\hat{y}\in N$ and $r\in(0,1)$ such that $|\hat{y}|=1$ and $\lambda\cdot\hat{y}\leq -r|\lambda|$ for all $\lambda\in N'$. We observe that if $y\in N$ and $|y|\leq A$, then, given $k\in\mathbb{N}$, there exist $\alpha_k>0$ such that $\alpha_k\to 0$ as $k\to\infty$ and $y+\alpha_k\hat{y}\in G_k$ if $k\geq k_0=\lfloor(A+1)/r\rfloor+1$. Indeed, since $\lambda\cdot\hat{y}\leq -r|\lambda|$ and $\lambda\cdot y\leq 0$ if $\lambda\in N'$, $|\hat{y}|=1$ and $|y|\leq A$, we have, for $\lambda\in N'$ and $0<\alpha_k<1$, that $\lambda\cdot(y+\alpha_k\hat{y})\leq-\alpha_k r|\lambda|\leq-\alpha_k r|\lambda||y+\alpha_k\hat{y}|/(1+A)$, so if $\alpha_k=(1+A)/(rk)$, then $y+\alpha_k\hat{y}\in G_k$ for $k\geq k_0$.

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We define next $\mathbf{x}_s^k = \hat{\mathbf{x}}_s + \alpha_k \hat{y} s,\ s \in \mathbb{R}_+$. Since $\dot{\hat{\mathbf{x}}}_s \in N$, we have by (2.8.47) that $\dot{\mathbf{x}}_s^k \in G_k$, so $\mathbf{x}^k \in D$ by (2.8.30). By Lemma 2.7.12, (2.8.6), and Remark 2.8.25, for (2.8.48) we need to prove that

$$\limsup_{k\to\infty} \int_0^t \sup_{\lambda\in\mathbb{R}^d} (\lambda\cdot\dot{\mathbf{x}}_s^k - g_s(\lambda;\mathbf{x}^k))ds \le \int_0^t \sup_{\lambda\in\mathbb{R}^d} (\lambda\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\hat{\mathbf{x}}))ds.$$

Since for $y \in \operatorname{ri} N$ the left equality in (2.8.46) holds and $\dot{\mathbf{x}}_s^k \in G_k \subset \operatorname{ri} N$, it suffices to prove that

$$\limsup_{k\to\infty} \int_0^t \sup_{\lambda\in \operatorname{aff} N} (\lambda\cdot\dot{\mathbf{x}}_s^k - g_s(\lambda;\mathbf{x}^k))\,ds \le \int_0^t \sup_{\lambda\in \operatorname{aff} N} (\lambda\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\hat{\mathbf{x}}))\,ds. \quad (2.8.49)$$

We denote $\lambda_s^k = \Lambda(s, \mathbf{x}^k, \dot{\mathbf{x}}_s^k)$ and observe that $\lambda_s^k \in \operatorname{aff} N$. Since $\dot{\mathbf{x}}_s^k \in N$, it follows that $\lambda_s^k \cdot \dot{\mathbf{x}}_s^k \le \operatorname{proj}_N \lambda_s^k \cdot \dot{\mathbf{x}}_s^k$, so, by the definition of $\Lambda(s, \mathbf{x}, y)$

$$0 \le \sup_{\lambda\in \operatorname{aff} N} (\lambda\cdot\dot{\mathbf{x}}_s^k - g_s(\lambda;\mathbf{x}^k)) \le |\operatorname{proj}_N \lambda_s^k||\dot{\mathbf{x}}_s^k| - g_s(\lambda_s^k;\mathbf{x}^k),$$

which implies by the facts that $\dot{\mathbf{x}}_s^k$ is bounded (see (2.8.47)), $\lambda_s^k \in \operatorname{aff} N$ and condition 1 holds that

$$B = \sup_{k\ge k_0} \sup_{s\le t} |\operatorname{proj}_N \lambda_s^k| < \infty. \quad (2.8.50)$$

Next, by the definitions of λ_s^k and $\mathbf{x}_s^k$

$$\sup_{\lambda\in \operatorname{aff} N} (\lambda\cdot\dot{\mathbf{x}}_s^k - g_s(\lambda;\mathbf{x}^k)) = (\lambda_s^k\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda_s^k;\hat{\mathbf{x}})) + \alpha_k \lambda_s^k\cdot\hat{y} - (g_s(\lambda_s^k;\mathbf{x}^k) - g_s(\lambda_s^k;\hat{\mathbf{x}})). \quad (2.8.51)$$

Since $|\hat{y}| = 1$ and $\hat{y} \in N$, we have that $\lambda_s^k \cdot \hat{y} \le |\operatorname{proj}_N \lambda_s^k| \le B$. Also, by (2.8.50) for $k \ge k_0$ and $s \le t$

$$|g_s(\lambda_s^k;\mathbf{x}^k) - g_s(\lambda_s^k;\hat{\mathbf{x}})| \le \sup_{\substack{\lambda\in \operatorname{aff} N:\\ |\operatorname{proj}_N \lambda|\le B}} |g_s(\lambda;\mathbf{x}^k) - g_s(\lambda;\hat{\mathbf{x}})|.$$

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Therefore, by (2.8.51) and (2.8.50) for $k \geq k_0$ and $s \leq t$

$$\sup_{\lambda \in \operatorname{aff} N} (\lambda \cdot \dot{\mathbf{x}}_s^k - g_s(\lambda; \mathbf{x}^k)) \leq \sup_{\lambda \in \operatorname{aff} N} (\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda; \hat{\mathbf{x}})) + B\alpha_k + \sup_{\substack{\lambda \in \operatorname{aff} N: \\ |\operatorname{proj}_N \lambda| \leq B}} |g_s(\lambda; \mathbf{x}^k) - g_s(\lambda; \hat{\mathbf{x}})|$$

which implies (2.8.49) by the convergence $\alpha_k \to 0$ and condition 3, finishing the proof of (2.8.48).

We consider now the second assertion of the lemma. There is something to prove only if $N \neq \mathbb{R}^d$. If $y \in \mathbb{R}^d \setminus N$, then $\lambda \cdot y \geq \varepsilon |\lambda||y|$ for some $\lambda \in N^\perp$ and $\varepsilon > 0$, so by condition 4 of the lemma, for every $\mathbf{x} \in \mathbb{C}$,

$$\limsup_{\substack{\lambda \in N^\perp: \\ |\lambda| \to \infty}} \frac{\lambda \cdot y - g_s(\lambda; \mathbf{x})}{|\lambda|} \geq \varepsilon |y|,$$

and therefore $\sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot y - g_s(\lambda; \mathbf{x})) = \infty$. By Lemma 2.7.12, (2.7.6) and (2.7.8) this implies that $\dot{\mathbf{x}}_s \in N$ if $\mathbf{\Pi}_x(\mathbf{x}) > 0$.

□

We now apply Theorem 2.8.27 and Lemma 2.8.31 to a proof of the following uniqueness result.

Theorem 2.8.32. *Let conditions I and II hold, and $g_s(\lambda; \mathbf{x})$ be differentiable and strictly convex in $\lambda \in \mathbb{R}^d$. Let, in addition, the following hold:*

1. *for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$*

$$\lim_{|\lambda| \to \infty} \inf_{s \leq t} \inf_{\mathbf{x} \in K} \frac{g_s(\lambda; \mathbf{x})}{|\lambda|} = \infty,$$

2. *for every $t \in \mathbb{R}_+$, compact $K \subset \mathbb{C}$ and $A \in \mathbb{R}_+$*

$$\inf_{|\lambda| \leq A} \inf_{s \leq t} \inf_{\mathbf{x} \in K} g_s(\lambda; \mathbf{x}) > -\infty,$$

3. *for every $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$ there exists $l > 1$ such that*

$$\liminf_{|\lambda| \to \infty} \inf_{s \leq t} \frac{g_s(l\lambda; \mathbf{x})}{l g_s(\lambda; \mathbf{x})} > 1,$$

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4. for every $t \in \mathbb{R}_+$, compact $K \subset \mathbb{C}$ and $\mathbf{x} \in K$ there exist $\eta > 0$ and $\gamma > 0$ such that

$$\liminf_{|\lambda|\to\infty} \inf_{s\le t} \inf_{\substack{\mathbf{x}'\in K:\\ \sup_{r\le t} |\mathbf{x}_r - \mathbf{x}'_r|\le\gamma}} \frac{g_s(\lambda;\mathbf{x}')}{g_s(\eta\lambda;\mathbf{x})} > 0.$$

Then uniqueness holds for the maxingale problem (x, G) with $\Pi = \mathbf{\Pi}_x$.

Proof. We first observe that conditions 1 and 2 of the theorem imply that for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$

$$\inf_{\lambda\in\mathbb{R}^d} \inf_{s\le t} \inf_{\mathbf{x}\in K} g_s(\lambda;\mathbf{x}) > -\infty. \tag{2.8.52}$$

It is easy to see that all the conditions of Lemma 2.8.31 hold with $N = \mathbb{R}^d$. (In particular, condition 3 follows by conditions I and II.) Hence, by Lemma 2.8.31 $\Pi(p_t^{-1} \circ p_t\,\mathbf{x}) = \mathbf{\Pi}_{x,t}(\mathbf{x})$, $t \in \mathbb{R}_+$, if, in addition, $\mathbf{x}_0 = x$ and $\sup_{s\in\mathbb{R}_+} |\dot{\mathbf{x}}_s| < \infty$. The proof of Lemma 2.8.31 also shows that there exists function $\Lambda(s, \mathbf{x}, y)$ satisfying the conditions of Theorem 2.8.27 with $G_m = \{y \in \mathbb{R}^d : |y| \le m\}$ so that by Theorem 2.8.27 it suffices to prove that the set $D = \{\mathbf{x} : \sup_{s\in\mathbb{R}_+} |\dot{\mathbf{x}}_s| < \infty, \mathbf{x}_0 = x\}$ is $\mathbf{\Pi}_x$–dense in $\mathbb{C}$. Assuming with no loss of generality that $x = 0$, we fix $\hat{\mathbf{x}} \in \mathbb{C}$ such that $\mathbf{\Pi}_0(\hat{\mathbf{x}}) > 0$ and look for $\mathbf{x}^k \in D$ such that $\mathbf{x}^k \to \hat{\mathbf{x}}$ as $k \to \infty$ and

$$\lim_{k\to\infty} \mathbf{I}_t(\mathbf{x}^k) = \mathbf{I}_t(\hat{\mathbf{x}}),\ t \in \mathbb{R}_+. \tag{2.8.53}$$

We define

$$\mathbf{x}^k_s = \int_0^s \dot{\hat{\mathbf{x}}}_u\, \mathbf{1}(|\dot{\hat{\mathbf{x}}}_u| \le k)\, du. \tag{2.8.54}$$

The convergence $\mathbf{x}^k \to \hat{\mathbf{x}}$ follows by the fact that $\int_0^t |\dot{\hat{\mathbf{x}}}_u|\, du < \infty$ and Lebesgue's dominated convergence theorem. By Remark 2.8.25 for (2.8.53) it suffices to show that

$$\limsup_{k\to\infty} \mathbf{I}_t(\mathbf{x}^k) \le \mathbf{I}_t(\hat{\mathbf{x}}),\ t \in \mathbb{R}_+, \tag{2.8.55}$$

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where by Lemma 2.7.12 and (2.8.54)

$$\mathbf{I}_t(\hat{\mathbf{x}}) = \int_0^t \sup_{\lambda\in\mathbb{R}^d} \left(\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\hat{\mathbf{x}})\right) ds,$$

$$\mathbf{I}_t(\mathbf{x}^k) = \int_0^t \sup_{\lambda\in\mathbb{R}^d} \left(\lambda \cdot \dot{\hat{\mathbf{x}}}_s \ \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| \le k) - g_s(\lambda;\mathbf{x}^k)\right) ds.$$

Noting that by (2.8.54)

$$\mathbf{I}_t(\mathbf{x}^k) = \int_0^t \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > k) \sup_{\lambda\in\mathbb{R}^d} \left(-g_s(\lambda;\mathbf{x}^k)\right) ds + \int_0^t \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| \le k) \sup_{\lambda\in\mathbb{R}^d} \left(\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\mathbf{x}^k)\right) ds$$

and that condition 1 of the lemma and conditions I and II easily yield the convergence

$$\lim_{k\to\infty} \int_0^t \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > k) \sup_{\lambda\in\mathbb{R}^d} \left(-g_s(\lambda;\mathbf{x}^k)\right) ds = 0,$$

we have that (2.8.55) would follow by

$$\limsup_{k\to\infty} \int_0^t \sup_{\lambda\in\mathbb{R}^d} \left(\lambda\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\mathbf{x}^k)\right) ds \le \int_0^t \sup_{\lambda\in\mathbb{R}^d} \left(\lambda\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\hat{\mathbf{x}})\right) ds. \tag{2.8.56}$$

By the definition of $\Lambda(s,\mathbf{x},y)$

$$\Lambda(s,\hat{\mathbf{x}},\dot{\hat{\mathbf{x}}}_s)\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\Lambda(s,\hat{\mathbf{x}},\dot{\hat{\mathbf{x}}}_s);\hat{\mathbf{x}}) = \sup_{\lambda\in\mathbb{R}^d} \left(\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\hat{\mathbf{x}})\right), \tag{2.8.57a}$$

$$\Lambda(s,\mathbf{x}^k,\dot{\hat{\mathbf{x}}}_s)\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\Lambda(s,\mathbf{x}^k,\dot{\hat{\mathbf{x}}}_s);\mathbf{x}^k) = \sup_{\lambda\in\mathbb{R}^d} \left(\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\mathbf{x}^k)\right). \tag{2.8.57b}$$

We denote

$$\hat{\lambda}_s = \Lambda(s,\hat{\mathbf{x}},\dot{\hat{\mathbf{x}}}_s), \quad \hat{\lambda}_s^k = \Lambda(s,\mathbf{x}^k,\dot{\hat{\mathbf{x}}}_s)\ \mathbf{1}(|\Lambda(s,\mathbf{x}^k,\dot{\hat{\mathbf{x}}}_s)| \le a_k), \tag{2.8.58}$$

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where $a_k \uparrow \infty$ are chosen so that

$$\limsup_{k\to\infty} \int_0^t \sup_{\lambda\in\mathbb{R}^d} (\lambda\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\mathbf{x}^k))\,ds = \limsup_{k\to\infty} \int_0^t (\hat{\lambda}_s^k\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\hat{\lambda}_s^k;\mathbf{x}^k))\,ds.$$

Then (2.8.56) and the lemma would be proved if

$$\limsup_{k\to\infty} \int_0^t (\hat{\lambda}_s^k\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\hat{\lambda}_s^k;\mathbf{x}^k))\,ds \le \int_0^t (\hat{\lambda}_s\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\hat{\lambda}_s;\hat{\mathbf{x}}))\,ds. \quad (2.8.59)$$

We first note that by continuity of $\Lambda(s,\mathbf{x},y)$ in $\mathbf{x}$, the convergence $\mathbf{x}^k \to \hat{\mathbf{x}}$ and (2.8.58)

$$\lim_{k\to\infty} \hat{\lambda}_s^k = \hat{\lambda}_s,\ s\in\mathbb{R}_+. \quad (2.8.60)$$

We now show that for some $\alpha > 0$

$$\sup_k \int_0^t |g_s(\alpha\hat{\lambda}_s^k;\hat{\mathbf{x}})|\,ds < \infty. \quad (2.8.61)$$

By Young's inequality for every $\alpha > 0$

$$\int_0^t \hat{\lambda}_s^k\cdot\dot{\hat{\mathbf{x}}}_s\,ds \le \frac{1}{\alpha}\int_0^t g_s(\alpha\hat{\lambda}_s^k;\hat{\mathbf{x}})\,ds + \frac{1}{\alpha}\mathbf{I}_t(\hat{\mathbf{x}})$$

(the integrals are well defined since $\hat{\lambda}_s^k$ is bounded). Thus, since $\hat{\lambda}_s^k\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\hat{\lambda}_s^k;\mathbf{x}^k) \ge 0$ (see (2.8.57a) and (2.8.58)), we have that

$$\int_0^t g_s(\hat{\lambda}_s^k;\mathbf{x}^k)\,ds \le \frac{1}{\alpha}\int_0^t g_s(\alpha\hat{\lambda}_s^k;\hat{\mathbf{x}})\,ds + \frac{1}{\alpha}\mathbf{I}_t(\hat{\mathbf{x}}). \quad (2.8.62)$$

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Condition 3 of the lemma implies that, given $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$, there exist $l > 1$ and $\varepsilon > 0$ such that $g_s(l\lambda; \mathbf{x}) \geq (1+\varepsilon) l g_s(\lambda; \mathbf{x})$ for all $|\lambda|$ large enough and $s \leq t$ (by condition 1 of the lemma $g_s(\lambda; \mathbf{x})$ is non-negative if $|\lambda|$ is large enough). Hence, $g_s(l^p\lambda; \mathbf{x}) \geq (1+\varepsilon)^p l^p g_s(\lambda; \mathbf{x})$ for arbitrary $p \in \mathbb{N}$ so that for arbitrary $M > 1$ there exists $L > 1$ such that for all $|\lambda|$ large enough

$$g_s(\lambda; \mathbf{x}) \geq ML\, g_s\Bigl(\frac{\lambda}{L}; \mathbf{x}\Bigr), \ s \leq t. \tag{2.8.63}$$

Combining this with condition 4 and recalling condition 1, we conclude that there exist $\delta > 0, \eta > 0$ and $k_0 \in \mathbb{N}$ such that for arbitrary $M > 1$ there exist $L > 1$ and $A > 0$ for which

$$g_s(\lambda; \mathbf{x}^k) \geq \delta g_s(\eta\lambda; \hat{\mathbf{x}}) \geq \delta ML\, g_s\Bigl(\frac{\eta}{L}\lambda; \hat{\mathbf{x}}\Bigr), \ s \leq t,$$

for all $k \geq k_0$ and $|\lambda| \geq A$. Choosing now $\alpha = \eta/L$ and $M = 2/(\delta\eta)$, we have that $g_s(\lambda; \mathbf{x}^k) \geq 2g_s(\alpha\lambda; \hat{\mathbf{x}})/\alpha$, $s \leq t$, when $k \geq k_0$ and $|\lambda| \geq A$. Hence, for $k \geq k_0$

$$\int_0^t g_s(\hat{\lambda}_s^k; \mathbf{x}^k)\, ds \geq \int_0^t \inf_{|\lambda| \leq A} g_s(\lambda; \mathbf{x}^k)\, ds + \int_0^t g_s(\hat{\lambda}_s^k; \mathbf{x}^k)\ \mathbf{1}(|\hat{\lambda}_s^k| > A)\, ds$$

$$\geq \int_0^t \inf_{|\lambda| \leq A} g_s(\lambda; \mathbf{x}^k)\, ds + \frac{2}{\alpha} \int_0^t g_s(\alpha\hat{\lambda}_s^k; \hat{\mathbf{x}})\, ds - \frac{2}{\alpha} \int_0^t \sup_{|\lambda| \leq A} g_s(\alpha\lambda; \hat{\mathbf{x}})\, ds,$$

and (2.8.62) yields for $k \geq k_0$ after a simple algebra

$$\int_0^t g_s(\alpha\hat{\lambda}_s^k; \hat{\mathbf{x}})\, ds \leq \mathbf{I}_t(\hat{\mathbf{x}}) + 2\int_0^t \sup_{|\lambda| \leq A} g_s(\alpha\lambda; \hat{\mathbf{x}})\, ds - \alpha \int_0^t \inf_{|\lambda| \leq A} g_s(\lambda; \mathbf{x}^k)\, ds,$$

which yields, since $\mathbf{I}_t(\hat{\mathbf{x}}) < \infty$ and conditions I and II hold,

$$\sup_k \int_0^t g_s(\alpha\hat{\lambda}_s^k; \hat{\mathbf{x}})\, ds < \infty.$$

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Inequality (2.8.61) now follows since the functions $g_s(\alpha\hat{\lambda}_s^k;\hat{\mathbf{x}})$ are bounded from below uniformly in $s \in [0,t]$ and k in view of (2.8.52).

Inequalities (2.8.61) and (2.8.62) yield, since by (2.8.52) the functions $g_s(\hat{\lambda}_s^k;\mathbf{x}^k)$ are bounded from below uniformly in $s \le t$ and k,

$$\sup_k \int_0^t |g_s(\hat{\lambda}_s^k;\mathbf{x}^k)|\, ds < \infty. \tag{2.8.64}$$

Since condition I, the convergences $\mathbf{x}^k \to \hat{\mathbf{x}}$ and (2.8.60) imply that $g_s(\hat{\lambda}_s^k;\mathbf{x}^k) \to g_s(\hat{\lambda}_s;\hat{\mathbf{x}}), s \le t$, as $k \to \infty$, by Fatou's lemma and (2.8.64) $\int_0^t |g_s(\hat{\lambda}_s;\hat{\mathbf{x}})|\, ds < \infty$; hence, by Lemma 2.7.12

$$\int_0^t (\hat{\lambda}_s \cdot \dot{\hat{\mathbf{x}}}_s) \vee 0\, ds \le \mathbf{I}_t(\hat{\mathbf{x}}) + \int_0^t |g_s(\hat{\lambda}_s;\hat{\mathbf{x}})|\, ds < \infty. \tag{2.8.65}$$

On the other hand, since by (2.8.57a) and (2.8.58) $\hat{\lambda}_s \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\hat{\lambda}_s;\hat{\mathbf{x}}) \ge 0$ and by (2.8.52) the function $g_s(\hat{\lambda}_s;\hat{\mathbf{x}})$ is bounded from below on $[0,t]$, we conclude that $\hat{\lambda}_s \cdot \dot{\hat{\mathbf{x}}}_s$ is bounded from below on $[0,t]$. This fact and (2.8.65) show that

$$\int_0^t |\hat{\lambda}_s \cdot \dot{\hat{\mathbf{x}}}_s|\, ds < \infty, \tag{2.8.66}$$

in particular, $\int_0^t \hat{\lambda}_s \cdot \dot{\hat{\mathbf{x}}}_s\, ds$ is well defined and finite.

Since the convergence $g_s(\hat{\lambda}_s^k;\mathbf{x}^k) \to g_s(\hat{\lambda}_s;\hat{\mathbf{x}}), s \le t$, and uniform boundedness from below of the functions $g_s(\hat{\lambda}_s^k;\mathbf{x}^k)$, $s \in [0,t], k \in \mathbb{N}$, also imply by Fatou's lemma that

$$\liminf_{k\to\infty} \int_0^t g_s(\hat{\lambda}_s^k;\mathbf{x}^k)\, ds \ge \int_0^t g_s(\hat{\lambda}_s;\hat{\mathbf{x}})\, ds,$$

we conclude that (2.8.59) would follow by

$$\limsup_{k\to\infty} \int_0^t \hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s\, ds \le \int_0^t \hat{\lambda}_s \cdot \dot{\hat{\mathbf{x}}}_s\, ds.$$

To prove the latter, note that by La Valleé Poussin's theorem the sequence $\{(\hat{\lambda}_s^k,\ s \in [0,t]), k \in \mathbb{N}\}$ is uniformly integrable with respect

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to Lebesgue measure in view of (2.8.61) and condition 1 of the lemma. This fact and convergence (2.8.60) yield for arbitrary $m > 0$

$$\lim_{k\to\infty} \int_0^t \hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s \; \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| \le m)\, ds = \int_0^t \hat{\lambda}_s \cdot \dot{\hat{\mathbf{x}}}_s \; \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| \le m)\, ds.$$

In view of (2.8.66), we thus complete the proof by showing that

$$\limsup_{m\to\infty} \limsup_{k\to\infty} \int_0^t \hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s \; \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m)\, ds \le 0. \tag{2.8.67}$$

Given arbitrary $\varepsilon > 0$, by (2.8.61) we can choose $M_1 > 0$ such that

$$\frac{1}{M_1} \sup_k \int_0^t |g_s(\alpha \hat{\lambda}_s^k; \hat{\mathbf{x}})|\, ds \le \varepsilon. \tag{2.8.68}$$

By (2.8.63) we can choose $A_1 > 0$ and $L_1 > 0$ such that $g_s(\alpha\lambda; \hat{\mathbf{x}}) \ge M_1 L_1 \, g_s(\lambda / L_1; \hat{\mathbf{x}})$ for $s \in [0, t]$ and $|\lambda| > A_1$. Young's inequality then yields

$$\begin{aligned}
&\int_0^t \hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s \; \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m) \; \mathbf{1}(|\hat{\lambda}_s^k| > A_1)\, ds \\
&\le L_1 \Big[\int_0^t g_s\Big(\frac{\hat{\lambda}_s^k}{L_1}; \hat{\mathbf{x}}\Big)\, \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m)\, \mathbf{1}(|\hat{\lambda}_s^k| > A_1)\, ds \\
&+ \int_0^t \sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda; \hat{\mathbf{x}})) \; \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m)\, ds \Big] \\
&\le \frac{1}{M_1} \int_0^t |g_s(\alpha \hat{\lambda}_s^k; \hat{\mathbf{x}})|\, ds \\
&\qquad + L_1 \int_0^t \sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda; \hat{\mathbf{x}})) \; \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m)\, ds.
\end{aligned}$$

Inequality (2.8.68) and finiteness of $\mathbf{I}_t(\hat{\mathbf{x}})$ then imply that

$$\limsup_{m\to\infty} \limsup_{k\to\infty} \int_0^t \hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s \; \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m) \; \mathbf{1}(|\hat{\lambda}_s^k| > A_1)\, ds \le \varepsilon.$$

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Since

$$\int_0^t \hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s \, \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m)\, ds \le A_1 \int_0^t |\dot{\hat{\mathbf{x}}}_s| \, \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m)\, ds$$

$$+ \int_0^t \hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s \, \mathbf{1}(|\dot{\hat{\mathbf{x}}}_s| > m)\, \mathbf{1}(|\hat{\lambda}_s^k| > A_1)\, ds,$$

and $\int_0^t |\dot{\hat{\mathbf{x}}}_s|\, ds < \infty$, the proof of (2.8.67) is over. □

The following uniqueness result, which is stated in terms of $b_s(\mathbf{x})$ and $\hat{g}_s(\lambda; \mathbf{x})$, is a direct consequence of Theorem 2.8.32.

Theorem 2.8.33. *Let conditions I and II hold, and $\hat{g}_s(\lambda; \mathbf{x})$ be differentiable and strictly convex in λ. Let the following hold:*

1. *for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$*

$$\lim_{|\lambda|\to\infty} \inf_{s\le t} \inf_{\mathbf{x}\in K} \frac{\hat{g}_s(\lambda; \mathbf{x})}{|\lambda|} = \infty,$$

2. *for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$*

$$\sup_{s\le t} \sup_{\mathbf{x}\in K} |b_s(\mathbf{x})| < \infty,$$

3. *for every $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$ there exists $l > 1$ such that*

$$\liminf_{|\lambda|\to\infty} \inf_{s\le t} \frac{\hat{g}_s(l\lambda; \mathbf{x})}{l\hat{g}_s(\lambda; \mathbf{x})} > 1,$$

4. *for every $t \in \mathbb{R}_+$, compact $K \subset \mathbb{C}$ and $\mathbf{x} \in K$ there exist $\eta > 0$ and $\gamma > 0$ such that*

$$\liminf_{|\lambda|\to\infty} \inf_{s\le t} \inf_{\substack{\mathbf{x}'\in K: \\ \sup_{r\le t} |\mathbf{x}_r - \mathbf{x}'_r| \le \gamma}} \frac{\hat{g}_s(\lambda; \mathbf{x}')}{\hat{g}_s(\eta\lambda; \mathbf{x})} > 0.$$

Then $\Pi = \mathbf{\Pi}_x$.

As another consequence, we have the following.

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Theorem 2.8.34. *Let* Π *be a deviability on* $\mathbb{C}$ *under which the canonical idempotent process* X *is a semimaxingale starting at* x *with local characteristics* $(b, c, \nu, 0)$*. Let the following conditions hold:*

1. *the functions* $b_s(\mathbf{x})$ *and* $c_s(\mathbf{x})$ *are continuous in* $\mathbf{x} \in \mathbb{C}$ *and the function* $\int_{\mathbb{R}^d} (e^{\lambda \cdot x} - 1 - \lambda \cdot x)\, \nu_s(dx; \mathbf{x})$ *is continuous in* $(\lambda, \mathbf{x}) \in \mathbb{R}^d \times \mathbb{C}$*;*

2. *for every* $t \in \mathbb{R}_+$, $A > 0$ *and compact* $K \subset \mathbb{C}$

$$\sup_{s \le t} \sup_{\mathbf{x} \in K} |b_s(\mathbf{x})| < \infty, \quad \sup_{s \le t} \sup_{\mathbf{x} \in K} \|c_s(\mathbf{x})\| < \infty,$$
$$\sup_{s \le t} \sup_{\mathbf{x} \in K} \int_{\mathbb{R}^d} (e^{A|x|} - 1 - A|x|)\, \nu_s(dx; \mathbf{x}) < \infty$$

and

$$\lim_{\lambda \to 0} \sup_{s \le t} \sup_{\mathbf{x} \in K} \int_{\mathbb{R}^d} (e^{\lambda \cdot x} - 1 - \lambda \cdot x)\, \nu_s(dx; \mathbf{x}) = 0;$$

3. *for every* $t \in \mathbb{R}_+$ *and compact* $K \subset \mathbb{C}$ *there exists* $B \in \mathbb{R}_+$ *such that*

a)
$$\inf_{\substack{\lambda \in \mathbb{R}^d: \\ |\lambda| = 1}} \inf_{s \le t} \inf_{\mathbf{x} \in K} \int_{\mathbb{R}^d} \mathbf{1}(\lambda \cdot x > B)\, \nu_s(dx; \mathbf{x}) > 0,$$

b) for every $\mathbf{x} \in K$ *there exist* $\eta > 0$ *and* $\gamma > 0$ *such that*

$$\liminf_{v \to \infty} \inf_{\substack{\lambda \in \mathbb{R}^d: \\ |\lambda| = 1}} \inf_{s \le t} \inf_{\substack{\mathbf{x}' \in K: \\ \sup_{r \le t} |\mathbf{x}_r - \mathbf{x}'_r| \le \gamma}} \frac{\int_{\mathbb{R}^d} \exp(v \lambda \cdot x)\, \mathbf{1}(\lambda \cdot x > B)\, \nu_s(dx; \mathbf{x}')}{\int_{\mathbb{R}^d} \exp(\eta v \lambda \cdot x)\, \mathbf{1}(\lambda \cdot x > B)\, \nu_s(dx; \mathbf{x})} > 0.$$

Then $\Pi = \mathbf{\Pi}_x$.

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Proof. We check the conditions of Theorem 2.8.33. Conditions I and II are met by Lemma 2.8.16. Condition 2 of Theorem 2.8.33 holds by hypotheses. Condition 1 of Theorem 2.8.33 follows since $\hat{g}_s(\lambda;\mathbf{x})$ grows at least exponentially fast in λ, which results from the inequalities

$$\hat{g}_s(\lambda;\mathbf{x}) \geq \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda\cdot x)\,\nu_s(dx;\mathbf{x})$$
$$\geq (e^{|\lambda|B} - 1 - |\lambda|B)\int_{\mathbb{R}^d} \mathbf{1}(\lambda\cdot x > B|\lambda|)\,\nu_s(dx;\mathbf{x})$$

and condition 3a).

Let us consider condition 3 of Theorem 2.8.33. We have that

$$\hat{g}_s(\lambda;\mathbf{x}) = \frac{1}{2}\,\lambda\cdot c_s(\mathbf{x})\lambda + \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda\cdot x)\,\nu_s(dx;\mathbf{x})$$
$$\leq \frac{1}{2}\,\lambda\cdot c_s(\mathbf{x})\lambda + \frac{1}{3}\int_{\mathbb{R}^d} (e^{2\lambda\cdot x} - 1 - 2\lambda\cdot x)\ \mathbf{1}(\lambda\cdot x > 0)\,\nu_s(dx;\mathbf{x})$$
$$+ \frac{1}{2}\int_{\mathbb{R}^d} (\lambda\cdot x)^2\ \mathbf{1}(\lambda\cdot x \leq 0)\,\nu_s(dx;\mathbf{x})$$
$$\leq \frac{1}{3}\hat{g}_s(2\lambda;\mathbf{x}) + \frac{1}{2}\int_{\mathbb{R}^d} (\lambda\cdot x)^2\,\nu_s(dx;\mathbf{x}).$$

Therefore, by the fact that $\hat{g}_s(\lambda;\mathbf{x})$ grows at least exponentially fast in λ and conditions 2 and 3a) of the theorem

$$\liminf_{|\lambda|\to\infty} \inf_{s\leq t} \frac{\hat{g}_s(2\lambda;\mathbf{x})}{\hat{g}_s(\lambda;\mathbf{x})} \geq 3,$$

verifying condition 3 of Theorem 2.8.33.

Finally, condition 4 of Theorem 2.8.33 follows by the inequalities

$$\hat{g}_s(\lambda;\mathbf{x}) \geq \int_{\mathbb{R}^d} e^{\lambda\cdot x/2}\,\mathbf{1}(\lambda\cdot x > B|\lambda|)\,\nu_s(dx;\mathbf{x})$$

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when $|\lambda|$ is large enough, and

$$\hat{g}_s(\eta\lambda;\mathbf{x}) \leq \frac{1}{2}(\eta\lambda)\cdot c_s(\mathbf{x})(\eta\lambda) + \frac{1}{2}\int\limits_{\mathbb{R}^d}(\eta\lambda\cdot x)^2\,\nu_s(dx;\mathbf{x})$$

$$+\frac{1}{2}e^{B|\eta\lambda|}\int\limits_{\mathbb{R}^d}(\eta\lambda\cdot x)^2\,\nu_s(dx;\mathbf{x})+\int\limits_{\mathbb{R}^d}e^{\eta\lambda\cdot x}\,\mathbf{1}(\lambda\cdot x > B|\lambda|)\,\nu_s(dx;\mathbf{x}),$$

and conditions 2 and 3. ☐

Remark 2.8.35. *Condition 3b) holds if condition 3a) holds and*

$$\limsup_{u\to\infty}\sup_{s\leq t}\sup_{\mathbf{x}\in K}\frac{\ln\int_{\mathbb{R}^d}e^{u|x|}\,\mathbf{1}(|x|>B)\nu_s(dx;\mathbf{x})}{u} < \infty.$$

Theorems 2.8.21, 2.8.28, 2.8.32, 2.8.33, and 2.8.34 are concerned with "the nondegenerate case" singled out by condition 1 of Theorem 2.8.33. We now consider another degenerate case along with Theorem 2.8.29, which takes advantage of the generality of Lemma 2.8.31.

Theorem 2.8.36. *Let* Π *be a deviability on* $\mathbb{C}$ *under which the canonical process* X *is a semimaxingale starting at* x *with local characteristics* $(b,0,\nu,0)$ *such that for some* $l\in\mathbb{N}$ *and* $v_i\in\mathbb{R}^d$

$$b_s(\mathbf{x}) = \sum_{i=1}^{l} b_s^{(i)}(\mathbf{x})v_i, \qquad \nu_s(\Gamma;\mathbf{x}) = \sum_{i=1}^{l}\mathbf{1}(v_i\in\Gamma)b_s^{(i)}(\mathbf{x}),$$

where $\mathbb{R}_+$*-valued functions* $b_s^{(i)}(\mathbf{x})$ *are continuous in* $\mathbf{x}$ *and* $\mathbf{C}$*-progressively measurable.*

Let also for every $t\in\mathbb{R}_+$ *and compact* $K\subset\mathbb{C}$

$$\inf_{s\leq t}\inf_{\mathbf{x}\in K} b_s^{(i)}(\mathbf{x}) > 0,\ \ \sup_{s\leq t}\sup_{\mathbf{x}\in K} b_s^{(i)}(\mathbf{x}) < \infty, \quad 1\leq i\leq l.$$

Then $\Pi = \mathbf{\Pi}_x$ *.*

Proof. Let N denote the smallest closed convex cone containing $v_1,\ldots,v_l$. Noting that $g_s(\lambda;\mathbf{x}) = \sum_{i=1}^{l}\big(\exp(\lambda\cdot v_i)-1\big)b_s^{(i)}(\mathbf{x})$, one can see that the hypotheses of Lemma 2.8.31 are satisfied. Specifically, the following stronger versions of conditions 1 and 2 of Lemma 2.8.31 hold:

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$1'$. for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$

$$\lim_{\substack{\lambda \in \mathbb{R}^d: \\ |\mathrm{proj}_N \lambda| \to \infty}} \inf_{s \le t} \inf_{\mathbf{x} \in K} \frac{g_s(\lambda; \mathbf{x})}{|\mathrm{proj}_N \lambda|} = \infty;$$

$2'$. for every $t \in \mathbb{R}_+$ and compact $K \subset \mathbb{C}$

$$\inf_{\lambda \in \mathbb{R}^d} \inf_{s \le t} \inf_{\mathbf{x} \in K} g_s(\lambda; \mathbf{x}) > -\infty.$$

Property $2'$ is obvious, property $1'$ follows by the inequality

$$|\mathrm{proj}_N \lambda| \le c \max_{i=1,\dots,l} (\lambda \cdot v_i) \vee 0, \tag{2.8.69}$$

where c is a constant depending only on $v_1, v_2, \dots, v_l$.

By Lemma 2.8.31 $\Pi(p_t^{-1} \circ p_t \mathbf{x}) = \mathbf{\Pi}_{x,t}(\mathbf{x})$ when $\mathbf{x}_0 = x$, $\dot{\mathbf{x}}_s \in N$ and $\sup_{s \in \mathbb{R}_+} |\dot{\mathbf{x}}_s| < \infty$. Therefore, in analogy with the proof of Theorem 2.8.32 it suffices to show that the set $D = \{\mathbf{x} \in \mathbb{C} : \dot{\mathbf{x}}_s \in N, \sup_{s \in \mathbb{R}_+} |\dot{\mathbf{x}}_s| < \infty, \mathbf{x}_0 = x\}$ is $\mathbf{\Pi}_x$–dense in $\mathbb{C}$. Let $\hat{\mathbf{x}} \in \mathbb{C}$ be such that $\mathbf{\Pi}_x(\hat{\mathbf{x}}) > 0$ and $\mathbf{x}^k$ be defined by (2.8.54). By Lemma 2.8.31 $\dot{\hat{\mathbf{x}}}_s \in N$, so $\mathbf{x}^k \in D$. The argument of the proof of Theorem 2.8.32 with the use of property $2'$ implies that it suffices to establish (2.8.56) for $t \in \mathbb{R}_+$. Since $\dot{\hat{\mathbf{x}}}_s \in N$, we have that $\lambda \cdot \dot{\hat{\mathbf{x}}}_s \le \mathrm{proj}_N \lambda \cdot \dot{\hat{\mathbf{x}}}_s$ for $\lambda \in \mathbb{R}^d$, which implies by properties $1'$ and $2'$ that $\sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda; \mathbf{x}^k)) \le \sup_{\lambda \in \mathbb{R}^d} (\mathrm{proj}_N \lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda; \mathbf{x}^k)) < \infty$, $k \in \mathbb{N}$. Therefore, by a measurable-selection theorem there exist Lebesgue measurable functions $(\tilde{\lambda}_s^k,\ s \in \mathbb{R}_+)$ such that

$$\tilde{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\tilde{\lambda}_s^k; \mathbf{x}^k) \ge \left[\sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda; \mathbf{x}^k)) - \frac{1}{k} \right] \vee 0.$$

Then, for suitable $a_k > 0$, the functions $\hat{\lambda}_s^k = \tilde{\lambda}_s^k \mathbf{1}(|\tilde{\lambda}_s^k| \le a_k)$ are bounded and satisfy the equality

$$\limsup_{k \to \infty} \int_0^t \sup_{\lambda \in \mathbb{R}^d} (\lambda \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\lambda; \mathbf{x}^k))\, ds$$
$$= \limsup_{k \to \infty} \int_0^t (\hat{\lambda}_s^k \cdot \dot{\hat{\mathbf{x}}}_s - g_s(\hat{\lambda}_s^k; \mathbf{x}^k))\, ds$$

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so (2.8.56) and the theorem are proved if

$$\limsup_{k\to\infty}\int_0^t (\hat\lambda_s^k\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\hat\lambda_s^k;\mathbf{x}^k))\,ds \le \int_0^t \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\hat{\mathbf{x}}))\,ds.$$

Next, the hypotheses on $b_s^{(i)}(\mathbf{x})$ imply that, given arbitrary $\varepsilon\in(0,1)$, we have, for k large enough,

$$(1-\varepsilon)\int_0^t b_s^{(i)}(\mathbf{x}^k)\,ds \le \int_0^t b_s^{(i)}(\hat{\mathbf{x}})\,ds \le (1+\varepsilon)\int_0^t b_s^{(i)}(\mathbf{x}^k)\,ds.$$

Hence, for arbitrary $\lambda\in\mathbb{R}^d$

$$\begin{aligned}(1+\varepsilon)\int_0^t g_s(\lambda;\mathbf{x}^k)\,ds &\ge \sum_{i=1}^l e^{\lambda\cdot v_i}\int_0^t b_s^{(i)}(\hat{\mathbf{x}})\,ds - \frac{1+\varepsilon}{1-\varepsilon}\sum_{i=1}^l\int_0^t b_s^{(i)}(\hat{\mathbf{x}})\,ds\\ &= \int_0^t g_s(\lambda;\hat{\mathbf{x}})\,ds - \frac{2\varepsilon}{1-\varepsilon}\sum_{i=1}^l\int_0^t b_s^{(i)}(\hat{\mathbf{x}})\,ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^t (\hat\lambda_s^k\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\hat\lambda_s^k;\mathbf{x}^k))\,ds &\le \int_0^t \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot\dot{\hat{\mathbf{x}}}_s - g_s(\lambda;\hat{\mathbf{x}}))\,ds\\ &\quad + \varepsilon\int_0^t g_s(\hat\lambda_s^k;\mathbf{x}^k)\,ds + \frac{2\varepsilon}{1-\varepsilon}\sum_{i=1}^l\int_0^t b_s^{(i)}(\hat{\mathbf{x}})\,ds,\end{aligned}$$

and, since ε can be taken arbitrarily small, the proof is complete if

$$\sup_k\int_0^t |g_s(\hat\lambda_s^k;\mathbf{x}^k)|\,ds < \infty. \tag{2.8.70}$$

The proof of the latter inequality is similar to that of (2.8.64) in the proof of Theorem 2.8.32. More specifically, it is not difficult to check that under the assumptions of the theorem the following holds:

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1. for every $t \in \mathbb{R}_+$, compact $K \subset \mathbb{C}$ and $A \in \mathbb{R}_+$,

$$\int_0^t \sup_{\mathbf{x}\in K} \sup_{\substack{\lambda\in\mathbb{R}^d:\\ |\mathrm{proj}_N\lambda|\le A}} |g_s(\lambda;\mathbf{x})|\,ds < \infty,$$

2. for every $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{C}$, there exists $l > 1$ such that

$$\liminf_{\substack{\lambda\in\mathbb{R}^d:\\ |\mathrm{proj}_N\lambda|\to\infty}} \inf_{s\le t} \frac{g_s(l\lambda;\mathbf{x})}{l g_s(\lambda;\mathbf{x})} > 1,$$

3. for every $t \in \mathbb{R}_+$, compact $K \subset \mathbb{C}$ and $\mathbf{x} \in K$, there exist $\eta > 0$ and $\gamma > 0$ such that

$$\liminf_{\substack{\lambda\in\mathbb{R}^d:\\ |\mathrm{proj}_N\lambda|\to\infty}} \inf_{s\le t} \inf_{\substack{\mathbf{x}'\in K:\\ \sup_{r\le t}|\mathbf{x}_r-\mathbf{x}'_r|\le\gamma}} \frac{g_s(\lambda;\mathbf{x}')}{g_s(\eta\lambda;\mathbf{x})} > 0.$$

Note that part 2 follows from (2.8.69), the other two properties being obvious.

These conditions, along with properties $1'$ and $2'$ above, imply (2.8.70) in the same way as in the proof of Theorem 2.8.32 conditions I and II together with conditions 1–4 of Theorem 2.8.32 implied (2.8.64) with $|\mathrm{proj}_N\lambda|$ playing the role of $|\lambda|$. □

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Part II

Large Deviation Convergence of Semimartingales

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Chapter 3

Large deviation convergence

This chapter contains basic facts on large deviation convergence in Tihonov spaces and their adaptation to the setting of the Skorohod space.

3.1 Large deviation convergence in Tihonov spaces

In this section we develop the theory of large deviation convergence in Tihonov spaces. Our exposition is along the lines of the content of Section 1.9.

Let E be a topological space with Borel σ-algebra $\mathcal{B}(E)$. Let Φ be a directed set, $\{P_\phi,\, \phi \in \Phi\}$ be a net of probability measures on $(E, \mathcal{B}(E))$, and $\{r_\phi,\, \phi \in \Phi\}$ be a net of real numbers greater than 1 converging to ∞ as $\phi \in \Phi$. We recall that $C_b^+(E)$, $\overline{C}_b^+(E)$, and $\underline{C}_b^+(E)$ denote the respective sets of $\mathbb{R}_+$-valued bounded continuous functions on E, $\mathbb{R}_+$-valued bounded upper semi-continuous functions on E, and $\mathbb{R}_+$-valued bounded lower semi-continuous functions on E. Let Π be an $\mathcal{F}$-idempotent probability on E, where $\mathcal{F}$ denotes the collection of closed subsets of E.

Definition 3.1.1. *We say that the net $\{P_\phi,\, \phi \in \Phi\}$ large deviation*

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(LD) converges at rate r_ϕ *to* Π *if for every* $h \in C_b^+(E)$

$$\lim_{\phi\in\Phi}\Big(\int_E h(z)^{r_\phi}\, dP_\phi(z)\Big)^{1/r_\phi} = \bigvee_E h(z)\, d\Pi(z). \tag{3.1.1}$$

Remark 3.1.2. *One could also consider the version of the above definition where* h *ranges in the set of* $\mathbb{R}_+$*-valued bounded continuous functions on* E *of compact support. Then the definition we have given would refer to "weak large deviation convergence", while the case of compactly supported* h *would specify "vague large deviation convergence". Since our focus is on "weak large deviation convergence", we simply call it "large deviation convergence".*

Note that if E is a Tihonov topological space, then according to Theorem 1.7.27 the $\mathcal{F}$-idempotent probability Π is uniquely specified by the right-hand sides of (3.1.1). We generally denote the large deviation convergence by $P_\phi \xrightarrow[r_\phi]{ld} \Pi$. Since the net r_ϕ is fixed in the rest of the chapter, we simplify the notation by writing $P_\phi \xrightarrow{ld} \Pi$. We denote $P_\phi^{1/r_\phi}(A) = \big(P_\phi(A)\big)^{1/r_\phi}$ and $\|f\|_\phi = \Big(\int_E f(z)^{r_\phi}\, dP_\phi(z)\Big)^{1/r_\phi}$, where $f : E \to \mathbb{R}_+$.

We state a Portmanteau theorem for large deviation convergence.

Theorem 3.1.3. *Let* E *be a Tihonov topological space. The following conditions are equivalent.*

1. $P_\phi \xrightarrow{ld} \Pi$.
2. (*i*) $\liminf_\phi \|g\|_\phi \ge \bigvee_E g\, d\Pi$ *for all* $g \in \underline{C}_b^+(E)$,

   (*ii*) $\limsup_\phi \|f\|_\phi \le \bigvee_E f\, d\Pi$ *for all* $f \in \overline{C}_b^+(E)$.

2′. *The inequalities of part 2 hold for all lower semi-continuous relative to* Π*, bounded Borel-measurable functions* $g : E \to \mathbb{R}_+$ *and all upper semi-continuous relative to* Π*, bounded Borel-measurable functions* $f : E \to \mathbb{R}_+$*, respectively.*

3. (*i*) $\liminf_\phi P_\phi^{1/r_\phi}(G) \ge \Pi(G)$ *for all open sets* $G \subset E$,

   (*ii*) $\limsup_\phi P_\phi^{1/r_\phi}(F) \le \Pi(F)$ *for all closed sets* $F \subset E$.

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3′. *The inequalities of part 3 hold for all open relative to* Π *Borel-measurable sets* G *and closed relative to* Π *Borel-measurable sets* F*, respectively.*

4. $\lim_\phi P_\phi^{1/r_\phi}(H) = \Pi(H)$ *for all continuous relative to* Π *Borel-measurable sets* $H \subset E$.

5. $\lim_\phi \|h\|_\phi = \bigvee_E h\, d\Pi$ *for all continuous relative to* Π *bounded Borel-measurable functions* $h : E \to \mathbb{R}_+$.

6. $\lim_\phi \|h\|_\phi = \bigvee_E h\, d\Pi$ *for all bounded Borel-measurable functions* $h : E \to \mathbb{R}_+$ *that are uniformly continuous with respect to a given uniformity on* E.

Proof. The proof is almost identical to the one of Theorem 1.9.2. We give it here to make the reading easier. Clearly, $1 \Rightarrow 6$, $2 \Leftrightarrow 2'$, $2 \Rightarrow 1$, $2 \Rightarrow 3$, $2 \Rightarrow 5$, $3 \Leftrightarrow 3'$, $3' \Rightarrow 4$, and $5 \Rightarrow 1$.

We prove the implication $1 \Rightarrow 3$. To prove $1 \Rightarrow 3(i)$, we note that, since E is Tihonov and G is open, $\mathbf{1}(G) = \sup h$ over $h \in C_b^+(E)$ such that $h \le \mathbf{1}(G)$. Therefore, by Theorem 1.4.4 $\Pi(G) = \sup_h \bigvee_E h\, d\Pi$, so that if $h_\epsilon \le \mathbf{1}(G)$ is such that $\Pi(G) \le \bigvee_E h_\varepsilon\, d\Pi + \varepsilon$, then

$$\liminf_\phi P_\phi^{1/r_\phi}(G) \ge \lim_\phi \|h_\varepsilon\|_\phi = \bigvee_E h_\varepsilon d\Pi \ge \Pi(G) - \varepsilon.$$

The proof of $3(ii)$ is analogous if we note that $\mathbf{1}_F = \inf h$ over $h \in C_b^+(E)$ such that $h \ge \mathbf{1}(F)$ so that by Theorem 1.4.19 $\Pi(F) = \inf_h \bigvee_E h\, d\Pi$.

We prove that $3(i) \Rightarrow 2(i)$ and $3(ii) \Rightarrow 2(ii)$. For $g \in \underline{C}_b^+(E)$ such that $\|g\| = 1$ let

$$g_k(z) = \max_{i=0,\dots,k-1} \Big[\frac{i}{k}\, \mathbf{1}\Big(g(z) > \frac{i}{k}\Big)\Big],\ k \in \mathbb{N}.$$

Since the sets $\{z : g(z) > x\}$ are open by the lower semi-continuity of g, $3(i)$ yields

$$\begin{aligned}\liminf_\phi \|g_k\|_\phi &\ge \max_{i=0,\dots,k-1} \liminf_\phi \Big[\frac{i}{k}\, P_\phi^{1/r_\phi}\Big(g(z) > \frac{i}{k}\Big)\Big] \\ &\ge \max_{i=0,\dots,k-1} \Big[\frac{i}{k}\, \Pi\Big(g(z) > \frac{i}{k}\Big)\Big] = \bigvee_E g_k\, d\Pi \ge \bigvee_E g\, d\Pi - \frac{1}{k}.\end{aligned}$$

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The proof of $3(ii) \Rightarrow 2(ii)$ is similar if we consider $f_k(z) = \max_{i=0,\dots,k-1}\big[(i+1)/k\ \mathbf{1}(f(z)\ge i/k)\big]$.

Now we prove $4 \Rightarrow 3$. Let G be open and $\delta > 0$. Let h be a function from $C_b^+(E)$ such that $h \le \mathbf{1}(G)$ and $\bigvee_E h\,d\Pi \ge \Pi(G) - \delta$. Let $H_u = \{z \in E :\ h(z) \ge u\}$, $u \in [0,1]$. Then the function $\Pi(H_u)$ increases as $u \downarrow 0$. Therefore, it has at most countably many jumps. Also $\Pi(H_u) \ge \bigvee_E h\,d\Pi - u$, so $\Pi(H_u) \ge \Pi(G) - 2\delta$ for u small enough. Thus, there exists $\varepsilon > 0$ such that $\Pi(H_\varepsilon) \ge \Pi(G) - 2\delta$ and $\Pi(H_u)$ is continuous at ε. By τ-maxitivity of Π the latter is equivalent to H_ε being continuous relative to Π, so we conclude that

$$\liminf_\phi P_\phi^{1/r_\phi}(G) \ge \lim_\phi P_\phi^{1/r_\phi}(H_\varepsilon) = \Pi(H_\varepsilon) \ge \Pi(G) - 2\delta.$$

The proof of $3(ii)$ is similar.

We prove that $6 \Rightarrow 3(ii)$. Let $\mathcal{V}$ be a uniformity on E and F be a closed subset of E. Let $\{\rho_\alpha\}$ be a collection of uniformly continuous with respect to $\mathcal{V}$ pseudo-metrics on E, which is closed under the formation of maximums and such that $\mathbf{1}(F) = \inf_{\varepsilon>0}\inf_\alpha(1-\rho_\alpha(z,F)/\varepsilon)^+$. (As above, $\rho_\alpha(z,F) = \inf_{z'\in F}\rho_\alpha(z,z')$.) The functions $(1-\rho_\alpha(z,F)/\varepsilon)^+$ are bounded and uniformly continuous with respect to $\mathcal{V}$ so that by Theorem 1.7.7

$$\limsup_\phi P_\phi^{1/r_\phi}(F) \le \inf_{\varepsilon>0}\inf_\alpha \lim_\phi \big\|\big(1-\rho_\alpha(z,F)/\varepsilon\big)^+\big\|_\phi$$
$$= \inf_{\varepsilon>0}\inf_\alpha \bigvee_E (1-\rho_\alpha(z,F)/\varepsilon)^+\,d\Pi(z) = \Pi(F).$$

The implication $6 \Rightarrow 3(i)$ is proved in an analogous manner. □

Remark 3.1.4. *As the proof shows, in part 6 it is enough to require that the convergences hold for functions h that are Lipshitz continuous with respect to the pseudo-metrics specifying the uniformity.*

Remark 3.1.5. *Part 3 of the theorem can be used to define "narrow large deviation convergence", which is identical to the definition of the large deviation principle, see, e.g., Varadhan [128]. Thus, on Tihonov spaces large deviation convergence is equivalent to the large deviation principle.*

We recall that $B_r(z)$ denotes the closed r-ball about an element z of a metric space.

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Corollary 3.1.6. *Let E be a Tihonov topological space. If $P_\phi \stackrel{ld}{\to} \Pi$, then*

$$\Pi(z) = \lim_{U\in\mathcal{U}'_z} \liminf_{\phi\in\Phi} P_\phi^{1/r_\phi}(U) = \lim_{U\in\mathcal{U}'_z} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(cl\,U),$$

where $\mathcal{U}'_z$ is a collection of open neighbourhoods of z whose closures decrease to z. In particular, if E is a metric space, then

$$\Pi(z) = \lim_{r\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(B_r(z)).$$

Proof. The claim follows by the inequalities

$$\Pi(z) \le \Pi(U) \le \liminf_{\phi\in\Phi} P_\phi^{1/r_\phi}(U) \le \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\mathrm{cl}\,U) \le \Pi(\mathrm{cl\ U}),$$

and the fact that $\lim_{U\in\mathcal{U}'_z} \Pi(\mathrm{cl}\,U) = \Pi(z)$. □

The following fact follows by Theorem 3.1.3.

Corollary 3.1.7. *Let E be a Tihonov topological space. Let a Borel subset E_0 of E be equipped with relative topology. Let $P_\phi(E \setminus E_0) = \Pi(E\setminus E_0) = 0$ and the restriction of Π to E_0, which is denoted by $\tilde{\Pi}$, be τ-smooth relative to the collection of closed subsets of E_0. Then $P_\phi \stackrel{ld}{\to} \Pi$ if and only if $\tilde{P}_\phi \stackrel{ld}{\to} \tilde{\Pi}$, where $\tilde{P}_\phi$ denotes the restriction of P_ϕ to E_0.*

Remark 3.1.8. *The τ-smoothness property of $\tilde{\Pi}$ holds if either E_0 is a closed subset of E or Π is a deviability on E.*

Lemma 1.9.14 provides us with the following corollary.

Corollary 3.1.9. *Let E be a Tihonov topological space and $P_\phi \stackrel{ld}{\to} \Pi$, where Π is supported by $E_0 \subset E$. Then the following holds.*

1. $$\lim_\phi \|h\|_\phi = \bigvee_E h\, d\Pi$$ *for all E_0-continuous bounded Borel-measurable functions $h : E \to \mathbb{R}_+$,*

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2. (*i*) $\liminf_\phi \|g\|_\phi \geq \bigvee_E g\, d\Pi$

for all E_0*-lower-semi-continuous bounded Borel-measurable functions* $g : E \to \mathbb{R}_+$,

(*ii*) $\limsup_\phi \|f\|_\phi \leq \bigvee_E f\, d\Pi$

for all E_0*-upper-semi-continuous bounded Borel-measurable functions* $f : E \to \mathbb{R}_+$,

3. (*i*) $\liminf_\phi P_\phi^{1/r_\phi}(G) \geq \Pi(G)$

for all E_0*-open Borel-measurable sets* $G \subset E$,

(*ii*) $\limsup_\phi P_\phi^{1/r_\phi}(F) \leq \Pi(F)$

for all E_0*-closed Borel-measurable sets* $F \subset E$,

4. $\lim_\phi P_\phi^{1/r_\phi}(H) = \Pi(H)$

for all E_0*-continuous Borel-measurable sets* $H \subset E$.

The next corollary allows one to strengthen topology for which LD convergence can be proved. We say that two topologies on a topological space are locally equivalent at a given point if they have equivalent local bases at the point.

Corollary 3.1.10. *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be Tihonov topologies on* E, *and let* $\mathcal{O}_2$ *be finer than* $\mathcal{O}_1$. *Let* $E_0 \subset E$ *be such that* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are locally equivalent at every* $z \in E_0$. *If* $P_\phi \stackrel{ld}{\to} \Pi$ *for topology* $\mathcal{O}_1$, *the* P_ϕ *can be extended to probabilities on the Borel* σ*-algebra of* E *generated by* $\mathcal{O}_2$, Π *is supported by* E_0 *and the restriction of* Π *to* E_0 *is* τ*-smooth relative to the collection of closed subsets of* E_0 *for the topology induced on* E_0 *by* $\mathcal{O}_1$, *then* Π *is a* τ*-smooth idempotent probability relative to the collection of sets closed in topology* $\mathcal{O}_2$ *and* $P_\phi \stackrel{ld}{\to} \Pi$ *for topology* $\mathcal{O}_2$.

Proof. Since the topologies induced on E_0 by $\mathcal{O}_1$ and $\mathcal{O}_2$ coincide, $\Pi(E \setminus E_0) = 0$ and the restriction of Π to E_0 is τ-smooth relative to the collection of closed subsets of E_0 for the topology induced on E_0 by $\mathcal{O}_1$, Π is a τ-smooth idempotent probability relative to the collection of sets closed in topology $\mathcal{O}_2$. The required LD convergence follows by the fact that if $h : E \to \mathbb{R}_+$ is continuous for topology $\mathcal{O}_2$, then it is E_0-continuous for topology $\mathcal{O}_1$. ☐

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Below, we will need convergence of integrals of not necessarily bounded functions. This requires an analogue of uniform integrability.

Definition 3.1.11. *A Borel-measurable function $f : E \to \mathbb{R}_+$ is said to be uniformly exponentially integrable (of order r_ϕ) with respect to the net $\{P_\phi\}$ if*

$$\lim_{a\to\infty} \limsup_{\phi\in\Phi} \Bigl(\int\limits_E f(z)^{r_\phi} \mathbf{1}(f(z) > a)\, dP_\phi(z) \Bigr)^{1/r_\phi} = 0.$$

In analogy to uniform integrability by Chebyshev's inequality the uniform exponential integrability holds if for some $\epsilon > 0$

$$\limsup_{\phi} \Bigl(\int\limits_E f(z)^{r_\phi(1+\varepsilon)}\, dP_\phi(z) \Bigr)^{1/r_\phi} < \infty.$$

Lemma 3.1.12. *Let E be Tihonov. Let $P_\phi \overset{ld}{\to} \Pi$ as $\phi \in \Phi$ and Π be supported by $E_0 \subset E$. Then the following holds.*

1. *For all E_0–continuous and uniformly exponentially integrable with respect to $\{P_\phi\}$ Borel-measurable functions $h : E \to \mathbb{R}_+$*

$$\lim_{\phi\in\Phi} \Bigl(\int\limits_E h(z)^{r_\phi} dP_\phi(z) \Bigr)^{1/r_\phi} = \bigvee_E h(z)\, d\Pi(z).$$

2. *For all E_0–lower-semi-continuous Borel-measurable functions $g : E \to \mathbb{R}_+$*

$$\liminf_{\phi\in\Phi} \Bigl(\int\limits_E g(z)^{r_\phi} dP_\phi(z) \Bigr)^{1/r_\phi} \geq \bigvee_E g(z)\, d\Pi(z).$$

Proof. The second part, being "a Fatou lemma for LD convergence", is proved by a similar means: for $a \in \mathbb{R}_+$ by Corollary 3.1.9

$$\liminf_{\phi\in\Phi} \Bigl(\int\limits_E g(z)^{r_\phi} dP_\phi(z) \Bigr)^{1/r_\phi}$$
$$\geq \liminf_{\phi\in\Phi} \Bigl(\int\limits_E (g(z) \wedge a)^{r_\phi} dP_\phi(z) \Bigr)^{1/r_\phi}$$
$$\geq \bigvee_E (g(z) \wedge a)\, d\Pi(z).$$

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Since the latter converges to $\bigvee_E g(z)\,d\Pi(z)$ as $a \to \infty$, the proof of part 2 is over.

Part 1 follows by part 2 and the inequalities

$$\begin{aligned}
&\limsup_{\phi\in\Phi}\Big(\int\limits_E h(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi}\\
&\le \limsup_{\phi\in\Phi}\Big(\int\limits_E h(z)^{r_\phi}\,\mathbf{1}(h(z)\le a)dP_\phi(z)\Big)^{1/r_\phi}\\
&+ \limsup_{\phi\in\Phi}\Big(\int\limits_E h(z)^{r_\phi}\,\mathbf{1}(h(z)> a)dP_\phi(z)\Big)^{1/r_\phi}\\
&\le \bigvee_E h(z)\,\mathbf{1}(h(z)\le a)\,d\Pi(z)\\
&\qquad\qquad + \limsup_{\phi\in\Phi}\Big(\int\limits_E h(z)^{r_\phi}\,\mathbf{1}(h(z)> a)dP_\phi(z)\Big)^{1/r_\phi},
\end{aligned}$$

where the latter inequality holds by Corollary 3.1.9. □

The following lemma gives an extension in a different direction.

Lemma 3.1.13. *Let E be a Tihonov topological space. Let $P_\phi \overset{ld}{\to} \Pi$. Let $h_\phi : E \to \mathbb{R}_+$ be uniformly bounded and Borel-measurable functions such that for a function $h : E \to \mathbb{R}_+$*

$$\lim_{\phi\in\Phi} h_\phi(z_\phi) = h(z)$$

for Π-almost every $z \in E$ and every net $z_\phi \to z$ as $\phi \in \Phi$. Then

$$\lim_{\phi\in\Phi}\Big(\int\limits_E h_\phi(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi} = \bigvee_E h(z)\,d\Pi(z).$$

Proof. The proof is similar to the one of Lemma 1.10.2 so we only sketch it. Defining $\overline{h}_\phi(z) = \inf_{U\in\mathcal{U}_z}\sup_{z'\in U}\sup_{\phi'\ge\phi} h_\phi(z')$, where as we recall $\mathcal{U}_z$ denotes the collection of open neighbourhoods of z, and $\overline{h}(z) = \inf_{\phi\in\Phi}\overline{h}_\phi(z)$, we can write for arbitrary $\epsilon > 0$ and suitable ϕ_0

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by Theorem 3.1.3 applied to $\overline{h}_{\phi_0}$ that

$$\limsup_{\phi\in\Phi}\Big(\int\limits_E h_\phi(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi}$$
$$\le \limsup_{\phi\in\Phi}\Big(\int\limits_E \overline{h}_{\phi_0}(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi} \le \bigvee_E \overline{h}_{\phi_0}(z)\,d\Pi(z)$$
$$\le \bigvee_E \overline{h}(z)\,d\Pi(z)+\epsilon \le \bigvee_E h(z)\,d\Pi(z)+\epsilon.$$

The complementary inequality

$$\liminf_{\phi\in\Phi}\Big(\int\limits_E h_\phi(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi} \ge \bigvee_E h(z)\,d\Pi(z)-\epsilon$$

is proved by a mirror argument. Specifically, defining $\underline{h}_\phi(z) = \sup_{U\in\mathcal{U}_z}\inf_{z'\in U}\inf_{\phi'\ge\phi} h_\phi(z')$ and $\underline{h}(z) = \sup_{\phi\in\Phi}\underline{h}_\phi(z)$, we have for arbitrary $\epsilon>0$ and suitable ϕ_1

$$\liminf_{\phi\in\Phi}\Big(\int\limits_E h_\phi(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi}$$
$$\ge \liminf_{\phi\in\Phi}\Big(\int\limits_E \underline{h}_{\phi_1}(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi} \ge \bigvee_E \underline{h}_{\phi_1}(z)\,d\Pi(z)$$
$$\ge \bigvee_E \underline{h}(z)\,d\Pi(z)-\epsilon \ge \bigvee_E h(z)\,d\Pi(z)-\epsilon.$$

□

As a consequence of Lemma 3.1.13, we obtain the following version of the contraction principle on preservation of LD convergence under mappings.

Theorem 3.1.14. *Let E be a Hausdorff topological space and E' be a Tihonov topological space. Let Π be a deviability on E. Let Borel-measurable functions $f_\phi : E\to E'$, $\phi\in\Phi$, and a Π-Luzin-measurable function $f : E\to E'$ be such that for Π-almost every $z\in E$ and every net $z_\phi\to z$ we have that $f_\phi(z_\phi)\to f(z)$. If $P_\phi \overset{ld}{\to} \Pi$, then $P_\phi\circ f_\phi^{-1} \overset{ld}{\to} \Pi\circ f^{-1}$.*

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Proof. The proof is similar to the one of Theorem 1.10.3. We first note that $\Pi\circ f^{-1}$ is a deviability on E' by Theorem 1.7.11. Next, for an $\mathbb{R}_+$-valued bounded continuous function h on E' by a change of variables and Lemma 3.1.13

$$\lim_{\phi\in\Phi}\Big(\int\limits_{E'} h(z')^{r_\phi}\,dP_\phi\circ f_\phi^{-1}(z')\Big)^{1/r_\phi}$$
$$=\lim_{\phi\in\Phi}\Big(\int\limits_{E} h\circ f_\phi(z)^{r_\phi}\,dP_\phi(z)\Big)^{1/r_\phi}$$
$$=\bigvee_E h\circ f(z)\,d\Pi(z)=\bigvee_{E'} h(z')\,d\Pi\circ f^{-1}(z').$$

□

The following consequence is often used below.

Corollary 3.1.15. *Let E be a Hausdorff topological space, E' be a Tihonov topological space, and Π be a deviability on E. If $P_\phi \stackrel{ld}{\to} \Pi$ as $\phi\in\Phi$ and $f: E\to E'$ is Borel measurable and continuous Π-a.e., then $P_\phi\circ f^{-1} \stackrel{ld}{\to} \Pi\circ f^{-1}$.*

We now derive a criterion of "large deviation relative compactness" in the theme of Prohorov's one for weak convergence.

Definition 3.1.16. *An $\mathcal{F}$–idempotent probability Π on E is called a large deviation (LD) accumulation point of $\{P_\phi,\phi\in\Phi\}$ (for rate r_ϕ) if there exists a subnet $\{P_{\phi'},\phi'\in\Phi'\}$ of $\{P_\phi,\phi\in\Phi\}$ that LD converges (at rate $r_{\phi'}$) to Π.*

Definition 3.1.17. *The net $\{P_\phi,\phi\in\Phi\}$ is called large deviation (LD) relatively compact (for rate r_ϕ) if every subnet $\{P_{\phi'},\phi'\in\Phi'\}$ of $\{P_\phi,\phi\in\Phi\}$ has an LD accumulation point (for rate $r_{\phi'}$).*

We recall that $\mathcal{K}$ denotes the collection of compact subsets of E.

Definition 3.1.18. *The net $\{P_\phi,\phi\in\Phi\}$ is called exponentially tight (of order r_ϕ) if $\inf_{K\in\mathcal{K}}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(K^c)=0$.*

Theorem 3.1.19. *Let E be a Tihonov topological space.*

1. *If the net $\{P_\phi,\phi\in\Phi\}$ is exponentially tight, then it is LD relatively compact, the accumulation points being deviabilities.*

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2. *Let E be, in addition, a locally compact Hausdorff topological space. If the net $\{P_\phi, \phi \in \Phi\}$ is LD relatively compact, then it is exponentially tight.*

Proof. The proof is analogous to the proof of Theorem 1.9.17. We start with part 1. Let $C_{b,1}^+(E) = \{f \in C_b^+(E) : \|f\| \le 1\}$. For a given $\phi \in \Phi$, the mapping $V_\phi : f \to \|f\|_\phi$, $f \in C_{b,1}^+(E)$, is an element of the space $[0,1]^{C_{b,1}^+(E)}$. The latter space, endowed with product topology, is compact and Hausdorff. Therefore, the net $\{V_\phi, \phi \in \Phi\}$ is relatively compact on $[0,1]^{C_{b,1}^+(E)}$ so that there exists a subnet $\{V_{\phi'}, \phi' \in \Phi'\}$ that converges to an element V of $[0,1]^{C_{b,1}^+(E)}$. We extend V to a functional on $C_b^+(E)$ by letting $V(cf) = cV(f)$, $c \in \mathbb{R}_+$. By the definition of topology on $[0,1]^{C_{b,1}^+(E)}$

$$\lim_{\phi' \in \Phi'} \|f\|_{\phi'} = V(f),\ f \in C_b^+(E). \tag{3.1.2}$$

The latter implies that V satisfies conditions $(V0)$, $(V1)$ and $(V2)$ of Theorem 1.7.25, i.e.,

$(V0)$ $V(1) = 1$,

$(V1)$ $V(cf) = cV(f)$, $c \in \mathbb{R}_+$,

$(V2)$ $V(f \vee g) = V(f) \vee V(g)$.

The first two properties directly follow by (3.1.2). The third property is valid in view of the inequalities $\|f\|_\phi \vee \|g\|_\phi \le \|f \vee g\|_\phi \le 2^{1/r_\phi}\big(\|f\|_\phi \vee \|g\|_\phi\big)$ and (3.1.2).

Also, exponential tightness of $\{P_\phi, \phi \in \Phi\}$ and (3.1.2) imply that V is tight in the sense of Theorem 1.7.25. Thus, the functional V satisfies all the conditions of Theorem 1.7.25, so according to the theorem $V(f) = \bigvee_E f\, d\Pi$, $f \in C_b^+(E)$, for some deviability Π, which implies that $P_{\phi'} \stackrel{ld}{\to} \Pi$ (at rate $r_{\phi'}$). This completes the proof of part 1.

Part 2 follows since by the argument of the proof of (1.9.5), where we use large deviation relative compactness of $\{P_\phi\}$ in place of weak relative compactness of $\{\Pi_\phi\}$, for arbitrary $\varepsilon > 0$ there exist open sets $A_1, \ldots, A_k$ with compact closures such that

$$\limsup_{\phi \in \Phi} P_\phi^{1/r_\phi}\Big(E \setminus \bigcup_{i=1}^k A_i\Big) \le \varepsilon.$$

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(Cf. also the proof of part 2 of Theorem 3.1.28 below.) □

Corollary 3.1.20. *Let* $\{P_{\phi,\upsilon},\ \phi \in \Phi\}$, $\upsilon \in \Upsilon$, *be nets of Borel measures on respective Tihonov topological spaces* E_υ. *If the nets* $\{P_{\phi,\upsilon},\ \phi \in \Phi\}$ *are exponentially tight (of order* r_ϕ*) for every* $\upsilon \in \Upsilon$, *then there exists a subnet* $\{(P_{\phi',\upsilon}, \upsilon \in \Upsilon),\ \phi' \in \Phi'\}$ *of* $\{(P_{\phi,\upsilon}, \upsilon \in \Upsilon),\ \phi \in \Phi\}$ *such that the nets* $\{P_{\phi',\upsilon},\ \phi' \in \Phi'\}$ *LD converge (at rate* $r_{\phi'}$*) to deviabilities on the* E_υ *for every* $\upsilon \in \Upsilon$.

Proof. For $f_\upsilon \in C_b^+(E_\upsilon)$, let $V_{\phi,\upsilon}(f_\upsilon) = \left(\int_{E_\upsilon} f_\upsilon^{r_\phi}\, dP_{\phi,\upsilon}\right)^{1/r_\phi}$. By Tihonov's theorem the set $\{(V_{\phi,\upsilon}(f_\upsilon),\ f_\upsilon \in C_{b,1}^+(E_\upsilon), \upsilon \in \Upsilon),\ \phi \in \Phi\}$ is a relatively compact subset of $\prod_{\upsilon\in\Upsilon}[0,1]^{C_{b,1}^+(E_\upsilon)}$, where the latter set is equipped with product topology. Thus, there exists a convergent subnet $\{(V_{\phi',\upsilon}(f_\upsilon),\ f_\upsilon \in C_{b,1}^+(E_\upsilon), \upsilon \in \Upsilon),\ \phi' \in \Phi'\}$. Now the required follows by the argument of the proof of Theorem 3.1.19. □

Theorem 3.1.19 allows us to introduce the following useful concept.

Definition 3.1.21. *Let* E *be a Tihonov topological space and* $E_0 \subset E$. *We say that the net* $\{P_\phi, \phi \in \Phi\}$ *is* E_0*-exponentially tight if it is exponentially tight and every LD accumulation point* Π *is supported by* E_0.

The following is a version of the contraction principle.

Corollary 3.1.22. *Let* E *be a Tihonov topological space,* $E_0 \subset E$, *and* E' *be a locally compact Hausdorff topological space. If the net* $\{P_\phi,\ \phi \in \Phi\}$ *is* E_0*–exponentially tight and a function* $f : E \to E'$ *is Borel measurable and* E_0*-continuous, then the net* $\{P_\phi \circ f^{-1},\ \phi \in \Phi\}$ *is exponentially tight.*

Proof. By part 1 of Theorem 3.1.19 the net $\{P_\phi,\ \phi \in \Phi\}$ is LD relatively compact. Since f is continuous a.e. with respect to every LD accumulation point of $\{P_\phi,\ \phi \in \Phi\}$, by Corollary 3.1.15 the net $\{P_\phi \circ f^{-1},\ \phi \in \Phi\}$ is LD relatively compact as well; hence, it is exponentially tight by part 2 of Theorem 3.1.19. □

We now assume that E is a metric space and introduce "metrics" for large deviation convergence. We first define an analogue of the

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Prohorov metric. We again assume as given a net $\{P_\phi,\, \phi \in \Phi\}$ of Borel measures on E, a net of real numbers $\{r_\phi,\, \phi \in \Phi\}$ greater than 1 converging to ∞, and an $\mathcal{F}$-idempotent probability Π on E. The analogue of the Prohorov metric is defined by

$$p^{ld}_\phi(P_\phi, \Pi) = \inf\bigl\{\epsilon > 0 :\; P_\phi^{1/r_\phi}(F) \le \Pi\bigl(F^\epsilon\bigr) + \epsilon,\\ \Pi(F) \le P_\phi^{1/r_\phi}\bigl(F^\epsilon\bigr) + \epsilon \;\text{ for all closed } F \subset E\bigr\}.$$

(As above, for $A \subset E$, we denote $A^\epsilon = \{z \in E :\, \rho(z, A) \le \epsilon\}$). The next lemma follows by regularity of Borel measures and τ-maxitivity of idempotent measures.

Lemma 3.1.23. *We can equivalently write*

$$p^{ld}_\phi(P_\phi, \Pi) = \inf\bigl\{\epsilon > 0 :\; P_\phi^{1/r_\phi}(A) \le \Pi\bigl(A^\epsilon\bigr) + \epsilon,\\ \Pi(A) \le P_\phi^{1/r_\phi}\bigl(A^\epsilon\bigr) + \epsilon \;\textit{ for all } A \in \mathcal{B}(E)\bigr\}.$$

Remark 3.1.24. *The distance p^{ld}_ϕ could equivalently be defined in terms of open ϵ-neighbourhoods.*

Theorem 3.1.25. *The net $\{P_\phi\}$ LD converges to Π if and only if $p^{ld}_\phi(P_\phi, \Pi) \to 0$ as $\phi \in \Phi$.*

Proof. The proof is analogous to the proof of Theorem 1.9.22. We first prove that if $p^{ld}_\phi(P_\phi, \Pi) \to 0$, then the P_ϕ LD converge to Π. By Theorem 3.1.3 it is sufficient to prove that, given a closed set F, an open set G, and $\epsilon > 0$, there exists $\delta > 0$ such that if $p^{ld}_\phi(P_\phi, \Pi) < \delta$, then $P_\phi^{1/r_\phi}(F) < \Pi(F) + \epsilon$ and $P_\phi^{1/r_\phi}(G) > \Pi(G) - \epsilon$. Since Π is τ-smooth relative to $\mathcal{F}$, there exists $\delta_1 \in (0, \epsilon/2)$ such that $\Pi(F) \ge \Pi(F^{\delta_1}) - \epsilon/2$. Therefore, if $p^{ld}_\phi(P_\phi, \Pi) < \delta_1$, then $P_\phi^{1/r_\phi}(F) < \Pi(F^{\delta_1}) + \delta_1 \le \Pi(F) + \epsilon$. Next, using τ-maxitivity of Π, we choose $\delta_2 \in (0, \epsilon/2)$ such that $\Pi(G) \le \Pi(G^{-\delta_2}) + \epsilon/2$. Then, if $p^{ld}_\phi(P_\phi, \Pi) < \delta_2$, then $\Pi(G) \le \Pi(G^{-\delta_2}) + \epsilon/2 < P_\phi^{1/r_\phi}(G) + \epsilon$. Taking $\delta = \delta_1 \wedge \delta_2$ proves the claim.

Conversely, let $P_\phi \overset{ld}{\to} \Pi$. We show using again Theorem 3.1.3 that given $\epsilon > 0$ there exists a collection of sets H_i, $i = 1, \ldots, k$ and $\delta > 0$ such that the H_i are continuous relative to Π and the fact that $|P_\phi^{1/r_\phi}(H_i) - \Pi(H_i)| < \delta$, $i = 1, \ldots, k$, implies that $p^{ld}_\phi(P_\phi, \Pi) < \epsilon$.

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Let $\delta < \epsilon/4$. Let closed $\delta/2$-balls $B_1, \ldots, B_l$ centred at $z_1, \ldots, z_l$, respectively, be such that $\Pi\big(E \setminus \cup_{i=1}^l B_i\big) < \delta$. By τ-maxitivity of Π for each $i = 1, 2, \ldots, l$ there exists a closed ball B_i' centred at z_i of radius not less than $\delta/2$ and not greater than δ, which is a continuous set relative to Π. Observing that a finite union of sets continuous relative to Π is also continuous relative to Π, we take as $H_1, \ldots, H_{k-1}$ the collection of arbitrary unions of the balls $B_1', \ldots, B_l'$. We also take $H_k = \big(E \setminus \cup_{i=1}^l B_i'\big)^{\delta'}$, where $\delta' > 0$ and is chosen such that H_k is continuous relative to Π and $\Pi(H_k) \le 2\delta$. Let $|P_\phi^{1/r_\phi}(H_i) - \Pi(H_i)| < \delta$, $i = 1, \ldots, k$. Let F be a closed subset of E and let H' be the largest set out of $H_1, \ldots, H_{k-1}$ such that F has non-empty intersection with each of the sets B_i' that make up H'. Then $H' \subset F^{2\delta}$ so that $\Pi(F) \le \Pi(F \cap H') + \Pi(F \cap H_k) \le \Pi(H') + \Pi(H_k) < P_\phi^{1/r_\phi}(H') + 3\delta \le P_\phi^{1/r_\phi}(F^{2\delta}) + 3\delta$. By a symmetric argument $P_\phi^{1/r_\phi}(F) \le P_\phi^{1/r_\phi}(F \cap H') + P_\phi^{1/r_\phi}(F \cap H_k) \le P_\phi^{1/r_\phi}(H') + P_\phi^{1/r_\phi}(H_k) < \Pi(H') + \Pi(H_k) + 2\delta \le \Pi(F^{2\delta}) + 4\delta$. Thus, $p_\phi^{ld}(P_\phi, \Pi) < \epsilon$. □

We now define an analogue of the Kantorovich-Wasserstein metric by

$$\rho_{\phi_{BL}}^{ld}(P_\phi, \Pi) = \sup_{\substack{f \in C_b^+(E): \\ \|f\|_{BL} \le 1}} \Big| \Big(\int_E f(z)^{r_\phi} \, dP_\phi(z) \Big)^{1/r_\phi} - \bigvee_E f(z) \, d\Pi(z) \Big|.$$

Theorem 3.1.26. *The net $\{P_\phi\}$ LD converges to Π if and only if $\rho_{\phi_{BL}}^{ld}(P_\phi, \Pi) \to 0$ as $\phi \in \Phi$.*

Proof. The fact that the convergence $\rho_{\phi_{BL}}^{ld}(P_\phi, \Pi) \to 0$ implies LD convergence of the P_ϕ to Π follows from Theorem 3.1.3 and Remark 3.1.4. For the converse, by Theorem 3.1.25 it suffices to prove that if $P_\phi \stackrel{ld}{\to} \Pi$, then

$$\limsup_{\phi \in \Phi} \big(\rho_{\phi_{BL}}^{ld}(P_\phi, \Pi) - 2 p_\phi^{ld}(P_\phi, \Pi) \big) \le 0.$$

Let $\|f\|_{BL} \le 1$. Given $\delta > 0$, we choose open δ-balls $A_\delta(z_k)$, $k = 1, 2, \ldots, l$, such that $\Pi\big(E \setminus \cup_{k=1}^l A_\delta(z_k)\big) < \delta$. Since $\limsup_{\phi \in \Phi} P_\phi^{1/r_\phi}\big(E \setminus \cup_{k=1}^l A_\delta(z_k)\big) \le \Pi\big(E \setminus \cup_{k=1}^l A_\delta(z_k)\big)$,we may assume that $P_\phi^{1/r_\phi}\big(E \setminus \cup_{k=1}^l A_\delta(z_k)\big) < \delta$. Abbreviating $p_\phi = p_\phi^{ld}(P_\phi, \Pi)$

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and recalling that $A_\delta(z)^{p_\phi}$ denotes the closed p_ϕ-neighbourhood of $A_\delta(z)$, we have

$$\begin{aligned}
\Big(\int_E f(z)^{r_\phi}\, dP_\phi(z)\Big)^{1/r_\phi} &\le \Big(l \max_{k=1,\dots,l} \int_{A_\delta(z_k)} f(z)^{r_\phi}\, dP_\phi(z)\Big)^{1/r_\phi} + \delta \\
&\le l^{1/r_\phi} \max_{k=1,\dots,l} \big(f(z_k)+\delta\big) P_\phi^{1/r_\phi}(A_\delta(z_k)) + \delta \\
&\le l^{1/r_\phi} \max_{k=1,\dots,l} f(z_k)\big(\Pi(A_\delta(z_k)^{p_\phi}) + p_\phi\big) + l^{1/r_\phi}\delta + \delta \\
&\le l^{1/r_\phi} \bigvee_E \big(f(z)+\delta+p_\phi\big)\, d\Pi(z) + l^{1/r_\phi} p_\phi + l^{1/r_\phi}\delta + \delta \\
&\le l^{1/r_\phi} \bigvee_E f(z)\, d\Pi(z) + 2l^{1/r_\phi} p_\phi + 2l^{1/r_\phi}\delta + \delta. \qquad (3.1.3)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bigvee_E f(z)\, d\Pi(z) &\le \max_{k=1,\dots,l} f(z_k)\Pi(A_\delta(z_k)) + \delta \\
&\le \max_{k=1,\dots,l} f(z_k)\big(P_\phi^{1/r_\phi}(A_\delta(z_k)^{p_\phi}) + p_\phi\big) + \delta \\
&\le \max_{k=1,\dots,l} \Big(\int_{A_\delta(z_k)^{p_\phi}} \big(f(z)+\delta+p_\phi\big)^{r_\phi} dP_\phi\Big)^{1/r_\phi} + p_\phi + \delta \\
&\le \Big(\int_E \big(f(z)+\delta+p_\phi\big)^{r_\phi} dP_\phi(z)\Big)^{1/r_\phi} + p_\phi + \delta \\
&\le \Big(\int_E f(z)^{r_\phi} dP_\phi(z)\Big)^{1/r_\phi} + 2p_\phi + 2\delta. \qquad (3.1.4)
\end{aligned}$$

Inequalities (3.1.3) and (3.1.4) imply that

$$\rho^{ld}_{\phi_{BL}}(P_\phi, \Pi) - 2p_\phi \le \big(l^{1/r_\phi} - 1\big)(1 + 2p_\phi) + 2l^{1/r_\phi}\delta + \delta$$

Since $l^{1/r_\phi} \to 1$ as $\phi \in \Phi$, $p_\phi \le 1$, and $\delta > 0$ is arbitrary, the proof is complete. □

We now consider sequential compactness for metric spaces. We thus assume that $\Phi = \mathbb{N}$ and replace general nets $\{r_\phi\}$ by sequences $\{r_n\}$. Probability measures are denoted by P_n.

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Definition 3.1.27. *A sequence $\{P_n,\, n \in \mathbb{N}\}$ is LD relatively sequentially compact (for rate r_n) if every subsequence $\{P_{n'}\}$ of $\{P_n\}$ contains a further subsequence $\{P_{n''}\}$ that LD converges (at rate $r_{n''}$) to an $\mathcal{F}$-idempotent probability on E.*

Theorem 3.1.28. *1. Let E be a metric space. If a sequence $\{P_n, n \in \mathbb{N}\}$ of probabilities on $(E, \mathcal{B}(E))$ is exponentially tight, then it is LD relatively sequentially compact, the LD accumulation points being deviabilities.*

2. Let E be homeomorphic to a complete separable metric space. If a sequence $\{P_n,\, n \in \mathbb{N}\}$ is LD relatively sequentially compact, then it is exponentially tight.

Proof. We prove part 1. Let us assume first that E is a separable metric space. Then it is embedded as a dense subspace into a compact metric space E'. We extend probabilities on $(E, \mathcal{B}(E))$ to probabilities on $(E', \mathcal{B}(E'))$ by letting $P'(A') = P\big(A' \cap E\big)$, $A' \in \mathcal{B}(E')$. The set $C^+_{b,1}(E')$ of $\mathbb{R}_+$-valued continuous functions on E' that are bounded by 1, endowed with the topology of uniform convergence, is a separable metric space. Let $C^+_{b,1,d}(E')$ denote a countable dense subset. The set $[0,1]^{C^+_{b,1,d}(E')}$ with product topology is sequentially compact, so the diagonal argument yields existence of a subsequence n_k such that the sequences $\{\, \|f\|'_{n_k}, k \in \mathbb{N}\}$ converge for all $f \in C^+_{b,1,d}(E')$, where $\|f\|'_{n_k}$ refers to the norm relative to P'_{n_k}. The inequality

$$| \, \|g\|'_n - \|g\|'_m| \le | \, \|f\|'_n - \|f\|'_m| + 2 \sup_{z' \in E'} |g(z') - f(z')|$$

and the fact that $C^+_{b,1,d}(E')$ is dense in $C^+_{b,1}(E')$ show that the sequences $\{\, \|f\|'_{n_k}, k \in \mathbb{N}\}$ converge for all $f \in C^+_{b,1}(E')$, which implies in analogy with the proof of Theorem 3.1.19 that there exists a deviability Π' on E' such that $P'_{n_k} \overset{ld}{\to} \Pi'$ as $k \to \infty$ at rate r_{n_k}. Exponential tightness of $\{P_n, n \in \mathbb{N}\}$ implies that $\inf_{K \in \mathcal{K}} \Pi'(E' \setminus K) = 0$ (where $\mathcal{K}$ is the collection of compact subsets of E) so that $\Pi'(E' \setminus E) = 0$ and the set function Π defined by $\Pi(A) = \Pi'(A), A \subset E$, is a deviability on E by Corollary 1.7.12. It is left to check that $\|f\|_{n_k} \to \bigvee_E f \, d\Pi$ for all $f \in C^+_b(E)$. By Theorem 3.1.3 we may assume that f is uniformly continuous on E so that it can be extended to $f' \in C^+_b(E')$, see, e.g., Engelking [47].

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The required follows since $\|f'\|'_{n_k} \to \bigvee_{E'} f' \, d\Pi'$, $\|f'\|'_{n_k} = \|f\|_{n_k}$ and $\bigvee_{E'} f' \, d\Pi' = \bigvee_E f \, d\Pi$.

Now, if E is an arbitrary metric space, then by the exponential tightness condition there exists a σ-compact space $E' \subset E$ such that $\lim_{n\to\infty} P_n^{1/r_n}(E \setminus E') = 0$. Applying the part just proved to the probabilities P'_n on the separable metric space E' defined by $P'_n(A) = P_n(A)/P_n(E')$, $A \in \mathcal{B}(E')$, we deduce existence of an LD convergent subsequence for the P'_n. This provides us with an LD convergent subsequence for the P_n. Part 1 is proved.

For part 2 we first check that for every $\delta > 0$ and $\epsilon > 0$ there exist open δ-balls $A_1, \ldots, A_k$ such that

$$\limsup_{n\to\infty} P_n^{1/r_n}\Bigl(E \setminus \bigcup_{i=1}^{k} A_i\Bigr) \le \epsilon. \tag{3.1.5}$$

Let open δ-balls A_i be such that $\cup_{i=1}^{\infty} A_i = E$. Let subsequences k_l and n_l be such that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} P_n^{1/r_n}\Bigl(E \setminus \bigcup_{i=1}^{k} A_i\Bigr) = \lim_{l\to\infty} P_{n_l}^{1/r_{n_l}}\Bigl(E \setminus \bigcup_{i=1}^{k_l} A_i\Bigr)$$

and $P_{n_l} \stackrel{ld}{\to} \Pi'$ for some $\mathcal{F}$-idempotent probability Π'. Then, for arbitrary k,

$$\begin{aligned}\Pi'\Bigl(E \setminus \bigcup_{i=1}^{k} A_i\Bigr) &\ge \limsup_{l\to\infty} P_{n_l}^{1/r_{n_l}}\Bigl(E \setminus \bigcup_{i=1}^{k} A_i\Bigr) \\ &\ge \lim_{l\to\infty} P_{n_l}^{1/r_{n_l}}\Bigl(E \setminus \bigcup_{i=1}^{k_l} A_i\Bigr).\end{aligned}$$

The required inequality (3.1.5) follows since $\lim_{k\to\infty} \Pi'\bigl(E \setminus \cup_{i=1}^{k} A_i\bigr) = 0$.

Since each P_n is tight by Ulam's theorem, (3.1.5) implies that for arbitrary $\varepsilon > 0$ and $k \in \mathbb{N}$, there exist open $1/k$-balls $A_{k1}, \ldots, A_{kn_k}$ such that for all $n \in \mathbb{N}$

$$P_n^{1/r_n}\Bigl(E \setminus \bigcup_{i=1}^{n_k} A_{ki}\Bigr) \le \frac{\varepsilon}{2^k}.$$

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The set $B = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded and hence relatively compact by completeness of E. Also for every $n \in \mathbb{N}$

$$P_n^{1/r_n}(E \backslash B) \leq \sum_{k=1}^{\infty} P_n^{1/r_n}\Big(E \backslash \bigcup_{i=1}^{n_k} A_{ki}\Big) \leq \varepsilon.$$

□

Remark 3.1.29. *Part 1 also follows by Theorem 3.1.19 and Theorem 3.1.25 (or Theorem 3.1.26).*

As a consequence, we have the following version of Corollary 3.1.22. The proof is similar.

Corollary 3.1.30. *Let E be a metric space and E' be homeomorphic to a complete separable metric space. Let $E_0 \subset E$. If a sequence $\{P_n, n \in \mathbb{N}\}$ of probabilities on $(E, \mathcal{B}(E))$ is E_0–exponentially tight and a function $f : E \to E'$ is Borel measurable and E_0-continuous, then the sequence $\{P_n \circ f^{-1}, n \in \mathbb{N}\}$ is exponentially tight.*

As an illustration of the use of LD relative compactness arguments, we prove Gärtner's theorem. Let $\mathcal{L}(X)$ denote the distribution of a random variable X and E_ϕ denote expectation with respect to a probability measure P_ϕ.

Theorem 3.1.31. *Let $\{X^\phi, \phi \in \Phi\}$ be a net of $\mathbb{R}^k$–valued random variables defined on respective probability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ such that for each $\lambda \in \mathbb{R}^k$*

$$\lim_{\phi \in \Phi} \frac{1}{r_\phi} \ln E_\phi \exp\big(r_\phi \lambda \cdot X^\phi\big) = G(\lambda),$$

where $G(\lambda)$ is an $\overline{\mathbb{R}}$-valued lower semi-continuous and essentially smooth convex function such that $0 \in$ int (dom G). Then $\mathcal{L}(X^\phi) \stackrel{ld}{\to} \Pi$, at rate r_ϕ, where Π is the deviability specified by the density $\Pi(x) = \exp\big(-\sup_{\lambda \in \mathbb{R}^k}(\lambda \cdot x - G(\lambda))\big)$.

Proof. We act as in the proof of Lemma 1.11.19. We first note that the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is exponentially tight. To see this, we write by Chebyshev's inequality, for $A > 0$ and $\eta > 0$, denoting by e_i,

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$i = 1, \dots, 2k$, the $2k$-vector, whose $\lfloor (k+1)/2 \rfloor$th entry is 1 if k is odd, -1 if k is even, and the rest of the entries are equal to 0,

$$P_\phi^{1/r_\phi}(|X^\phi| > A) \le \max_{i=1,\dots,2k} P_\phi^{1/r_\phi}(e_i \cdot X^\phi > A/k)$$
$$\le \exp(-\eta A/k) \max_{i=1,\dots,2k} \bigl(E_\phi \exp(r_\phi \eta e_i \cdot X^\phi)\bigr)^{1/r_\phi}.$$

The exponential tightness follows since by hypotheses

$$\lim_{\phi \in \Phi} \bigl(E_\phi \exp(r_\phi \eta e_i \cdot X^\phi)\bigr)^{1/r_\phi} = \exp(G(\eta e_i)),$$

where the right-hand side is finite if η is small enough by the fact that $G(\lambda)$ is finite in a neighbourhood of the origin.

Therefore, by Theorem 3.1.19 there exists a subnet $\{X^{\phi'}, \phi' \in \Phi'\}$ of $\{X^\phi, \phi \in \Phi\}$ and a deviability $\tilde{\Pi}$ on $\mathbb{R}^k$ such that $\mathcal{L}(X^{\phi'}) \xrightarrow{ld} \tilde{\Pi}$. Next, it follows from Chebyshev's inequality that if $\lambda \in \text{int}(\text{dom } G)$, then the function $\bigl(\exp(\lambda \cdot x),\, x \in \mathbb{R}^k\bigr)$ is uniformly exponentially integrable with respect to $\{\mathcal{L}(X^{\phi'}), \phi' \in \Phi'\}$, so by Lemma 3.1.12

$$\lim_{\phi'} \bigl(E \exp(r_{\phi'} \lambda \cdot X^{\phi'})\bigr)^{1/r_{\phi'}} = \bigvee_{\mathbb{R}^k} \exp(\lambda \cdot x)\, d\tilde{\Pi}(x),$$
$$\lambda \in \text{int}(\text{dom } G).$$

Thus, $\bigvee_{\mathbb{R}^k} \exp(\lambda \cdot x)\, d\tilde{\Pi}(x) = \exp(G(\lambda))$ for all $\lambda \in \text{int}(\text{dom } G)$, which as in the proof of Lemma 1.11.19 implies that $\tilde{\Pi} = \Pi$. □

The following result is proved similarly to Theorem 1.9.28.

Theorem 3.1.32. *Let E be a Tihonov space. Let $\mathcal{G}$ be a subset of $C_b^+(E)$ consisting of uniformly bounded and pointwise equicontinuous functions, i.e., $\sup_{f \in \mathcal{G}} \sup_{z \in E} f(z) < \infty$ and for every $\epsilon > 0$ and $z \in E$ there exists an open neighbourhood U_z of z such that $\sup_{f \in \mathcal{G}} \sup_{z' \in U_z} |f(z) - f(z')| \le \epsilon$. If $P_\phi \xrightarrow{ld} \Pi$, then*

$$\lim_\phi \sup_{f \in \mathcal{G}} \Bigl| \Bigl(\int_E f^{r_\phi}\, dP_\phi \Bigr)^{1/r_\phi} - \bigvee_E f\, d\Pi \Bigr| = 0.$$

As we have mentioned, if we replace space $C_b^+(E)$ in the definition of weak LD convergence by space $C_\mathcal{K}^+(E)$ of $\mathbb{R}_+$-valued continuous functions with compact support, then we obtain the notion of vague LD convergence.

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Definition 3.1.33. *We say that a net $\{P_\phi,\ \phi \in \Phi\}$ of probabilities on $(E, \mathcal{B}(E))$ vaguely LD converges at rate r_ϕ to a $\mathcal{K}$-idempotent probability Π on E if for every $f \in C_{\mathcal{K}}^+(E)$*

$$\lim_{\phi\in\Phi}\Bigl(\int_E f(z)^{r_\phi}\, dP_\phi(z)\Bigr)^{1/r_\phi} = \bigvee_E f(z)\, d\Pi(z).$$

If E is locally compact and Hausdorff, the vague LD convergence has properties similar to the properties of the weak LD convergence. For instance, there is an easy analogue of Theorem 1.9.2. A distinctive feature of this type of LD convergence is that nets of probability measures are LD relatively compact.

Theorem 3.1.34. *Let E be a locally compact Hausdorff topological space. Then a net $\{P_\phi,\ \phi \in \Phi\}$ of probabilities on $(E, \mathcal{B}(E))$ is vaguely LD relatively compact.*

The proof is similar to the proof of part 1 of Theorem 3.1.19, the main distinction being the use of Theorem 1.7.21 in place of Theorem 1.7.25.

At times it is more intuitive to formulate large deviation convergence of probability measures as large deviation convergence in distribution of the associated random variables.

Definition 3.1.35. *Let $\{X_\phi,\ \phi \in \Phi\}$ be a net of random variables defined on respective probability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ and assuming values in a topological space E and X be an idempotent variable defined on an idempotent probability space (Ω, Π) and assuming values in E, whose idempotent distribution is τ-smooth relative to the collection of closed subsets of E. We say that the net $\{X_\phi,\ \phi \in \Phi\}$ large deviation converges in distribution to X if $P_\phi \circ X_\phi^{-1} \stackrel{ld}{\to} \Pi \circ X^{-1}$.*

We denote large deviation convergence in distribution by $\stackrel{ld}{\to}$ as well. Whether this notation refers to large deviation convergence of probability measures or large deviation convergence in distribution of random variables should be clear from the context. We will also occasionally say that a net of random variables is LD relatively compact if the associated net of laws is LD relatively compact.

We have the following version of Lemma 3.1.12. Let us say that a net $\{\xi_\phi,\ \phi \in \Phi\}$ of $\mathbb{R}_+$-valued random variables on respective prob-

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ability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ is uniformly exponentially integrable relative to $\{P_\phi,\, \phi \in \Phi\}$ (with rate r_ϕ) if

$$\lim_{A\to\infty} \limsup_{\phi\in\Phi} \bigl(E_\phi \xi_\phi^{r_\phi}\, \mathbf{1}(\xi_\phi > A)\bigr)^{1/r_\phi} = 0.$$

Lemma 3.1.36. *Let* $\xi_\phi \overset{ld}{\to} \xi$, *where* ξ *is an* $\mathbb{R}_+$*-valued idempotent variable on an idempotent probability space* (Ω, Π)*. If the net* $\{\xi_\phi,\, \phi \in \Phi\}$ *is uniformly exponentially integrable relative to* $\{P_\phi,\, \phi \in \Phi\}$*, then* $\lim_{\phi\in\Phi} \bigl(E_\phi \xi_\phi^{r_\phi}\bigr)^{1/r_\phi} = S\xi$.

We now prove a number of technical lemmas. Let us recall that if X and Y are random variables with values in respective separable metric spaces E and E' with Borel σ-algebras, then (X, Y) is a random variable in $E \times E'$ with product topology and Borel σ-algebra; in particular, if $E = E'$ and ρ denotes the metric on E, then $\rho(X, Y)$ is a random variable. The following result is an analogue of Lemma 1.10.5 and admits a proof along the same lines. We give another proof that illustrates the use of metrics.

Lemma 3.1.37. *Let* E *be a separable metric space with metric* ρ*, and let* X_ψ^ϕ *and* Y^ϕ*, where* $\phi \in \Phi$*,* $\psi \in \Psi$*,* Φ *and* Ψ *are directed sets, be nets of random variables with values in* E*, defined on respective probability spaces* $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$*.*

Let

$$\lim_{\psi\in\Psi} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\bigl(\rho(X_\psi^\phi, Y^\phi) \ge \varepsilon\bigr) = 0, \varepsilon > 0,$$

and

$$\mathcal{L}\bigl(X_\psi^\phi\bigr) \overset{ld}{\to} \Pi_\psi \text{ as } \phi \in \Phi,$$

where Π_ψ, $\psi \in \Psi$, *are* $\mathcal{F}$*-idempotent probabilities on* E*.*

Then, for an $\mathcal{F}$*-idempotent probability* Π *on* E*, we have that*

$$\mathcal{L}(Y^\phi) \overset{ld}{\to} \Pi \text{ as } \phi \in \Phi$$

if and only if

$$\Pi_\psi \overset{iw}{\to} \Pi \text{ as } \psi \in \Psi.$$

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Proof. The claims follow by Theorem 1.9.25 and Theorem 3.1.26 since in view of the definitions of ρ_{BL} and $\rho^{ld}_{\phi_{BL}}$

$$\begin{aligned}
&\big|\rho_{BL}(\Pi_\psi,\Pi)-\rho^{ld}_{\phi_{BL}}\big(\mathcal{L}(Y^\phi),\Pi\big)\big| \le \rho^{ld}_{\phi_{BL}}\big(\mathcal{L}(X^\phi_\psi),\Pi_\psi\big)\\
&+\sup_{\substack{f\in C_b^+(E):\\ \|f\|_{BL}\le 1}}\Big|\Big(\int\limits_{\Omega_\phi} f(X^\phi_\psi)^{r_\phi}\,dP_\phi\Big)^{1/r_\phi}-\Big(\int\limits_{\Omega_\phi} f(Y^\phi)^{r_\phi}\,dP_\phi\Big)^{1/r_\phi}\Big|\\
&\qquad\le \rho^{ld}_{\phi_{BL}}\big(\mathcal{L}(X^\phi_\psi),\Pi_\psi\big)+\Big(\int\limits_{\Omega_\phi}\big(1\wedge\rho(X^\phi_\psi,Y^\phi)\big)^{r_\phi}\,dP_\phi\Big)^{1/r_\phi}.
\end{aligned}$$

□

We will often use the case where the X^ϕ_ψ do not depend on ψ.

Lemma 3.1.38. *Let E be a separable metric space with metric ρ, and let X^ϕ and Y^ϕ, where $\phi\in\Phi$, be nets of random variables defined on respective probability spaces $(\Omega_\phi,\mathcal{F}_\phi,P_\phi)$ with values in E. If $\mathcal{L}(X^\phi)\overset{ld}{\to}\Pi$, where Π is an $\mathcal{F}$-idempotent probability on E, and $\rho\big(X^\phi,Y^\phi\big)\overset{P_\phi^{1/r_\phi}}{\to}0$ as $\phi\in\Phi$, then $\mathcal{L}(Y^\phi)\overset{ld}{\to}\Pi$.*

We give another application of metrics.

Definition 3.1.39. *We say that a net $\{X^\phi,\phi\in\Phi\}$ of random variables on $(\Omega_\phi,\mathcal{F}_\phi,P_\phi)$ assuming values in a metric space E with metric ρ converges to $z\in E$ super-exponentially in probability at rate r_ϕ (or simply super-exponentially in probability if the rate is understood) and write $X^\phi\overset{P_\phi^{1/r_\phi}}{\to}z$ if $\lim_{\phi\in\Phi}P_\phi^{1/r_\phi}(\rho(X^\phi,z)>\epsilon)=0$ for every $\epsilon>0$.*

Remark 3.1.40. *Note that $X^\phi\overset{P_\phi^{1/r_\phi}}{\to}z$ if and only if*

$$\lim_{\phi\in\Phi}\Big(\int\limits_{\Omega_\phi}\big(1\wedge\rho(X_\phi,z)\big)^{r_\phi}\,dP_\phi\Big)^{1/r_\phi}=0.$$

Lemma 3.1.41. *Let $\{X^\phi,\phi\in\Phi\}$ be a net of random variables on respective probability spaces $(\Omega_\phi,\mathcal{F}_\phi,P_\phi)$ assuming values in a metric space E with metric ρ. Then $X^\phi\overset{P_\phi^{1/r_\phi}}{\to}z$ if and only if $\mathcal{L}(X^\phi)\overset{ld}{\to}\mathbf{1}_z$, where $\mathbf{1}_z$ denotes the unit mass at z.*

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Proof. If $\mathcal{L}(X^\phi) \overset{ld}{\to} \mathbf{1}_z$, then by the definition of LD convergence

$$\lim_{\phi\in\Phi}\Bigl(\int_{\Omega_\phi}\bigl(1\wedge\rho(X^\phi,z)\bigr)^{r_\phi}\,dP_\phi\Bigr)^{1/r_\phi} = \bigvee_E 1\wedge\rho(z',z)\,d\,\mathbf{1}_z(z') = 0.$$

The converse follows since as in the proof of Lemma 3.1.37

$$\rho^{ld}_{\phi_{BL}}(\mathcal{L}(X^\phi),\,\mathbf{1}_z) \le \Bigl(\int_{\Omega_\phi}\bigl(1\wedge\rho(X^\phi,z)\bigr)^{r_\phi}\,dP_\phi\Bigr)^{1/r_\phi}.$$

□

The next lemma considers joint LD convergence. We formulate the results in the language of LD convergence in distribution.

Lemma 3.1.42. *Let E and E' be separable metric spaces, and let X^ϕ and Y^ϕ, where $\phi\in\Phi$, be nets of random variables on $(\Omega_\phi,\mathcal{F}_\phi,P_\phi)$ with values in E and E', respectively. Let X and Y be idempotent variables on an idempotent probability space (Ω,Π) with values in E and E', respectively, whose idempotent distributions are τ-smooth relative to the associated collections of closed sets. Let $E\times E'$ be equipped with product topology.*

1. *If $X^\phi \overset{ld}{\to} X$, $Y^\phi \overset{ld}{\to} Y$, X and Y are independent, and X^ϕ and Y^ϕ are independent, then $(X^\phi,Y^\phi) \overset{ld}{\to} (X,Y)$.*
2. *If $X^\phi \overset{ld}{\to} X$ and $Y^\phi \overset{P_\phi^{1/r_\phi}}{\to} z$, then $(X^\phi,Y^\phi) \overset{ld}{\to} (X,z)$.*

Proof. The proof of part 1 uses Theorem 3.1.32 and is analogous to the proof of part 1 of Lemma 1.10.8. In more detail, let P_ϕ^X and P_ϕ^Y denote the respective distributions of X^ϕ and Y^ϕ, and Π^X and Π^Y denote the respective idempotent distributions of X and Y; Theorem 3.1.32 implies that for an $\mathbb{R}_+$-valued bounded uniformly continuous function $h(z,z')$ on $E\times E'$

$$\lim_{\phi\in\Phi}\Bigl|\Bigl(\int_E\Bigl(\int_{E'} h(z,z')^{r_\phi}\,dP_\phi^Y(z')\Bigr)\,dP_\phi^X(z)\Bigr)^{1/r_\phi} - \Bigl(\int_E\bigl(\sup_{z'\in E'} h(z,z')\,\Pi^Y(z')\bigr)^{r_\phi}\,dP_\phi^X(z)\Bigr)^{1/r_\phi}\Bigr| = 0. \quad (3.1.6)$$

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Also, since $\sup_{z'\in E'} h(z,z')\,\Pi^Y(z')$ is continuous in $z \in E$,

$$\lim_{\phi\in\Phi}\Big(\int\limits_E \big(\sup_{z'\in E'} h(z,z')\,\Pi^Y(z')\big)^{r_\phi}\, dP_\phi^X(z)\Big)^{1/r_\phi}$$
$$= \sup_{z\in E}\big(\sup_{z'\in E'} h(z,z')\,\Pi^Y(z')\big)\,\Pi^X(z). \quad (3.1.7)$$

The required follows by (3.1.6) and (3.1.7).

The proof of part 2 is also analogous: we observe that $(\rho \times \rho')((X^\phi, Y^\phi),(X^\phi, z)) \overset{P_\phi^{1/r_\phi}}{\to} 0$, where $\rho\times\rho'$ is a product metric on $E\times E'$, so that the required follows by part 1 and Lemma 3.1.38. $\square$

3.2 Large deviation convergence in the Skorohod space

The purpose of this section is to lay groundwork for deriving large deviation convergence results for semimartingales.

We begin by introducing basic notation for the Skorohod space. We denote by $\mathbb{D} = \mathbb{D}(\mathbb{R}_+,\mathbb{R}^d), d \in \mathbb{N}$, the space of $\mathbb{R}^d$–valued, right-continuous with left-hand limits functions $\mathbf{x} = (\mathbf{x}_t,\ t\in\mathbb{R}_+)$. We equip it with the Skorohod J_1 topology and metrise it by the Skorohod–Prohorov–Lindvall metric denoted by ρ_S, under which it is a complete separable metric space. Let $\mathcal{D}$ denote the Borel σ–algebra on $\mathbb{D}$, $\mathcal{D}_t$, for $t\in\mathbb{R}_+$, denote the sub–σ–algebra generated by the coordinate maps $\mathbf{x}\to\mathbf{x}_s$, $s\le t$, and $\mathbf{D} = (\mathcal{D}_t, t\in\mathbb{R}_+)$. (Note that the flow $\mathbf{D}$ is not right-continuous.) Given $\mathbf{x}\in\mathbb{D}$, we denote $\mathbf{x}^*_t = \sup_{s\le t}|\mathbf{x}_s|$, $\mathbf{x}^*_{t-} = \sup_{s<t}|\mathbf{x}_s|$, $\mathbf{x}^*_\infty = \sup_{s\in\mathbb{R}_+}|\mathbf{x}_s|$, $\Delta\mathbf{x}_t = \mathbf{x}_t - \mathbf{x}_{t-}$, where $\mathbf{x}_{t-}$ denotes the left-hand limit of $\mathbf{x}$ at t, $\mathbf{x}_{0-} = \mathbf{x}_0$.

As above, we denote by $\mathbb{C} = \mathbb{C}(\mathbb{R}_+,\mathbb{R}^d)$ the subspace of $\mathbb{D}$ of continuous functions with induced topology, which is the locally uniform topology with the metric defined in Section 2.2. By contrast with $\mathbb{D}$ and as we did in part I, we equip $\mathbb{C}$ with τ-algebras rather than σ-algebras. Keeping to the notation of part I, $\mathcal{C}$ denotes the discrete τ–algebra on $\mathbb{C}$, which is the power set of $\mathbb{C}$; $\mathcal{C}_t$ denotes the sub–τ–algebra generated by the coordinate maps $\mathbf{x}\to\mathbf{x}_s, s\le t$, and $\mathbf{C} = (\mathcal{C}_t, t\in\mathbb{R}_+)$.

If a deviability Π on $\mathbb{D}$ is supported by $\mathbb{C}$, i.e., $\Pi(\mathbb{D}\setminus\mathbb{C}) = 0$, then, since $\mathbb{C}$ is closed in $\mathbb{D}$, by Corollary 1.8.7 the restriction of Π to $\mathbb{C}$ is

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a deviability on $\mathbb{C}$, and, conversely, the image of a deviability on $\mathbb{C}$ under the natural embedding is a deviability on $\mathbb{D}$ with support in $\mathbb{C}$ by Corollary 1.7.12. Therefore, we do not distinguish between deviabilities on $\mathbb{D}$ with support in $\mathbb{C}$ and deviabilities on $\mathbb{C}$. Similarly, a Luzin-continuous idempotent process with paths in $\mathbb{C}$ can be considered as an idempotent variable with values in $\mathbb{D}$, whose idempotent distribution is a deviability on $\mathbb{D}$.

Our main object in the rest of the book is a net $\{X^\phi,\, \phi \in \Phi\}$ of stochastic processes with paths in $\mathbb{D}$ defined on respective stochastic bases $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$, where $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ are complete probability spaces and $\mathbf{F}_\phi = (\mathcal{F}_t^\phi,\, t \in \mathbb{R}_+)$ are filtrations, which we define as right-continuous increasing flows of complete sub-σ-algebras of $\mathcal{F}$. Our approach for establishing LD convergence results for semimartingales consists in proving LD relative compactness by checking the conditions of Theorem 3.1.19 and identifying the LD accumulation point. Therefore, we proceed with deriving exponential tightness results for probability distributions on the Skorohod space.

We first give a necessary and sufficient condition for exponential tightness in $\mathbb{D}$. For $\mathbf{x} = (\mathbf{x}_t,\, t \in \mathbb{R}_+) \in \mathbb{D}$, $T > 0$ and $\delta > 0$, we define the modulus of continuity

$$w'_T(\mathbf{x}, \delta) = \inf_{(t_j)} \max_{j=1,\ldots,k} w_{\mathbf{x}}\big([t_{j-1}, t_j)\big),$$

where $w_{\mathbf{x}}\big([s,t)\big) = \sup_{u,v \in [s,t)} |\mathbf{x}_u - \mathbf{x}_v|,\ s < t$, and the infimum is taken over all collections (t_j) such that $0 = t_0 < t_1 < \ldots < t_k = T$ and $t_j - t_{j-1} > \delta$ for $j < k$. The next theorem routinely follows by characterisation of compacts in $\mathbb{D}$, see, e.g., Jacod and Shiryaev [67], and is analogous to tightness conditions for sequences of probabilities in $\mathbb{D}$, cf. Ethier and Kurtz [48].

Theorem 3.2.1. *Let $\{X^\phi,\, \phi \in \Phi\}$, where $X^\phi = (X_t^\phi,\, t \in \mathbb{R}_+)$, be a net of stochastic processes with paths in $\mathbb{D}$ defined on respective probability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$. The net $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ is exponentially tight if and only if for all $T > 0$ and $\eta > 0$*

(i) $$\lim_{A\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\sup_{t\le T} |X_t^\phi| > A\big) = 0,$$

(ii) $$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(w'_T(X^\phi, \delta) > \eta\big) = 0.$$

The concept of E_0-exponential tightness with $E_0 = \mathbb{C}$ plays an important role in the developments below, so we repeat it here.

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Definition 3.2.2. *We say that a net* $\{P_\phi, \phi \in \Phi\}$ *of probability measures on* $\mathbb{D}$ *is* $\mathbb{C}$*-exponentially tight if it is exponentially tight and every LD accumulation point* Π *is supported by* $\mathbb{C}$*.*

The next result gives conditions for $\mathbb{C}$-exponential tightness.

Theorem 3.2.3. *Let* $\{X^\phi,\ \phi \in \Phi\}$*, where* $X^\phi = (X_t^\phi,\ t \in \mathbb{R}_+)$*, be a net of stochastic processes with paths in* $\mathbb{D}$*, defined on respective stochastic bases* $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$*. The net* $\{\mathcal{L}(X^\phi),\ \phi \in \Phi\}$ *is* $\mathbb{C}$*–exponentially tight if and only if either one of the following equivalent conditions* I *or* II *holds.*

I (*i*)
$$\lim_{A\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(|X_0^\phi| > A\big) = 0,$$

(*ii*)
$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} |X_t^\phi - X_s^\phi| > \eta\Big) = 0,$$

II (*i*)
$$\lim_{A\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\sup_{t\le T} |X_t^\phi| > A\big) = 0,$$

(*ii*)
$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} \sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}\big(\sup_{0\le t\le\delta} |X_{\tau+t}^\phi - X_\tau^\phi| > \eta\big) = 0,$$

where $T > 0$ *and* $\eta > 0$ *are arbitrary, and* $\mathbf{S}_T(\mathbf{F}_\phi)$ *denotes the set of all* $\mathbf{F}_\phi$*–stopping times not greater than* T*.*

Remark 3.2.4. *A similar result holds if the* X^ϕ *assume values in a complete separable metric space* E*. Then instead of conditions I(i) and II(i) one should require that the nets* $\{\mathcal{L}(X_t^\phi),\ \phi \in \Phi\}$ *be exponentially tight in* E *for all* $t \in \mathbb{R}_+$*, and in I(ii) and II(ii) replace the moduli of the increments of the* X^ϕ *by the distances between the values of the* X^ϕ*, cf. Ethier and Kurtz [48].*

We precede the proof with a lemma. In the rest of the book we use E to denote expectation and $\mathcal{F}$ to denote σ-algebras.

Lemma 3.2.5. *Let* ξ_i*,* $i \in \mathbb{N}$*, be positive random variables on a probability space* $(\Omega, \mathcal{F}, P)$ *and let*

$$A_t = \begin{cases} \max\big\{k \in \mathbb{N} :\ \sum_{i=1}^k \xi_i \le t\big\}, & \text{if } \xi_1 \le t,\\ 0, & \text{otherwise.}\end{cases}$$

If $\delta > 0$*,* $t \in \mathbb{R}_+$ *and* $N \in \mathbb{N}$ *are such that* $N\delta/t > 1$*, then*

$$P(A_t \ge N) \le \Big(1 - \frac{t}{N\delta}\Big)^{-1} \max_{k=1,\dots,N} P(\xi_k \le \delta,\ A_t \ge k).$$

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Proof. In view of Chebyshev's inequality for $n \le N$

$$P(A_t \ge N) \le P(A_t \ge N,\, \xi_n \le \delta) + P(A_t \ge N,\, \xi_n \ge \delta)$$
$$\le \max_{k=1,\dots,N} P(A_t \ge N,\, \xi_k \le \delta) + \frac{1}{\delta} E\big(\xi_n\, \mathbf{1}(A_t \ge N)\big). \quad (3.2.1)$$

Choosing n such that $E\big(\xi_n\, \mathbf{1}(A_t \ge N)\big) = \min_{k \le N} E\big(\xi_k\, \mathbf{1}(A_t \ge N)\big)$, we estimate the expectation on the right of (3.2.1) as follows

$$E\big(\xi_n\, \mathbf{1}(A_t \ge N)\big) \le \frac{1}{N} \sum_{k=1}^{N} E\big(\xi_k\, \mathbf{1}(A_t \ge N)\big)$$
$$= \frac{1}{N} E\Big(\sum_{k=1}^{N} \xi_k\, \mathbf{1}(A_t \ge N)\Big) \le \frac{t}{N} P(A_t \ge N).$$

Substituting the estimate into the right-hand side of (3.2.1) gives the required inequality. ☐

Proof of Theorem 3.2.3. It is obvious that part I implies part II. Let $\mathbb{C}$-exponential tightness of $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ hold. We prove that the assertion of part I holds. The argument is fairly standard. We derive part I(ii). Let us denote $x_{\phi,\delta} = P_\phi^{1/r_\phi}\big(\sup_{\substack{s,t \in [0,T]:\\ |s-t| \le \delta}} |X_t^\phi - X_s^\phi| > \eta\big)$. In analogy with the diagonal argument in the proof of Theorem 1.9.17 there exists a subnet $\{P_{\phi'}, x_{\phi'}, \phi' \in \Phi'\}$ of $\{P_\phi, x_{\phi,\delta},\, \phi \in \Phi, \delta > 0\}$ such that $P_{\phi'} \xrightarrow{ld} \Pi$ for some deviability Π and $\lim_{\phi' \in \Phi'} x_{\phi'} = \limsup_{\delta \to 0} \limsup_{\phi \in \Phi} x_{\phi,\delta}$. Then Corollary 3.1.9 and the fact that $\Pi(\mathbb{D} \setminus \mathbb{C}) = 0$ yield for arbitrary $\delta' > 0$

$$\limsup_{\delta \to 0} \limsup_{\phi \in \Phi} P_\phi^{1/r_\phi}\big(\sup_{\substack{s,t \in [0,T]:\\ |s-t| \le \delta}} |X_t^\phi - X_s^\phi| > \eta\big)$$
$$\le \limsup_{\phi' \in \Phi'} P_{\phi'}^{1/r_{\phi'}}\big(\sup_{\substack{s,t \in [0,T]:\\ |s-t| \le \delta'}} |X_t^{\phi'} - X_s^{\phi'}| > \eta\big)$$
$$\le \Pi\big(\mathbf{x} \in \mathbb{C} : \sup_{\substack{s,t \in [0,T]:\\ |s-t| \le \delta'}} |\mathbf{x}_t - \mathbf{x}_s| \ge \eta\big).$$

The claim follows since the right-hand side can be made arbitrarily small by choosing δ' in view of the τ-smoothness property of a deviability with respect to decreasing nets of closed sets. Part I(i) is derived similarly.

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Thus, it remains to prove that part II implies the $\mathbb{C}$-exponential tightness. We first prove that under the hypotheses the net $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ is exponentially tight in $\mathbb{D}$. We apply Theorem 3.2.1. The first condition of the theorem holds by hypotheses. We check the second. We define stopping times

$$\sigma_0^\phi = 0,\, \sigma_k^\phi = \inf\{t \in \mathbb{R}_+ : |X_t^\phi - X^\phi_{\sigma^\phi_{k-1}}| \ge \eta/2\},\, k \in \mathbb{N}.$$

Introducing $A_T^\phi = \max\{k \in \mathbb{N} : \sigma_k^\phi \le T\}$ and $\xi_k^\phi = \sigma_k^\phi - \sigma_{k-1}^\phi$, we have for $\delta < T$ by the fact that $w_{X^\phi}\big([\sigma^\phi_{k-1}, \sigma^\phi_k)\big) < \eta$

$$\begin{aligned}
P_\phi\big(w'_T(X^\phi, \delta) \ge \eta\big) &\le P_\phi\big(A_T^\phi \ge 1,\, \min_{k \le A_T^\phi} \xi_k^\phi \le \delta\big) \\
&\le P_\phi(A_T^\phi \ge N) + P_\phi\big(\min_{k \le A_T^\phi} \xi_k^\phi \le \delta,\, 1 \le A_T^\phi < N\big) \\
&\le P_\phi(A_T^\phi \ge N) + \sum_{i=1}^{N-1} \sum_{k=1}^{i} P_\phi\big(\xi_k^\phi \le \delta,\, A_T^\phi = i\big) \\
&\le P_\phi(A_T^\phi \ge N) + N^2 \max_{k=1,\dots,N} P_\phi\big(\xi_k^\phi \le \delta,\, A_T^\phi \ge k\big).
\end{aligned}$$

Since by Lemma 3.2.5, for $N \in \mathbb{N}$,

$$P_\phi(A_T^\phi \ge N) \le 2 \max_{k=1,\dots,N} P_\phi\Big(\xi_k^\phi \le \frac{2T}{N},\, A_T^\phi \ge k\Big),$$

we obtain the estimate

$$\begin{aligned}
P_\phi\big(w'_T(X^\phi, \delta) \ge \eta\big) &\le 2 \max_{k=1,\dots,N} P_\phi\Big(\xi_k^\phi \le \frac{2T}{N},\, A_T^\phi \ge k\Big) \\
&\quad + N^2 \max_{k=1,\dots,N} P_\phi\big(\xi_k^\phi \le \delta,\, A_T^\phi \ge k\big). \qquad (3.2.2)
\end{aligned}$$

Next, by right-continuity of X^ϕ

$$\begin{aligned}
P_\phi\big(\xi_k^\phi \le \delta,\, A_T^\phi \ge k\big) &= P_\phi\Big(\sup_{t \le \delta} |X^\phi_{\sigma^\phi_{k-1}+t} - X^\phi_{\sigma^\phi_{k-1}}| \ge \frac{\eta}{2},\, \sigma_k^\phi \le T\Big) \\
&\le P_\phi\Big(\sup_{t \le \delta} |X^\phi_{\sigma^\phi_{k-1}\wedge T+t} - X^\phi_{\sigma^\phi_{k-1}\wedge T}| \ge \frac{\eta}{2}\Big) \\
&\le \sup_{\tau \in \mathbf{S}_T(\mathbf{F}_\phi)} P_\phi\Big(\sup_{t \le \delta} |X^\phi_{\tau+t} - X^\phi_\tau| \ge \frac{\eta}{2}\Big). \qquad (3.2.3)
\end{aligned}$$

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Similarly,

$$P_\phi\Big(\xi_k^\phi \le \frac{2T}{N},\, A_T^\phi \ge k\Big) \le \sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi\Big(\sup_{t\le 2T/N}|X^\phi_{\tau+t}-X^\phi_\tau| \ge \frac{\eta}{2}\Big). \tag{3.2.4}$$

Substituting (3.2.3) and (3.2.4) into (3.2.2) yields

$$\begin{aligned} &P_\phi^{1/r_\phi}\big(w'_T(X^\phi,\delta)\ge\eta\big)\\ &\le 2^{1/r_\phi}\sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}\Big(\sup_{t\le\delta}|X^\phi_{\tau+t}-X^\phi_\tau|\ge\frac{\eta}{2}\Big)\\ &\qquad + N^{2/r_\phi}\sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}\Big(\sup_{t\le 2T/N}|X^\phi_{\tau+t}-X^\phi_\tau|\ge\frac{\eta}{2}\Big). \end{aligned}$$

Taking on the left-hand side the limits, firstly, as $\phi\in\Phi$, then as $N\to\infty$, and finally as $\delta\to 0$, checks the required condition.

We now prove the $\mathbb{C}$-exponential tightness. Let Π be deviability on $\mathbb{D}$, which is an LD accumulation point for $\{\mathcal{L}(X^\phi)\}$, and let $\mathbf{x}\in\mathbb{D}$ be a discontinuous function. We show that $\Pi(\mathbf{x})=0$. Let $t\in\mathbb{R}_+$ and $\eta>0$ be such that $|\Delta\mathbf{x}_t|\ge\eta$. Using Proposition VI.2.1 of Jacod and Shiryaev [67] (see also Liptser and Pukhalskii [78] for details), we have that if $\delta>0$ is small enough, then

$$\begin{aligned} &\{\mathbf{y}\in\mathbb{D}:\,\rho_S(\mathbf{x},\mathbf{y})\le\delta\}\subset\Big\{\mathbf{y}\in\mathbb{D}:\,\sup_{|t-s|\le\delta}|\Delta\mathbf{y}_s|\ge\frac{\eta}{2}\Big\}\\ &\subset\Big\{\mathbf{y}\in\mathbb{D}:\,\sup_{|t-s|\le\delta}|\mathbf{y}_t-\mathbf{y}_s|\ge\frac{\eta}{4}\Big\}\\ &\subset\Big\{\mathbf{y}\in\mathbb{D}:\,\sup_{t<s\le t+\delta}|\mathbf{y}_s-\mathbf{y}_t|\ge\frac{\eta}{4}\Big\}\bigcup\Big\{\mathbf{y}\in\mathbb{D}:\,|\mathbf{y}_t-\mathbf{y}_{t-\delta}|\ge\frac{\eta}{8}\Big\}\\ &\qquad\qquad\bigcup\Big\{\mathbf{y}\in\mathbb{D}:\,\sup_{t-\delta<s\le t}|\mathbf{y}_s-\mathbf{y}_{t-\delta}|\ge\frac{\eta}{8}\Big\}. \end{aligned}$$

Then by Corollary 3.1.6

$$\begin{aligned} \Pi(\mathbf{x})&\le\limsup_{\delta\to0}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\big(\rho_S(\mathbf{x},X^\phi)\le\delta\big)\\ &\le\limsup_{\delta\to0}\limsup_{\phi\in\Phi}3^{1/r_\phi}\sup_{u\le t}P_\phi^{1/r_\phi}\Big(\sup_{0<s\le\delta}|X^\phi_{u+s}-X^\phi_u|\ge\frac{\eta}{8}\Big). \end{aligned}$$

The latter limit equals 0 by hypotheses. □

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The following analogue of the Lenglart–Rebolledo inequality will allow us to estimate the probabilities in part II(ii) "in predictable terms".

Lemma 3.2.6. *Let* $X = (X_t,\, t \in \mathbb{R}_+)$ *and* $Y = (Y_t,\, t \in \mathbb{R}_+)$ *be positive processes on a stochastic basis* $(\Omega, \mathcal{F}, \mathbf{F}, P)$*. If* $E(X_\tau / Y_\tau) \le 1$ *for every* $\mathbf{F}$*-stopping time* $\tau < \infty$*, then for every* $\mathbf{F}$*-stopping time* $\sigma \le \infty$*,* $a > 0$ *and* $b > 0$*,*

$$P\big(\sup_{t\le\sigma} X_t \ge a\big) \le \frac{b}{a} + P\big(\sup_{t\le\sigma} Y_t > b\big)$$

(here $\sup_{t\le\infty} = \sup_{t\in\mathbb{R}_+}$*).*

Proof. We define the stopping time $\tau = \inf\{t \in \mathbb{R}_+ : X_t \ge a\} \le \infty$. If $P(\sigma < \infty) = 1$, then

$$P(\sup_{t\le\sigma} X_t > a) \le P(X_{\sigma\wedge\tau} \ge a) \le P(Y_{\sigma\wedge\tau} > b)$$
$$+P(X_{\sigma\wedge\tau} \ge a, Y_{\sigma\wedge\tau} \le b) \le P(Y_{\sigma\wedge\tau} > b) + P(X_{\sigma\wedge\tau}/Y_{\sigma\wedge\tau} \ge a/b).$$

By Chebyshev's inequality

$$P\big(\sup_{t\le\sigma} X_t > a\big) \le P(Y_{\sigma\wedge\tau} > b) + \frac{b}{a} \le P\big(\sup_{t\le\sigma} Y_t > b\big) + \frac{b}{a}.$$

To obtain the required, note that

$$P\big(\sup_{t\le\sigma} X_t \ge a\big) = \lim_{N\to\infty} P\Big(\sup_{t\le\sigma} X_t > a - \frac{1}{N}\Big).$$

If $P(\sigma < \infty) < 1$, then by the part just proved we have for $N > 0$

$$P\big(\sup_{t\le\sigma} X_t \ge a\big) \le P\Big(\sup_{t\le\sigma} X_t > a - \frac{1}{N}\Big)$$
$$= \lim_{M\to\infty} P\Big(\sup_{t\le\sigma\wedge M} X_t > a - \frac{1}{N}\Big) \le P\big(\sup_{t\le\sigma} Y_t > b\big) + \frac{b}{a - 1/N}.$$

Since N is arbitrary, the proof is over. □

The following useful fact is a direct consequence of Theorem 3.2.3.

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Corollary 3.2.7. *Let* $\{X^\phi,\, \phi \in \Phi\}$ *and* $\{Y^\phi,\, \phi \in \Phi\}$ *be nets of stochastic processes with paths in respective Skorohod spaces* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ *and* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^k)$ *such that* X^ϕ *and* Y^ϕ *are defined on a common probability space* $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ *for every* $\phi \in \Phi$. *If the net* $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ *is* $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$*-exponentially tight and the net* $\{\mathcal{L}(Y^\phi),\, \phi \in \Phi\}$ *is* $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^k)$*-exponentially tight, then the net* $\{\mathcal{L}(X^\phi, Y^\phi),\, \phi \in \Phi\}$ *of distributions on* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^k)$ *is* $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^k)$*-exponentially tight.*

The next two results concern methods of identifying LD limits. The following theorem presents the method of finite-dimensional distributions.

Theorem 3.2.8. *Let* $\{X^\phi,\, \phi \in \Phi\}$*, where* $X^\phi = (X^\phi_t,\, t \in \mathbb{R}_+)$*, be a net of stochastic processes with paths in* $\mathbb{D}$ *defined on respective probability spaces* $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$*. Let the net* $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ *of the distributions of the* X^ϕ *on* $\mathbb{D}$ *be* $\mathbb{C}$*–exponentially tight. Let for all* $k \in \mathbb{N}$*, and* $t_1 < \ldots < t_k \in U$*, where* U *is a dense subset of* $\mathbb{R}_+$*, as* $\phi \in \Phi$*,*

$$\mathcal{L}(X^\phi_{t_1}, \ldots, X^\phi_{t_k}) \xrightarrow{ld} \Pi_{t_1,\ldots,t_k},$$

where $\Pi_{t_1,\ldots,t_k}$ *are deviabilities on* $(\mathbb{R}^d)^k$*.*

Then idempotent probability Π *on* $\mathbb{D}$ *with density* $\Pi(\mathbf{x}) = \inf_{t_1,\ldots,t_k \in U} \Pi_{t_1,\ldots,t_k}(\mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_k})$ *if* $\mathbf{x} = (\mathbf{x}_t,\, t \in \mathbb{R}_+) \in \mathbb{C}$*, and* $\Pi(\mathbf{x}) = 0$ *if* $\mathbf{x} = (\mathbf{x}_t,\, t \in \mathbb{R}_+) \in \mathbb{D} \setminus \mathbb{C}$*, is a deviability on* $\mathbb{D}$*, and* $\mathcal{L}(X^\phi) \xrightarrow{ld} \Pi$*.*

Proof. By Theorem 3.1.19 $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ is LD relatively compact. Let Π' be an LD accumulation point. It suffices to prove that $\Pi' = \Pi$. By $\mathbb{C}$-exponential tightness of $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ this is true on $\mathbb{D} \setminus \mathbb{C}$. Let $\mathbf{x} \in \mathbb{C}$. By the contraction principle (Corollary 3.1.15) we have that $\Pi' \circ \pi^{-1}_{t_1,\ldots,t_k} = \Pi_{t_1,\ldots,t_k}$. By Theorem 2.2.2 and Remark 2.2.3 $\Pi'(\mathbf{x}) = \inf_{t_1,\ldots,t_k \in U} \Pi_{t_1,\ldots,t_k}(\mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_k}) = \Pi(\mathbf{x})$. □

The following result, which roughly shows that LD limits in distribution of non-negative martingales are exponential maxingales, lays a foundation for the maxingale problem method of proving LD convergence. We denote $E_\phi^{1/r_\phi}\xi = (E_\phi \xi)^{1/r_\phi}$, where, as above, E_ϕ denotes expectation with respect to P_ϕ.

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Theorem 3.2.9. *Let X^ϕ, $\phi \in \Phi$, be processes with paths in $\mathbb{D}$ defined on respective stochastic bases $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$. Let the net $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ be $\mathbb{C}$–exponentially tight and Π be a deviability on $\mathbb{D}$ supported by $\mathbb{C}$, which is an LD accumulation point of $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$. Let $M^\phi = (M_t^\phi,\, t \in \mathbb{R}_+),\, \phi \in \Phi$, be $\mathbb{R}_+$–valued martingales on $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$ such that the net $\{(M_t^\phi)^{1/r_\phi},\, \phi \in \Phi\}$ is uniformly exponentially integrable relative to the net $\{P_\phi,\, \phi \in \Phi\}$ for each $t \in \mathbb{R}_+$, and let $M_t(\mathbf{x})$, $t \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{D}$, be an $\mathbb{R}_+$–valued function, which, for every $t \in \mathbb{R}_+$, is $\mathbb{C}$–continuous, Borel measurable and, if restricted to $\mathbb{C}$, $\mathcal{C}_t$–measurable in $\mathbf{x}$.*

If, for every $t \in \mathbb{R}_+$, as $\phi \in \Phi$,

$$(M_t^\phi)^{1/r_\phi} - M_t(X^\phi) \overset{P_\phi^{1/r_\phi}}{\to} 0,$$

then $(M_t(\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ is a $\mathbf{C}$-exponential maxingale on $(\mathbb{C}, \Pi)$.

Proof. Since $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight, we can assume, by taking a subnet if necessary, that $\mathcal{L}(X^\phi) \overset{ld}{\to} \Pi$.

Consider a function $f(\mathbf{x}) = g(\mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_k})$, where $0 \le t_1 < \ldots < t_k$ and $g : (\mathbb{R}^d)^k \to \mathbb{R}_+$ is continuous and bounded. Since f and M_t are Borel measurable, $\mathbb{C}$–continuous and $\Pi(\mathbb{D}\setminus\mathbb{C}) = 0$, by the contraction principle for every $t \in \mathbb{R}_+$

$$\mathcal{L}\big(M_t(X^\phi) f(X^\phi)\big) \overset{ld}{\to} \Pi \circ h^{-1},$$

where $h : \mathbb{D} \to \mathbb{R}_+$ is defined by $h(\mathbf{x}) = M_t(\mathbf{x}) f(\mathbf{x})$. Since by hypotheses and boundedness of f

$$(M_t^\phi)^{1/r_\phi} f(X^\phi) - M_t(X^\phi) f(X^\phi) \overset{P_\phi^{1/r_\phi}}{\to} 0,$$

it follows by Lemma 3.1.38 that

$$\mathcal{L}\big((M_t^\phi)^{1/r_\phi} f(X^\phi)\big) \overset{ld}{\to} \Pi \circ h^{-1}. \tag{3.2.5}$$

By uniform exponential integrability of $\{(M_t^\phi)^{1/r_\phi},\, \phi \in \Phi\}$ relative to $\{P_\phi,\, \phi \in \Phi\}$, boundedness of f and Lemma 3.1.36 we then have

$$\begin{aligned}\lim_{\phi \in \Phi} E_\phi^{1/r_\phi}\big(M_t^\phi f(X^\phi)^{r_\phi}\big) &= \sup_{x \in \mathbb{R}_+} x\, \Pi \circ h^{-1}(x) \\ &= \sup_{\mathbf{x} \in \mathbb{C}} M_t(\mathbf{x}) f(\mathbf{x}) \Pi(\mathbf{x}). \end{aligned} \tag{3.2.6}$$

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(The last equality is the change-of-variables formula from Theorem 1.4.6.)

Now let $0 \le s < t$ and $t_i \le s$, $i = 1, \ldots, k$. By the martingale property

$$E_\phi\big(M_t^\phi f(X^\phi)^{r_\phi}\big) = E_\phi\big(M_s^\phi f(X^\phi)^{r_\phi}\big),$$

so (3.2.6) yields the maxingale property

$$\sup_{\mathbf{x}\in\mathbb{C}} f(\mathbf{x}) M_t(\mathbf{x})\Pi(\mathbf{x}) = \sup_{\mathbf{x}\in\mathbb{C}} f(\mathbf{x}) M_s(\mathbf{x})\Pi(\mathbf{x}).$$

The collection of functions $g(\mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_k})$ such that $0 \le t_1 < \ldots < t_k \le s$ and $g : (\mathbb{R}^d)^k \to \mathbb{R}_+$ are continuous and bounded satisfies the requirements on $\mathcal{H}_s$ in part 2 of Lemma 2.3.5 and also generates $\mathcal{C}_s$. An application of Lemma 2.3.5 shows that M is an exponential maxingale on $(\mathbb{C}, \mathbf{C}, \Pi)$.

Also taking $f = 1$ in (3.2.5), we obtain by Lemma 3.1.12 for $a > 0$

$$\liminf_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(M_t^\phi\,\mathbf{1}\big((M_t^\phi)^{1/r_\phi} > a\big)\big) \ge \sup_{\mathbf{x}\in\mathbb{C}} M_t(\mathbf{x})\,\mathbf{1}(M_t(\mathbf{x}) > a)\Pi(\mathbf{x}).$$

The latter implies, since $\{(M_t^\phi)^{1/r_\phi}, \phi \in \Phi\}$ is uniformly exponentially integrable relative to $\{P_\phi, \phi \in \Phi\}$, that $(M_t(\mathbf{x}), \mathbf{x} \in \mathbb{C})$ is Π–maximable.

□

In certain cases $\mathbb{C}$-exponential tightness allows one to establish LD convergence for the locally uniform topology on $\mathbb{D}$. The following result is an adaptation of Theorem 3.1.10.

Theorem 3.2.10. *Let $X^\phi \stackrel{ld}{\to} X$ for the Skorohod topology, where X is a Luzin-continuous idempotent process. If the X^ϕ are random variables on $\mathbb{D}$ relative to the locally uniform topology on $\mathbb{D}$, then $X^\phi \stackrel{ld}{\to} X$ for the locally uniform topology.*

Proof. Since convergence in the Skorohod topology to a continuous function is equivalent to locally uniform convergence, the Skorohod and locally uniform topologies are locally equivalent at every $\mathbf{x} \in \mathbb{C}$ so Theorem 3.1.10 applies. □

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We now discuss composition and first-passage-time mappings. If $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^d) \in \mathbb{D}$ and the component functions of $\mathbf{y} = (\mathbf{y}^1, \dots, \mathbf{y}^d) \in \mathbb{D}$ are increasing and $\mathbb{R}_+$-valued, we define the composition $\mathbf{x} \circ \mathbf{y}$ by $\mathbf{x} \circ \mathbf{y} = ((\mathbf{x}^1_{\mathbf{y}^1_t}, \dots, \mathbf{x}^d_{\mathbf{y}^d_t}),\, t \in \mathbb{R}_+)$.

Lemma 3.2.11. *Let $X^\phi \stackrel{ld}{\to} X$, where X is a Luzin-continuous idempotent process. Let Y^ϕ be stochastic processes with paths in $\mathbb{D}$, whose component processes are $\mathbb{R}_+$-valued and increasing, such that $Y^\phi \stackrel{P_\phi^{1/r_\phi}}{\to} \hat{\mathbf{y}} \in \mathbb{C}$. Then $X^\phi \circ Y^\phi \stackrel{ld}{\to} X \circ \hat{\mathbf{y}}$.*

Proof. Clearly, $\hat{\mathbf{y}}$ is component-wise $\mathbb{R}_+$-valued and increasing so that $X \circ \hat{\mathbf{y}}$ is well defined. By Lemma 3.1.42 $(X^\phi, Y^\phi) \stackrel{ld}{\to} (X, \hat{\mathbf{y}})$. The claim now follows by Corollary 3.1.15 and the fact that the composition map $(\mathbf{x}, \mathbf{y}) \to \mathbf{x} \circ \mathbf{y}$ is continuous at $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}$ and $\mathbf{y}$ are continuous, Billingsley [11], Whitt [135, Theorem 3.1]. ☐

Definition 3.2.12. *Given an $\mathbb{R}_+$-valued function $\mathbf{x} = (\mathbf{x}_t,\ t \in \mathbb{R}_+)$ from $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ with sample paths that are unbounded above, the associated first-passage-time function $\mathbf{x}^{(-1)} = (\mathbf{x}^{(-1)}_t, t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is defined by*

$$\mathbf{x}^{(-1)}_t = \inf\{s \in \mathbb{R}_+ :\, \mathbf{x}_s > t\},\ t \in \mathbb{R}_+ \,.$$

If $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^d) \in \mathbb{D}$ is such that the $\mathbf{x}^i$, $i = 1, \dots, d$, are $\mathbb{R}_+$-valued and unbounded above, we define $\mathbf{x}^{(-1)} = (\mathbf{x}^{1\,(-1)}, \dots, \mathbf{x}^{d\,(-1)})$.

In the next lemma $c_\phi \to \infty$ as $\phi \in \Phi$. We also denote $\mathbf{e} = (t, t \in \mathbb{R}_+)$ and, for a vector $\alpha = (\alpha^1, \dots, \alpha^d)$, we let $\alpha\mathbf{e} = (\alpha t, t \in \mathbb{R}_+)$ and $\alpha^{-1} = (1/\alpha^1, \dots, 1/\alpha^d)$ if α has positive entries. For vectors $\alpha = (\alpha^1, \dots, \alpha^d)$ and $\beta = (\beta^1, \dots, \beta^d)$, we denote $\alpha \otimes \beta = (\alpha^1\beta^1, \dots, \alpha^d\beta^d)$.

Lemma 3.2.13. *Let $\{X^\phi,\, \phi \in \Phi\}$ be a net of stochastic processes with paths in $\mathbb{D}$ defined on respective probability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ such that the component processes are $\mathbb{R}_+$-valued and unbounded above. Let X be an $\mathbb{R}^d$-valued Luzin-continuous idempotent process defined on an idempotent probability space (Ω, Π).*

1. *Let, in addition, the component idempotent processes of X be $\mathbb{R}_+$-valued, unbounded above and strictly increasing Π-a.e. If $X^\phi \stackrel{ld}{\to} X$, then $X^{\phi(-1)} \stackrel{ld}{\to} X^{(-1)}$.*

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2. Let $Y^\phi = (Y_t^\phi,\, t \in \mathbb{R}_+)$ be stochastic processes with paths from $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{d'})$ and Y be an $\mathbb{R}^{d'}$-valued Luzin-continuous idempotent process. Let $\alpha_\phi \in \mathbb{R}^{d'}$ be such that $\alpha_\phi \to \alpha$, where α is entrywise positive. Let, in addition, $X_0 = 0$. If $(c_\phi(X^\phi - \alpha_\phi \mathbf{e}), Y^\phi) \overset{ld}{\to} (X, Y)$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{d+d'})$, then $(c_\phi(X^{\phi(-1)} - \alpha_\phi{}^{-1}\mathbf{e}), c_\phi(X^\phi - \alpha_\phi \mathbf{e}), Y^\phi) \overset{ld}{\to} (-\alpha^{-1} \otimes X \circ (\alpha^{-1}\mathbf{e}), X, Y)$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d+d'})$.

Proof. Part 1 follows by Corollary 3.1.15 and the fact that the map $\mathbf{x} \to \mathbf{x}^{(-1)}$ is continuous at strictly increasing $\mathbf{x} \in \mathbb{D}$, Whitt [135, Theorem 7.2].

We prove part 2. We first consider the case $\alpha_\phi = \mathbf{1}_d$, where $\mathbf{1}_d$ is a d-vector with unity entries. Since $c_\phi(X^\phi - \mathbf{1}_d\mathbf{e}) \overset{ld}{\to} X$ as $\phi \in \Phi$, $c_\phi \to \infty$, and X is Luzin-continuous, it follows that $X^\phi \overset{P_\phi^{1/r_\phi}}{\to} \mathbf{1}_d\mathbf{e}$ in $\mathbb{D}$ so by part 1 and Lemma 3.1.41 $X^{\phi(-1)} \overset{P_\phi^{1/r_\phi}}{\to} \mathbf{1}_d\mathbf{e}$ in $\mathbb{D}$. By Lemma 3.2.11 $(c_\phi(X^\phi - \mathbf{1}_d\mathbf{e}) \circ X^{\phi(-1)}, c_\phi(X^\phi - \mathbf{1}_d\mathbf{e}), Y^\phi) \overset{ld}{\to} (X, X, Y)$. Since

$$c_\phi(X^{\phi(-1)} - \mathbf{1}_d\mathbf{e}) = c_\phi(\mathbf{1}_d\mathbf{e} - X^\phi) \circ X^{\phi(-1)} + c_\phi(X^\phi \circ X^{\phi(-1)} - \mathbf{1}_d\mathbf{e}) ,$$

by Lemma 3.1.38 it suffices to prove that $c_\phi(X^\phi \circ X^{\phi(-1)} - \mathbf{1}_d\mathbf{e}) \overset{P_\phi^{1/r_\phi}}{\to} 0$ in $\mathbb{D}$, which would follow by

$$\sup_{t\in[0,T]} c_\phi |X^\phi \circ X^{\phi(-1)}_t - t| \overset{P_\phi^{1/r_\phi}}{\to} 0\,, \quad T \in \mathbb{R}_+. \tag{3.2.7}$$

Since

$$0 \le \sup_{t\in[0,T]} (X^\phi \circ X^{\phi(-1)}_t - t) \le (X_0^\phi)^+ \vee \sup_{t\in[0, X^{\phi(-1)}_T]} (\Delta X_t^\phi)^+ ,$$

we have for $A \in \mathbb{R}_+$ and $\epsilon > 0$ that

$$P_\phi^{1/r_\phi}\big(\sup_{t\in[0,T]} c_\phi |X^\phi \circ X^{\phi(-1)}_t - t| > \epsilon\big) \le P_\phi^{1/r_\phi}\big(X^{\phi(-1)}_T > A\big) + P_\phi^{1/r_\phi}\big(\sup_{t\in[0,A]} c_\phi((X_0^\phi)^+ \vee (\Delta X_t^\phi)^+) > \epsilon\big)\,. \tag{3.2.8}$$

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Since the function $\mathbf{x} \to \sup_{t\in[0,A]}(\Delta\mathbf{x}_t)^+$ is continuous at continuous $\mathbf{x}$, Liptser and Shiryaev [79], and $\mathbf{x} \to \mathbf{x}_0$ is continuous, by Corollary 3.1.15 and the LD convergence $c_\phi(X^\phi - \mathbf{1}_d\mathbf{e}) \stackrel{ld}{\to} X$

$$\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\sup_{t\in[0,A]} c_\phi((X_0^\phi)^+ \vee (\Delta X_t^\phi)^+) \ge \epsilon\big) \le \Pi\big(\sup_{t\in[0,A]} X_0^+ \vee (\Delta X_t)^+ \ge \epsilon\big) = 0 \ ,$$

proving that the second term on the right of (3.2.8) tends to 0 as $\phi \in \Phi$. The first term goes to 0 as $\phi \in \Phi$ and $A \to \infty$ since $X^{\phi(-1)} \stackrel{P_\phi^{1/r_\phi}}{\to} \mathbf{1}_d\mathbf{e}$. The limit (3.2.7) has been proved so that the claim for the case $\alpha_\phi = \mathbf{1}_d$ has been proved.

For general α_ϕ the hypotheses imply by Lemma 3.2.11 that $\big(c_\phi(X^\phi \circ (\alpha_\phi^{-1}\mathbf{e}) - \mathbf{1}_d\mathbf{e}), Y^\phi\big) \stackrel{ld}{\to} (X \circ (\alpha^{-1}\mathbf{e}), Y)$ so by the part proved $\big(c_\phi\big((X^\phi \circ (\alpha_\phi^{-1}\mathbf{e}))^{(-1)} - \mathbf{1}_d\mathbf{e}\big), c_\phi(X^\phi \circ (\alpha_\phi^{-1}\mathbf{e}) - \mathbf{1}_d\mathbf{e}), Y^\phi\big) \stackrel{ld}{\to} \big(-X \circ (\alpha^{-1}\mathbf{e}), X \circ (\alpha^{-1}\mathbf{e}), Y\big)$. Since $(X^\phi \circ (\alpha_\phi^{-1}\mathbf{e}))^{(-1)} - \mathbf{1}_d\mathbf{e} = \alpha_\phi \otimes \big(X^{\phi(-1)} - \alpha_\phi^{-1}\mathbf{e}\big)$, by Corollary 3.1.15 and Lemma 3.2.11 $\big(c_\phi\big(X^{\phi(-1)} - \alpha_\phi^{-1}\mathbf{e}\big), c_\phi(X^\phi - \alpha_\phi\mathbf{e}), Y^\phi\big) \stackrel{ld}{\to} \big(-\alpha^{-1} \otimes X \circ (\alpha^{-1}\mathbf{e}), X, Y\big)$. □

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Chapter 4

The method of finite-dimensional distributions

In this chapter we consider the method of finite-dimensional distributions of identifying LD accumulation points. It is best suited for studying LD convergence to idempotent processes with independent increments and is based on Theorem 3.2.8. As in the preceding section, we consider a net $\{X^\phi,\ \phi \in \Phi\}$ of stochastic processes, which for the most part are semimartingales defined on respective stochastic bases $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$ and having paths in $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. The filtrations $\mathbf{F}_\phi = (\mathcal{F}_t^\phi,\ t \in \mathbb{R}_+)$ are assumed to be complete and right-continuous; E_ϕ denotes expectation with respect to P_ϕ. We assume as fixed a net $\{r_\phi,\ \phi \in \Phi\}$ of real numbers greater than 1 converging to ∞ as $\phi \in \Phi$, which is used as a rate for LD convergences below; the latter refer to the Skorohod topology. We retain the rest of the notation of Section 3.2, e.g., we write $E_\phi^{1/r_\phi}\xi$ for $(E_\phi \xi)^{1/r_\phi}$. Section 4.1 formulates conditions for LD convergence in distribution in terms of convergence of the stochastic exponentials of the semimartingales, Section 4.2 gives conditions on convergence of the predictable characteristics, Sections 4.3 and 4.4 consider implications of the general results.

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4.1 Convergence of stochastic exponentials

In this section we use the method of finite dimensional distributions to derive conditions for LD convergence of semimartingales in a Skorohod space in terms of convergence of the associated stochastic exponentials. We start by introducing the general setting for both this chapter and the next one. For the notions and facts from stochastic calculus used below we refer the reader to Jacod and Shiryaev [67] and Liptser and Shiryaev [79].

Let $X^\phi = (X^\phi_t,\, t \in \mathbb{R}_+)$, $\phi \in \Phi$, be $\mathbb{R}^d$–valued semimartingales defined on stochastic bases $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$, where $\mathbf{F}_\phi = (\mathcal{F}^\phi_t, t \in \mathbb{R}_+)$. All the X^ϕ, as well as all the processes we consider below, have paths in an appropriate Skorohod space (which is $\mathbb{D}$ for the X^ϕ).

We recall that a Borel function $h : \mathbb{R}^d \to \mathbb{R}^d$ is said to be a limiter if it is bounded and $h(x) = x$ in a neighbourhood of the origin. Every truncation function as defined in Jacod and Shiryaev [67] is a limiter. In the same way as it is done for truncation functions one can define the triplet of the predictable characteristics of a semimartingale associated with a limiter. This slight extension of the class of truncation functions is convenient for technical reasons in that it allows us to consider characteristics associated with limiters that do not vanish at infinity by contrast with truncation functions.

Let $h(x)$ be a limiter. Then

$$h^\phi(x) = \frac{1}{r_\phi}\, h(r_\phi x) \tag{4.1.1}$$

is also a limiter, and we denote by $(B^\phi, C^\phi, \nu^\phi)$ the triplet of the predictable characteristics of X^ϕ associated with $h^\phi(x)$ (i.e., defined as if $h^\phi(x)$ were a truncation function). We also say that $(B^\phi, C^\phi, \nu^\phi)$ corresponds to $h(x)$. We recall that this is equivalent to X^ϕ having the following canonical representation:

$$X^\phi_t = X^\phi_0 + B^\phi_t + X^{\phi,c}_t + h^\phi(x) * (\mu^\phi - \nu^\phi)_t + (x - h^\phi(x)) * \mu^\phi_t, \tag{4.1.2}$$

where

$B^\phi = (B^\phi_t,\, t \in \mathbb{R}_+), B^\phi_0 = 0$, is an $\mathbb{R}^d$–valued $\mathbf{F}_\phi$–predictable process with bounded variation over bounded intervals;

$X^{\phi,c} = (X^{\phi,c}_t, t \in \mathbb{R}_+), X^{\phi,c}_0 = 0$, is an $\mathbb{R}^d$–valued continuous local martingale with respect to $\mathbf{F}_\phi$ that is the continuous martingale part of X^ϕ;

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μ^ϕ is the measure associated with jumps of X^ϕ, i.e.,

$$\mu^\phi([0,t],\Gamma) = \sum_{0<s\le t} \mathbf{1}(\Delta X_s^\phi \in \Gamma\backslash\{0\}),\ \Gamma \in \mathcal{B}(\mathbb{R}^d);$$

ν^ϕ is an $\mathbf{F}_\phi$–predictable random measure on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d))$ that is the $\mathbf{F}_\phi$–compensator of μ^ϕ.

We use $*$ to denote integration so that

$$\begin{aligned} h^\phi(x) * (\mu^\phi - \nu^\phi)_t &= \int_0^t \int_{\mathbb{R}^d} h^\phi(x)(\mu^\phi(ds,dx) - \nu^\phi(ds,dx)), \\ (x - h^\phi(x)) * \mu_t^\phi &= \int_0^t \int_{\mathbb{R}^d} (x - h^\phi(x))\mu^\phi(ds,dx), \\ f(x) * \nu_t^\phi &= \int_0^t \int_{\mathbb{R}^d} f(x)\nu^\phi(ds,dx). \end{aligned}$$

In analogy with earlier notation for semimaxingales we also denote

$$f(x) \bullet \nu_t^\phi = \int_{\mathbb{R}^d} f(x)\nu^\phi(\{t\},dx).$$

An $\mathbb{R}^{d\times d}$–valued continuous process $C^\phi = (C_t^\phi,\, t \in \mathbb{R}_+), C_0^\phi = 0$, is defined to be the $\mathbf{F}_\phi$–predictable quadratic-variation process of $X^{\phi,c}$. We also define the continuous part of the predictable measure of jumps by

$$\nu^{\phi,c}(ds,dx) = \mathbf{1}(\nu^\phi(\{s\},\mathbb{R}^d) = 0)\nu^\phi(ds,dx).$$

We consider the version $(B^\phi, C^\phi, \nu^\phi)$ of the characteristics for which identically:
$C_t^\phi - C_s^\phi,\ 0 \le s < t$, is a symmetric positive semi-definite $d \times d$–matrix,

$$\nu^\phi(\{0\},\mathbb{R}^d) = 0,\ \nu^\phi(\mathbb{R}_+,\{0\}) = 0,\ \nu^\phi(\{t\},\mathbb{R}^d) \le 1, \tag{4.1.3a}$$

$$(|x|^2 \wedge 1) * \nu_t^\phi < \infty, \tag{4.1.3b}$$

$$\Delta B_t^\phi = h^\phi(x) \bullet \nu_t^\phi. \tag{4.1.3c}$$

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We recall that C^ϕ and ν^ϕ do not depend on the choice of h, while if $\overline{B}^\phi = (\overline{B}^\phi_t,\, t \in \mathbb{R}_+)$ is the first characteristic corresponding to another limiter $\overline{h}(x)$, then

$$\overline{B}^\phi_t - B^\phi_t = (\overline{h}^\phi(x) - h^\phi(x)) * \nu^\phi_t. \tag{4.1.4}$$

Along with C^ϕ, we introduce $\mathbb{R}^{d\times d}$–valued processes $C^{\phi,\delta} = (C^{\phi,\delta}_t, t \in \mathbb{R}_+), \delta > 0$, and $\tilde{C}^\phi = (\tilde{C}^\phi_t,\, t \in \mathbb{R}_+)$ that are the respective $\mathbf{F}_\phi$–predictable quadratic-variation processes of the locally square-integrable martingales

$$M^{\phi,\delta}_t = X^{\phi,c}_t + x\,\mathbf{1}(r_\phi|x| \le \delta) * (\mu^\phi - \nu^\phi)_t, t \in \mathbb{R}_+, \tag{4.1.5}$$

and

$$\tilde{M}^\phi_t = X^{\phi,c}_t + h^\phi(x) * (\mu^\phi - \nu^\phi)_t, t \in \mathbb{R}_+, \tag{4.1.6}$$

and are specified by the equalities

$$\lambda \cdot C^{\phi,\delta}_t \lambda = \lambda \cdot C^\phi_t \lambda + \big(\lambda \cdot x\ \mathbf{1}(r_\phi|x| \le \delta)\big)^2 * \nu^\phi_t - \sum_{s\le t} \big(\lambda \cdot x\ \mathbf{1}(r_\phi|x| \le \delta) \bullet \nu^\phi_s\big)^2 \tag{4.1.7}$$

and

$$\lambda \cdot \tilde{C}^\phi_t \lambda = \lambda \cdot C^\phi_t \lambda + (\lambda \cdot h^\phi(x))^2 * \nu^\phi_t - \sum_{s\le t} \big(\lambda \cdot h^\phi(x) \bullet \nu^\phi_s\big)^2, \tag{4.1.8}$$

where $\lambda \in \mathbb{R}^d$. The processes $C^{\phi,\delta}$ and $\tilde{C}^\phi$ are referred to as modified second characteristics.

We recall that X^ϕ is a special semimartingale if

$$|x|\,\mathbf{1}(|x| > 1) * \nu^\phi_t < \infty, \ \ t \in \mathbb{R}_+. \tag{4.1.9}$$

Then one can consider the predictable triplet of X^ϕ "without truncation", i.e., assume that in (4.1.2) $h^\phi(x) = x$. We denote the process B^ϕ corresponding to this "nontruncation" by $B'^\phi = (B'^\phi_t,\, t \in \mathbb{R}_+)$, so the predictable triplet without truncation is $(B'^\phi, C^\phi, \nu^\phi)$. As it follows by (4.1.3c) and (4.1.4),

$$\Delta B'^\phi_t = x \bullet \nu^\phi_t, \tag{4.1.10}$$

$$B'^\phi_t = B^\phi_t + (x - h^\phi(x)) * \nu^\phi_t. \tag{4.1.11}$$

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If, moreover,

$$|x|^2\,\mathbf{1}(|x|>1)*\nu_t^\phi<\infty,\ \ t\in\mathbb{R}_+, \tag{4.1.12}$$

then X^ϕ is said to be a locally square integrable semimartingale. In that case one can define "nontruncated modified second characteristics" $\tilde{C}_t'^\phi=(\tilde{C}_t'^\phi,\,t\in\mathbb{R}_+)$ by

$$\lambda\cdot\tilde{C}_t'^\phi\lambda=\lambda\cdot C_t^\phi\lambda+(\lambda\cdot x)^2*\nu_t^\phi-\sum_{s\le t}(\lambda\cdot x\bullet\nu_s^\phi)^2. \tag{4.1.13}$$

The following stronger condition on ν^ϕ plays an important role below and is further referred to as the Cramér condition:

$$(Cr)\qquad e^{\alpha|x|}\,\mathbf{1}(|x|>1)*\nu_t^\phi<\infty,\ \ \text{for all } t\in\mathbb{R}_+,\alpha\in\mathbb{R}_+.$$

Under (Cr), we can define the stochastic cumulant

$$G_t^\phi(\lambda)=\lambda\cdot B'^\phi_t+\frac{1}{2}\lambda\cdot C_t^\phi\lambda+(e^{\lambda\cdot x}-1-\lambda\cdot x)*\nu_t^\phi,\ \lambda\in\mathbb{R}^d,t\in\mathbb{R}_+. \tag{4.1.14}$$

The process $G^\phi(\lambda)=(G_t^\phi(\lambda),t\in\mathbb{R}_+)$ is a real-valued $\mathbf{F}_\phi$–predictable process with bounded variation over bounded intervals, in particular, a semimartingale, so that we can define the associated stochastic (or Dolèans–Dade) exponential $\mathcal{E}^\phi(\lambda)=(\mathcal{E}_t^\phi(\lambda),t\in\mathbb{R}_+),\lambda\in\mathbb{R}^d$, by

$$\mathcal{E}_t^\phi(\lambda)=e^{G_t^\phi(\lambda)}\prod_{0<s\le t}(1+\Delta G_s^\phi(\lambda))e^{-\Delta G_s^\phi(\lambda)}. \tag{4.1.15}$$

(The right-hand side is well defined and is a semimartingale with paths in $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$, see Liptser and Shiryaev [79, Theorem 2.4.1], Jacod and Shiryaev [67, Theorem I.4.61].)

By (4.1.14), (4.1.10) and (4.1.3a)

$$\Delta G_s^\phi(\lambda)=\int_{\mathbb{R}^d}(e^{\lambda\cdot x}-1)\,\nu^\phi(\{s\},dx)>-1, \tag{4.1.16}$$

so, as we are going to see,

$$\mathcal{E}_t^\phi(\lambda)>0,\ t\in\mathbb{R}_+. \tag{4.1.17}$$

Therefore, we can define the processes $Y^\phi(\lambda)=(Y_t^\phi(\lambda),t\in\mathbb{R}_+),\lambda\in\mathbb{R}^d$, by

$$Y_t^\phi(\lambda)=e^{\lambda\cdot(X_t^\phi-X_0^\phi)}\mathcal{E}_t^\phi(\lambda)^{-1}. \tag{4.1.18}$$

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A fundamental property of the stochastic exponential is expressed by the following lemma.

Lemma 4.1.1. *Under the Cramér condition the process* $Y^\phi(\lambda) = (Y_t^\phi(\lambda), t \in \mathbb{R}_+), \lambda \in \mathbb{R}^d$, *is a well-defined local martingale on* $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$.

Proof. We first check $Y^\phi(\lambda)$ is well defined by showing that (4.1.17) holds. Since $G^\phi(\lambda)$ has bounded variation over bounded intervals,

$$\sum_{0<s\le t} |\Delta G_s^\phi(\lambda)| < \infty. \tag{4.1.19}$$

Next,

$$\prod_{0<s\le t} (1 + \Delta G_s^\phi(\lambda)) \\ \ge \exp\Big(-2 \sum_{0<s\le t} |\Delta G_s^\phi(\lambda)|\Big) \prod_{\substack{0<s\le t:\\ |\Delta G_s^\phi(\lambda)|>1/2}} (1 + \Delta G_s^\phi(\lambda)).$$

By (4.1.19) the product on the right has finitely many terms, which are positive, and is itself positive.

We now check the local martingale property of $Y^\phi(\lambda)$. Let for $k = 2, 3, \dots$

$$\tau_k = \inf\{t \in \mathbb{R}_+ : 1+\Delta G_t^\phi(\lambda) < 1/k\}.$$

Since by (4.1.19) G^ϕ has a finite number of jumps less than $1/k - 1$ on a bounded interval, it follows that $1 + \Delta G_{\tau_k}^\phi(\lambda) < 1/k$. Thus, the τ_k are $\mathbf{F}_\phi$-predictable stopping times as the débuts of predictable sets whose graphs belong to the sets, see Dellacherie [34, IV-T.16], Jacod and Shiryaev [67, I.2.13]. Also $\tau_k \to \infty$ as $k \to \infty$. Hence, there exist $\mathbf{F}_\phi$-stopping times $\sigma_k < \tau_k$ such that $\sigma_k \to \infty$ as $k \to \infty$. It is sufficient to show that the $(Y_{t\wedge\sigma_k}^\phi(\lambda), t \in \mathbb{R}_+)$ are local martingales relative to $\mathbf{F}_\phi$. Since $\sigma_k < \tau_k$, we have that

$$\inf_{t\le\sigma_k} (1+\Delta G_t^\phi(\lambda)) \ge \frac{1}{k}. \tag{4.1.20}$$

By (4.1.19) $\sum_{0<s\le t} |\ln(1 + \Delta G_s^\phi(\lambda)| < \infty$, so the process

$$\ln \mathcal{E}_t^\phi(\lambda) = G_t^\phi(\lambda) - \sum_{0<s\le t} \Delta G_s^\phi(\lambda) + \sum_{0<s\le t} \ln(1+\Delta G_s^\phi(\lambda)) \tag{4.1.21}$$

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is well defined and is a semimartingale. Let

$$U_t = \lambda \cdot (X_t^\phi - X_0^\phi) - \ln \mathcal{E}_t^\phi(\lambda).$$

Then $Y_t^\phi(\lambda) = \exp U_t$ so that by the Ito formula

$$Y_{t\wedge\sigma_k}^\phi(\lambda) = 1 + \int_0^{t\wedge\sigma_k} e^{U_{s-}} dU_s + \frac{1}{2} \int_0^{t\wedge\sigma_k} e^{U_{s-}} d\langle U^c \rangle_s + \sum_{0<s\le t\wedge\sigma_k} \left(e^{U_s} - e^{U_{s-}} - e^{U_{s-}} \Delta U_s\right), \quad (4.1.22)$$

where $\langle U^c \rangle$ denotes the predictable quadratic-variation process of the continuous martingale part of U. Noting that $\langle U^c \rangle = \lambda \cdot C^\phi \lambda$ and invoking the canonical decomposition of the special semimartingale X^ϕ with no truncation $X_t^\phi = X_0^\phi + B_t'^\phi + M_t^\phi$, where M^ϕ is a local martingale relative to $\mathbf{F}_\phi$, we derive from (4.1.22) after some algebra that

$$Y_{t\wedge\sigma_k}^\phi(\lambda) = 1 + \lambda \cdot \int_0^{t\wedge\sigma_k} e^{U_{s-}} dM_s^\phi + \int_0^{t\wedge\sigma_k} \int_{\mathbb{R}^d} e^{U_{s-}} \left(e^{\lambda\cdot x} - 1 - \lambda \cdot x\right) (\mu^\phi - \nu^\phi)(ds, dx) - \sum_{0<s\le t\wedge\sigma_k} e^{U_{s-}} \left(\frac{e^{\lambda\cdot\Delta X_s^\phi}}{1 + \Delta G_s^\phi(\lambda)} - 1\right) \Delta G_s^\phi(\lambda). \quad (4.1.23)$$

The first integral on the right of (4.1.23) is a stochastic integral of a locally bounded predictable process with respect to a local martingale, hence, it is a local martingale. The second integral is an integral with respect to a martingale measure and is a local martingale since $\int_0^t \int_{\mathbb{R}^d} e^{U_{s-}} \left(e^{\lambda\cdot x} - 1 - \lambda \cdot x\right) \nu^\phi(ds, dx) < \infty$, Jacod and Shiryaev [67, II.1.28]. Let us consider the sum on the right of (4.1.23). Firstly, it is

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absolutely convergent by (4.1.19) and (4.1.20). Secondly, by (4.1.16)

$$\sum_{0<s\le t\wedge\sigma_k} e^{U_{s-}}\Big(\frac{e^{\lambda\cdot\Delta X_s^\phi}}{1+\Delta G_s^\phi(\lambda)}-1\Big)\Delta G_s^\phi(\lambda)$$

$$=\sum_{0<s\le t\wedge\sigma_k} e^{U_{s-}}\big(e^{\lambda\cdot\Delta X_s^\phi}-1\big)\frac{\Delta G_s^\phi(\lambda)}{1+\Delta G_s^\phi(\lambda)}$$

$$-\sum_{0<s\le t\wedge\sigma_k} e^{U_{s-}}\frac{\big(\Delta G_s^\phi(\lambda)\big)^2}{1+\Delta G_s^\phi(\lambda)}$$

$$=\int_0^{t\wedge\sigma_k}\int_{\mathbb{R}^d} e^{U_{s-}}\big(e^{\lambda\cdot x}-1\big)\frac{\Delta G_s^\phi(\lambda)}{1+\Delta G_s^\phi(\lambda)}(\mu-\nu)(ds,dx),$$

which is a local martingale by Jacod and Shiryaev [67, II.1.28] and the fact that

$$\sum_{0<s\le t\wedge\sigma_k} e^{U_{s-}}\frac{\big(\Delta G_s^\phi(\lambda)\big)^2}{1+\Delta G_s^\phi(\lambda)}<\infty.$$

□

Let $(G_t(\lambda),\, t\in\mathbb{R}_+,\lambda\in\mathbb{R}^d)$ be an $\mathbb{R}$-valued function, which is continuous in t, differentiable in λ, and such that the increments $G_t(\lambda)-G_s(\lambda)$ are convex functions of $\lambda\in\mathbb{R}^d$ for $0\le s<t$ and $G_0(\lambda)=G_t(0)=0$. Let $\mathbf{\Pi}_{x_0}$, where $x_0\in\mathbb{R}^d$, be defined as in Section 2.7 (see (2.7.6)). We recall that it is a deviability on $\mathbb{C}$ by Lemma 2.8.3. We denote its extension to a deviability on $\mathbb{D}$ with support in $\mathbb{C}$ as $\mathbf{\Pi}_{x_0}$ as well. Let X be the canonical idempotent process on $(\mathbb{D},\mathbf{\Pi}_{x_0})$. We state the central result of the section.

Theorem 4.1.2. *Let the X^ϕ satisfy the Cramér condition. If, as $\phi\in\Phi$,*

(0) $$X_0^\phi \overset{P_\phi^{1/r_\phi}}{\to} x_0,$$

and for all $T>0$ and $\lambda\in\mathbb{R}^d$

(sup $\mathcal{E}$) $$\sup_{t\le T}\Big|\frac{1}{r_\phi}\ln\mathcal{E}_t^\phi(r_\phi\lambda)-G_t(\lambda)\Big| \overset{P_\phi^{1/r_\phi}}{\to} 0\,,$$

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then $X^\phi \stackrel{ld}{\to} X$.

We begin the proof with preliminary results. The hypotheses of Theorem 4.1.2 are assumed to hold. We also assume with no loss of generality that $x_0 = 0$. We introduce the $\mathbf{F}_\phi$-stopping times ($\lambda \in \mathbb{R}^d$)

$$\sigma^\phi(\lambda) = \inf\bigl\{t \in \mathbb{R}_+ : \mathcal{E}_t^\phi(r_\phi\lambda)^{1/r_\phi} \vee \mathcal{E}_t^\phi(r_\phi\lambda)^{-1/r_\phi} \geq 2e^{G_t^*(\lambda)} \\ \text{or } \mathcal{E}_t^\phi(2r_\phi\lambda)^{1/r_\phi} \vee \mathcal{E}_t^\phi(2r_\phi\lambda)^{-1/r_\phi} \geq 2e^{G_t^*(2\lambda)}\bigr\}. \quad (4.1.24)$$

By condition (sup $\mathcal{E}$)

$$\lim_{\phi\in\Phi} P_\phi^{1/r_\phi}\bigl(\sigma^\phi(\lambda) \leq t\bigr) = 0,\ t \in \mathbb{R}_+, \lambda \in \mathbb{R}^d. \quad (4.1.25)$$

Being the début of a predictable set whose graph belongs to the set, $\sigma^\phi(\lambda)$ is an $\mathbf{F}_\phi$–predictable stopping time. Thus, $\sigma^\phi(\lambda)$ is P_ϕ-a.s. announced by an increasing sequence of $\mathbf{F}_\phi$–stopping times. Since $\sigma^\phi(\lambda) > 0$, there exist $\mathbf{F}_\phi$–stopping times $\tau^\phi(\lambda) < \sigma^\phi(\lambda)$ such that

$$P_\phi^{1/r_\phi}\Bigl(\tau^\phi(\lambda) + \frac{1}{r_\phi} \leq \sigma^\phi(\lambda) < \infty\Bigr) \\ + P_\phi^{1/r_\phi}\bigl(\tau^\phi(\lambda) \leq r_\phi,\, \sigma^\phi(\lambda) = \infty\bigr) \leq \frac{1}{r_\phi}. \quad (4.1.26)$$

In view of the inequality

$$P_\phi(\tau^\phi(\lambda) \leq t) \leq P_\phi(\sigma^\phi(\lambda) \leq t+1) \\ + P_\phi\Bigl(\tau^\phi(\lambda) + \frac{1}{r_\phi} \leq (t+1) \wedge \sigma^\phi(\lambda)\Bigr),$$

we have from (4.1.25) and (4.1.26) that

$$\lim_{\phi\in\Phi} P_\phi^{1/r_\phi}\bigl(\tau^\phi(\lambda) \leq t\bigr) = 0, t \in \mathbb{R}_+. \quad (4.1.27)$$

Note also that by (4.1.24) and the inequality $\tau^\phi(\lambda) < \sigma^\phi(\lambda)$

$$\mathcal{E}_{t\wedge\tau^\phi(\lambda)}^\phi(r_\phi\lambda)^{1/r_\phi} \vee \mathcal{E}_{t\wedge\tau^\phi(\lambda)}^\phi(r_\phi\lambda)^{-1/r_\phi} < 2e^{G_t^*(\lambda)}, \quad (4.1.28)$$

$$\mathcal{E}_{t\wedge\tau^\phi(\lambda)}^\phi(2r_\phi\lambda)^{1/r_\phi} \vee \mathcal{E}_{t\wedge\tau^\phi(\lambda)}^\phi(2r_\phi\lambda)^{-1/r_\phi} < 2e^{G_t^*(2\lambda)}. \quad (4.1.29)$$

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Lemma 4.1.3. *For all* $\lambda \in \mathbb{R}^d$, *the process* $Y^\phi(\lambda)$ *is a positive supermartingale relative to* $\mathbf{F}_\phi$. *The process* $(Y^\phi_{t\wedge\tau^\phi(\lambda)}(\lambda),\, t \in [0,T])$ *is a square integrable martingale relative to* $\mathbf{F}_\phi$ *for every* $T > 0$ *and*

$$E_\phi\big(Y^\phi_{t\wedge\tau^\phi(\lambda)}(r_\phi\lambda)^2\big) \le 2^{3r_\phi} e^{r_\phi[G^*_t(2\lambda)+2G^*_t(\lambda)]},\ \lambda \in \mathbb{R}^d, t \in \mathbb{R}_+. \tag{4.1.30}$$

Proof. By Lemma 4.1.1 $Y^\phi(\lambda)$ is a positive local martingale relative to $\mathbf{F}_\phi$. Hence it is a supermartingale. So we have to prove only (4.1.30). By the supermartingale property of $Y^\phi(\lambda)$ and the fact that $Y^\phi_0(\lambda) = 1$, for all finite $\mathbf{F}_\phi$-stopping times τ,

$$E_\phi Y^\phi_\tau(\lambda) \le 1,\ \ \lambda \in \mathbb{R}^d. \tag{4.1.31}$$

In view of (4.1.29) and the definition of $Y^\phi(\lambda)$, (4.1.31) with $2r_\phi\lambda$ implies that

$$E_\phi \exp\big(2r_\phi\lambda\cdot(X^\phi_{t\wedge\tau^\phi(\lambda)} - X^\phi_0)\big) \le 2^{r_\phi} \exp\big(r_\phi G^*_t(2\lambda)\big),\ \ \lambda \in \mathbb{R}^d,$$

which by (4.1.28) yields

$$E_\phi\big(Y^\phi_{t\wedge\tau^\phi(\lambda)}(r_\phi\lambda)^2\big) \le 2^{r_\phi} \exp\big(r_\phi G^*_t(2\lambda)\big) 2^{2r_\phi} \exp\big(2r_\phi G^*_t(\lambda)\big),$$

proving the lemma. □

Lemma 4.1.4. *Let for* $0 = t_0 < t_1 < \ldots < t_k$

$$Z^\phi_i(\lambda) = X^\phi_{t_i\wedge\tau^\phi(\lambda)} - X^\phi_{t_{i-1}\wedge\tau^\phi(\lambda)},\ \lambda \in \mathbb{R}^d.$$

Then, for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^d$, *the net* $\Big\{\exp\big(\sum_{i=1}^k \lambda_i\cdot Z^\phi_i(\lambda_i)\big), \phi \in \Phi\Big\}$ *is uniformly exponentially integrable with respect to the* P_ϕ *and*

$$\lim_{\phi\in\Phi} E_\phi^{1/r_\phi} \exp\Big(r_\phi \sum_{i=1}^k \lambda_i\cdot Z^\phi_i(\lambda_i)\Big) = \exp\Big(\sum_{i=1}^k \big(G_{t_i}(\lambda_i) - G_{t_{i-1}}(\lambda_i)\big)\Big).$$

Proof. Let

$$\zeta^\phi_i = \sum_{j=1}^i \lambda_j\cdot Z^\phi_j(\lambda_j),\ i = 1, \ldots, k,\ \zeta^\phi_0 = 0. \tag{4.1.32}$$

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We first prove that for $i = 1, \ldots, k$

$$E_\phi \exp(2r_\phi \zeta_i^\phi) \le 2^{2r_\phi i} \prod_{j=1}^{i} \exp\bigl(r_\phi\bigl(G^*_{t_j}(2\lambda_j) + G^*_{t_{j-1}}(2\lambda_j)\bigr)\bigr). \tag{4.1.33}$$

The proof of Lemma 4.1.3 implies that $E_\phi \exp(2r_\phi \zeta_i^\phi) < \infty$. In view of (4.1.32) and the definitions of $Z_i^\phi(\lambda)$ and $Y^\phi(\lambda)$, we have by Lemma 4.1.3 and (4.1.29) for $i = 1, \ldots, k$

$$\begin{aligned}
&E_\phi \left[\exp(2r_\phi \zeta_i^\phi) \mid \mathcal{F}^\phi_{t_{i-1}}\right] \\
&\le \exp(2r_\phi \zeta_{i-1}^\phi) \exp\bigl(-2r_\phi \lambda_i \cdot (X^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)} - X_0^\phi)\bigr) \\
&Y^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)}(2r_\phi\lambda_i) 2^{r_\phi} e^{r_\phi G^*_{t_i}(2\lambda_i)} \\
&\qquad = \exp(2r_\phi \zeta_{i-1}^\phi) \mathcal{E}^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)}(2r_\phi\lambda_i)^{-1} 2^{r_\phi} e^{r_\phi G^*_{t_i}(2\lambda_i)}.
\end{aligned}$$

Applying to the latter (4.1.29) again we deduce

$$\begin{aligned}
&E_\phi \left[\exp(2r_\phi \zeta_i^\phi) \mid \mathcal{F}^\phi_{t_{i-1}}\right] \\
&\qquad \le \exp(2r_\phi \zeta_{i-1}^\phi) 2^{2r_\phi} \exp\bigl(r_\phi\bigl(G^*_{t_i}(2\lambda_i) + G^*_{t_{i-1}}(2\lambda_i)\bigr)\bigr).
\end{aligned}$$

This proves (4.1.33). Uniform exponential integrability of $\bigl(\exp\bigl(\sum_{i=1}^k \lambda_i \cdot Z_i^\phi(\lambda_i)\bigr), \phi \in \Phi\bigr)$ is implied by (4.1.33) with $i = k$.

We prove the convergence required in the lemma by proving that for $i = 1, \ldots, k$

$$\lim_{\phi\in\Phi} E_\phi^{1/r_\phi} \exp(r_\phi \zeta_i^\phi) = e^{g_i}, \tag{4.1.34}$$

where

$$g_i = \sum_{j=1}^{i} \bigl(G_{t_j}(\lambda_j) - G_{t_{j-1}}(\lambda_j)\bigr),\ i = 1, \ldots, k,\ g_0 = 0,$$

provided (4.1.34) holds for $(i-1)$.

For $\delta \in (0, 1/2)$ we define the sets

$$\begin{aligned}
B_\delta^\phi = \Bigl\{\omega \in \Omega_\phi : &\left|\mathcal{E}^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i)^{1/r_\phi} e^{-G_{t_{i-1}}(\lambda_i)} - 1\right| \ge \delta \\
&\text{or } \left|\mathcal{E}^\phi_{t_i\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i)^{-1/r_\phi} e^{G_{t_i}(\lambda_i)} - 1\right| \ge \delta\Bigr\}
\end{aligned}$$

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and $A_\delta^\phi = \Omega_\phi \setminus B_\delta^\phi$. By (sup $\mathcal{E}$) and (4.1.27), as $\phi \in \Phi$,

$$P_\phi^{1/r_\phi}\big(B_\delta^\phi\big) \to 0,\ P_\phi^{1/r_\phi}\big(A_\delta^\phi\big) \to 1\,. \tag{4.1.35}$$

Applying the Cauchy-Schwarz inequality we have by (4.1.33) and (4.1.35)

$$\lim_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(\exp(r_\phi\zeta_i^\phi)\mathbf{1}(B_\delta^\phi)\big) = 0,$$

which implies that (4.1.34) would follow from

$$\liminf_{\delta\to 0}\liminf_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(\exp(r_\phi\zeta_i^\phi)\mathbf{1}(A_\delta^\phi)\big) = \limsup_{\delta\to 0}\limsup_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(\exp(r_\phi\zeta_i^\phi)\mathbf{1}(A_\delta^\phi)\big) = e^{g_i}. \tag{4.1.36}$$

Let

$$R_\delta^\phi = \exp(r_\phi\zeta_{i-1}^\phi)Y^\phi_{t_i\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i)Y^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i)^{-1}. \tag{4.1.37}$$

By Lemma 4.1.3 $E_\phi R_\delta^\phi = E_\phi \exp(r_\phi\zeta_{i-1}^\phi)$, so by our assumption

$$\lim_{\phi\in\Phi} E_\phi^{1/r_\phi} R_\delta^\phi = e^{g_{i-1}}. \tag{4.1.38}$$

On the other hand, (4.1.32), (4.1.37), and the definitions of $Y^\phi(\lambda)$ and $Z_i^\phi(\lambda)$ yield

$$R_\delta^\phi = \exp(r_\phi\zeta_i^\phi)\mathcal{E}^\phi_{t_i\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i)^{-1}\mathcal{E}^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i). \tag{4.1.39}$$

Therefore, applying the Cauchy-Schwarz inequality we have in view of (4.1.28), (4.1.35) and (4.1.33) that $\lim_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(R_\delta^\phi\mathbf{1}(B_\delta^\phi)\big) = 0$, which by (4.1.38) obtains

$$\lim_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(R_\delta^\phi\mathbf{1}(A_\delta^\phi)\big) = e^{g_{i-1}}. \tag{4.1.40}$$

By definition, we have that on A_δ^ϕ

$$\mathcal{E}^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i) \le (1+\delta)^{r_\phi}\exp\big(r_\phi G_{t_{i-1}}(\lambda_i)\big),$$
$$\mathcal{E}^\phi_{t_i\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i)^{-1} \le (1+\delta)^{r_\phi}\exp\big(-r_\phi G_{t_i}(\lambda_i)\big).$$

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Therefore, by (4.1.39)

$$R_\delta^\phi \mathbf{1}(A_\delta^\phi)$$
$$\le \exp(r_\phi \zeta_i^\phi)(1+\delta)^{2r_\phi} \exp\big(-r_\phi(G_{t_i}(\lambda_i) - G_{t_{i-1}}(\lambda_i))\big)\mathbf{1}(A_\delta^\phi).$$

The latter implies by (4.1.40) that

$$\liminf_{\delta\to 0} \liminf_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(\exp(r_\phi \zeta_i)\mathbf{1}(A_\delta^\phi)\big)$$
$$\ge e^{g_{i-1}} \exp\big(G_{t_i}(\lambda_i) - G_{t_{i-1}}(\lambda_i)\big) = e^{g_i}.$$

In an analogous manner, the inequalities

$$\mathcal{E}^\phi_{t_{i-1}\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i) \ge (1-\delta)^{r_\phi} \exp\big(r_\phi G_{t_{i-1}}(\lambda_i)\big),$$
$$\mathcal{E}^\phi_{t_i\wedge\tau^\phi(\lambda_i)}(r_\phi\lambda_i)^{-1} \ge (1-\delta)^{r_\phi} \exp\big(-r_\phi G_{t_i}(\lambda_i)\big)$$

yield the limit

$$\limsup_{\delta\to 0} \limsup_{\phi\in\Phi} E_\phi^{1/r_\phi}\big(\exp(r_\phi \zeta_i)\mathbf{1}(A_\delta^\phi)\big) \le e^{g_i}.$$

Limits (4.1.36) are proved. □

The last preliminary result needed for the proof of Theorem 4.1.2 is the following version of Theorem 3.1.31, which is proved in a similar manner.

Lemma 4.1.5. *Let $\{K^\phi, \phi \in \Phi\}$ be a net of $\mathbb{R}^m$–valued random variables defined on respective probability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ and K be an $\mathbb{R}^m$-valued idempotent variable defined on an idempotent probability space (Ω, Π). Let $S_\Pi \exp(\lambda\cdot K)$ be finite for and differentiable in $\lambda \in \mathbb{R}^m$. Let there exist nets $\{Z^\phi(\lambda),\ \phi\in\Phi\}$ of $\mathbb{R}^m$–valued random variables defined on $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ such that $K^\phi - Z^\phi(\lambda) \xrightarrow{P_\phi^{1/r_\phi}} 0$ and $E_\phi^{1/r_\phi} \exp\big(r_\phi \lambda \cdot Z^\phi(\lambda)\big) \to S_\Pi \exp(\lambda\cdot K)$ as $\phi \in \Phi$, and the net $\{\exp\big(\lambda\cdot Z^\phi(\lambda)\big), \phi\in\Phi\}$ is uniformly exponentially integrable for $\lambda \in \mathbb{R}^m$. Then $K^\phi \xrightarrow{ld} K$.*

Now we proceed with the proof of Theorem 4.1.2 itself. Recall that $x_0 = 0$.

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Proof of Theorem 4.1.2. We apply Theorem 3.2.8 to the net $\{\mathcal{L}(X^\phi),\ \phi \in \Phi\}$. We verify the $\mathbb{C}$-exponential tightness by checking the conditions of part II of Theorem 3.2.3. We begin with condition (i). In view of (4.1.31) and the definition of $Y^\phi(\lambda)$, we can apply to $\bigl(\exp\bigl(r_\phi \lambda \cdot (X_t^\phi - X_0^\phi)\bigr),\ t \in \mathbb{R}_+\bigr)$ and $\mathcal{E}^\phi(r_\phi \lambda)$ Lemma 3.2.6 to obtain for all $A > 0, B > 0$ and $L > 0$

$$P_\phi\bigl(\sup_{t \le L} \exp\bigl(r_\phi \lambda \cdot (X_t^\phi - X_0^\phi)\bigr) \ge e^{r_\phi A}\bigr) \le e^{r_\phi (B-A)} + P_\phi\bigl(\sup_{t \le L} \mathcal{E}_t^\phi(r_\phi \lambda) \ge e^{r_\phi B}\bigr),\ \lambda \in \mathbb{R}^d. \quad (4.1.41)$$

Taking $B > G_L^*(\lambda) + 1$, we have by (sup $\mathcal{E}$)

$$\lim_{\phi \in \Phi} P_\phi^{1/r_\phi}\bigl(\sup_{t \le T} \mathcal{E}_t^\phi(r_\phi \lambda) \ge e^{r_\phi B}\bigr) = 0,$$

and then (4.1.41) yields, for $\lambda \in \mathbb{R}^d$,

$$\limsup_{\phi \in \Phi} P_\phi^{1/r_\phi}\bigl(\sup_{t \le T} \lambda \cdot (X_t^\phi - X_0^\phi) > A\bigr) \le e^{B-A} \to 0 \text{ as } A \to \infty.$$

Since λ is arbitrary, this implies that

$$\lim_{A \to \infty} \limsup_{\phi \in \Phi} P_\phi^{1/r_\phi}\bigl(\sup_{t \le T} |X_t^\phi - X_0^\phi| > A\bigr) = 0.$$

As by hypotheses $X_0^\phi \overset{P_\phi^{1/r_\phi}}{\to} 0$, we obtain (i).

Turning to (ii) it is again sufficient to prove that for all $\lambda \in \mathbb{R}^d, \lambda \ne 0, \eta > 0$, and $T > 0$,

$$\lim_{\delta \to 0} \limsup_{\phi \in \Phi} \sup_{\tau \in \mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}\bigl(\sup_{t \le \delta} \frac{\lambda}{|\lambda|} \cdot (X_{t+\tau}^\phi - X_\tau^\phi) > \eta\bigr) = 0. \quad (4.1.42)$$

By Lemma 4.1.3 and Doob's stopping theorem we have for every finite $\mathbf{F}_\phi$– stopping times σ and τ such that $\sigma \ge \tau$

$$E_\phi\left[\frac{Y_\sigma^\phi(r_\phi \lambda)}{Y_\tau^\phi(r_\phi \lambda)}\right] \le 1. \quad (4.1.43)$$

Fixing $\tau \in \mathbf{S}_T(\mathbf{F}_\phi)$, let for $t \in \mathbb{R}_+$

$$X_t^{\phi,\tau} = X_{t+\tau}^\phi - X_\tau^\phi, \quad (4.1.44a)$$

$$\mathcal{E}_t^{\phi,\tau}(\lambda) = \frac{\mathcal{E}_{t+\tau}^\phi(\lambda)}{\mathcal{E}_\tau^\phi(\lambda)} \quad (4.1.44b)$$

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and introduce the filtration $\mathbf{F}_{\phi,\tau} = (\mathcal{F}^{\phi}_{t+\tau},\, t \in \mathbb{R}_+)$. Let σ be a finite $\mathbf{F}_{\phi,\tau}$–stopping time. Then $(\sigma+\tau)$ is an $\mathbf{F}_{\phi}$–stopping time so that by (4.1.43) (with $\sigma = \sigma+\tau$), (4.1.44a), (4.1.44b), and the definition of $Y^{\phi}(\lambda)$ we have

$$E_{\phi}\left[\frac{\exp(r_{\phi}\lambda\cdot X^{\phi,\tau}_{\sigma})}{\mathcal{E}^{\phi,\tau}_{\sigma}(r_{\phi}\lambda)}\right] \le 1.$$

As σ is an arbitrary finite $\mathbf{F}_{\phi,\tau}$–stopping time, by Lemma 3.2.6 we conclude that for all $\lambda \in \mathbb{R}^d, \eta > 0, \delta > 0$, and $\alpha > 0$

$$P_{\phi}\big(\sup_{t\le\delta}\lambda\cdot X^{\phi,\tau}_t \ge |\lambda|\eta\big) \le e^{r_{\phi}(\alpha-\eta)|\lambda|} + P_{\phi}\big(\sup_{t\le\delta}\mathcal{E}^{\phi,\tau}_t(r_{\phi}\lambda)^{1/r_{\phi}} \ge e^{\alpha|\lambda|}\big). \quad (4.1.45)$$

By (4.1.44b)

$$\sup_{t\le\delta}\frac{1}{r_{\phi}}\ln\mathcal{E}^{\phi,\tau}_t(r_{\phi}\lambda) \le \Big|\frac{1}{r_{\phi}}\ln\mathcal{E}^{\phi}_{\tau}(r_{\phi}\lambda) - G_{\tau}(\lambda)\Big| + \sup_{t\le\delta}\Big|\frac{1}{r_{\phi}}\ln\mathcal{E}^{\phi}_{t+\tau}(r_{\phi}\lambda) - G_{t+\tau}(\lambda)\Big| + \sup_{t\le\delta}|G_{t+\tau}(\lambda) - G_{\tau}(\lambda)|. \quad (4.1.46)$$

Since $G_t(\lambda)$ is continuous in t,

$$\sup_{\substack{|t-s|\le\delta\\ 0\le t,s\le T+\delta}}|G_t(\lambda) - G_s(\lambda)| \le \frac{1}{2}\alpha|\lambda|$$

for all sufficiently small δ. Thus, (4.1.46) yields for these δ by the fact that $\tau \le T$

$$P_{\phi}\big(\sup_{t\le\delta}\mathcal{E}^{\phi,\tau}_t(r_{\phi}\lambda)^{1/r_{\phi}} \ge e^{\alpha|\lambda|}\big) \le P_{\phi}\Big(\sup_{t\le T+\delta}\Big|\frac{1}{r_{\phi}}\ln\mathcal{E}^{\phi}_t(r_{\phi}\lambda) - G_t(\lambda)\Big| \ge \frac{\alpha|\lambda|}{4}\Big).$$

Substituting the right-hand side into (4.1.45) and using $(\sup\mathcal{E})$ we obtain for $\lambda \ne 0$

$$\limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_{\phi}^{1/r_{\phi}}\big(\sup_{t\le\delta}\frac{\lambda}{|\lambda|}\cdot X^{\phi,\tau}_t \ge \eta\big) \le e^{(\alpha-\eta)|\lambda|}.$$

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Taking $\alpha = \eta/2$ and $|\lambda| \to \infty$, and using (4.1.44a) we arrive at (4.1.42). $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(X^\phi)\}$ follows.

Let us check LD convergence of finite-dimensional distributions. Let $0 = t_0 < t_1 < \ldots < t_k$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^d$, and X denote the canonical idempotent process on $\mathbb{D}$. We also denote $m = d \times k, \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^m, K^\phi = (X^\phi_{t_1} - X^\phi_0, \ldots, X^\phi_{t_k} - X^\phi_{t_{k-1}}), K = (X_{t_1} - X_0, \ldots, X_{t_k} - X_{t_{k-1}})$, and $Z^\phi(\lambda) = (Z^\phi_1(\lambda_1), \ldots, Z^\phi_k(\lambda_k))$, where the $Z^\phi_i(\lambda_i)$ are defined in Lemma 4.1.4. Then by (4.1.27)

$$P_\phi^{1/r_\phi}\big(|K^\phi - Z^\phi(\lambda)| > \varepsilon\big) \le P_\phi^{1/r_\phi}\big(K^\phi \ne Z^\phi(\lambda)\big) \le \sum_{i=1}^k P_\phi^{1/r_\phi}\big(\tau^\phi(\lambda_i) \le t_i\big) \to 0.$$

By Theorem 2.8.5 if $\mathbb{D}$ is equipped with $\mathbf{\Pi}_0$, then X is an idempotent process with independent increments starting at 0 and such that $S_{\mathbf{\Pi}_0} \exp\big(\lambda \cdot (X_t - X_s)\big) = \exp\big(G_t(\lambda) - G_s(\lambda)\big)$. By Lemma 4.1.4 $\{K^\phi, \phi \in \Phi\}$ and $\{Z^\phi(\lambda), \phi \in \Phi\}, \lambda \in \mathbb{R}^m$, satisfy the conditions of Lemma 4.1.5 so that $K^\phi \stackrel{ld}{\to} K$. The contraction principle then implies by the definition of K^ϕ that $\big(X^\phi_{t_1} - X^\phi_0, \ldots, X^\phi_{t_k} - X^\phi_0\big) \stackrel{ld}{\to} \big(X_{t_1} - X_0, \ldots, X_{t_k} - X_0\big)$. Since $X^\phi_0 \stackrel{P_\phi^{1/r_\phi}}{\to} 0$ and $X_0 = 0$ $\mathbf{\Pi}_0$-a.e., by Lemma 3.1.42 and the contraction principle $\big(X^\phi_{t_0}, X^\phi_{t_1}, \ldots, X^\phi_{t_k}\big) \stackrel{ld}{\to} \big(X_{t_0}, X_{t_1}, \ldots, X_{t_k}\big)$ proving the finite dimensional LD convergence. $\square$

Remark 4.1.6. *By Theorem 2.8.5 under $\mathbf{\Pi}_{x_0}$ X is an idempotent process with independent increments starting at x_0 and having cumulant $G(\lambda)$. In particular, X satisfies "the Cramér condition" $S\exp\big(\alpha|X_t|\big) < \infty$ for $\alpha \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$.*

We now give two more versions of the theorem in which the X^ϕ do not have to be semimartingales. Inspection of the above proof shows that the critical property of $\mathcal{E}^\phi(\lambda)$ is the one stated in Lemma 4.1.1 that the process $Y^\phi(\lambda) = (Y^\phi_t(\lambda), t \in \mathbb{R}_+), \lambda \in \mathbb{R}^d$, is a local martingale. Therefore, the following extension of Theorem 4.1.2 holds. The function $G_t(\lambda)$ satisfies the same conditions as above.

Theorem 4.1.7. *Let $X^\phi, \phi \in \Phi$, be stochastic processes with paths in $\mathbb{D}$ defined on respective stochastic bases $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$. Let for*

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every $\lambda \in \mathbb{R}^d$ *and* $\phi \in \Phi$ *there exist* $\mathbf{F}_\phi$*–predictable positive processes* $\mathcal{E}^\phi(\lambda) = \big(\mathcal{E}^\phi_t(\lambda),\, t \in \mathbb{R}_+\big)$, $\mathcal{E}^\phi_0(\lambda) = 1$, *such that the processes* $Y^\phi(\lambda) = \big(Y^\phi_t(\lambda),\, t \in \mathbb{R}_+\big)$ *defined by*

$$Y^\phi_t(\lambda) = \exp\big(\lambda\cdot(X^\phi_t - X^\phi_0)\big)\mathcal{E}^\phi_t(\lambda)^{-1}$$

are $\mathbf{F}_\phi$*–local martingales.*

If $X^\phi_0 \overset{P_\phi^{1/r_\phi}}{\to} x_0$ *and, for all* $T > 0$ *and* $\lambda \in \mathbb{R}^d$,

$$\sup_{t\le T}\left|\frac{1}{r_\phi}\ln \mathcal{E}^\phi_t(r_\phi\lambda) - G_t(\lambda)\right| \overset{P_\phi^{1/r_\phi}}{\to} 0,$$

as $\phi \in \Phi$, *then* $\mathcal{L}(X^\phi) \overset{ld}{\to} \mathbf{\Pi}_{x_0}$.

As a consequence, we obtain the following result for processes with independent increments that are not necessarily semimartingales.

Theorem 4.1.8. *Let* X^ϕ *be processes with independent increments with paths in* $\mathbb{D}$ *such that* $E_\phi \exp\big(\lambda\cdot(X^\phi_t - X^\phi_0)\big) < \infty$ *for all* $t \in \mathbb{R}_+$ *and* $\lambda \in \mathbb{R}^d$.

If $X^\phi_0 \overset{P_\phi^{1/r_\phi}}{\to} x_0$ *and, for all* $T > 0$ *and* $\lambda \in \mathbb{R}^d$,

$$\sup_{t\le T}\left|\frac{1}{r_\phi}\ln E_\phi \exp\big(\lambda\cdot(X^\phi_t - X^\phi_0)\big) - G_t(\lambda)\right| \to 0,$$

then $\mathcal{L}(X^\phi) \overset{ld}{\to} \mathbf{\Pi}_{x_0}$.

Remark 4.1.9. *If the* X^ϕ *are semimartingales with independent increments, then the assertions of Theorems 4.1.2 and 4.1.8 coincide since the triplets* $(B'^\phi, C^\phi, \nu^\phi)$ *are deterministic and* $\mathcal{E}^\phi_t(\lambda) = E_\phi \exp\big(\lambda\cdot(X^\phi_t - X^\phi_0)\big)$ *(cf. Jacod and Shiryaev [67, II.4.15]).*

4.2 Convergence of characteristics

In this section we give results on LD convergence in terms of the characteristics of the semimartingales. This allows us to do without the Cramér condition; we require instead that the measure of “big

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jumps" be small. As in the preceding section, we consider a net $\{X^\phi, \phi \in \Phi\}$ of semimartingales on $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$ with predictable triplets $(B^\phi, C^\phi, \nu^\phi)$ corresponding to a limiter $h(x)$. We also consider as given a cumulant $G_t(\lambda)$, which does not depend on $\mathbf{x}$ and is defined as in (2.7.7) and (2.7.55) by

$$G_t(\lambda) = \int_0^t g_s(\lambda)\, ds,\ \lambda \in \mathbb{R}^d, t \in \mathbb{R}_+, \tag{4.2.1}$$

where

$$\begin{aligned} g_s(\lambda) = \lambda \cdot b_s + \frac{1}{2}\, \lambda \cdot c_s \lambda + \int_{\mathbb{R}^d} (e^{\lambda \cdot x} - 1 - \lambda \cdot x) \nu_s(dx) \\ + \Big(\ln\big(1 + \int_{\mathbb{R}^d} (e^{\lambda \cdot x} - 1) \hat{\nu}_s(dx)\big) - \int_{\mathbb{R}^d} (e^{\lambda \cdot x} - 1) \hat{\nu}_s(dx)\Big), \end{aligned} \tag{4.2.2}$$

$(b_s,\ s \in \mathbb{R}_+)$ is an $\mathbb{R}^d$-valued Lebesgue-measurable function such that $\int_0^t |b_s|\, ds < \infty$ for $t \in \mathbb{R}_+$,
$(c_s,\ s \in \mathbb{R}_+)$ is a Lebesgue-measurable function with values in the space of symmetric, positive semi-definite $d \times d$-matrices such that $\int_0^t \|c_s\|\, ds < \infty$ for $t \in \mathbb{R}_+$,
$(\nu_s(\Gamma),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d))$ is a transition kernel from $(\mathbb{R}_+, \overline{\mathcal{B}}(\mathbb{R}_+))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that for $t \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+$

$$\begin{gathered} \nu_t(\{0\}) = 0,\ \int_{\mathbb{R}^d} |x|^2 \wedge 1\, \nu_t(dx) < \infty, \\ \int_{\mathbb{R}^d} e^{\alpha |x|}\, \mathbf{1}(|x| > 1)\, \nu_t(dx) < \infty, \\ |x|^2 \wedge 1 * \nu_t < \infty,\ e^{\alpha|x|}\, \mathbf{1}(|x| > 1) * \nu_t < \infty, \end{gathered} \tag{4.2.3}$$

$\big(\hat{\nu}_s(\Gamma),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d)\big)$ is a transition kernel from $(\mathbb{R}_+, \overline{\mathcal{B}}(\mathbb{R}_+))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that for $s \in \mathbb{R}_+$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$\hat{\nu}_s(\Gamma) \le \nu_s(\Gamma),\ \hat{\nu}_s(\mathbb{R}^d) \le 1. \tag{4.2.4}$$

We also assume the following condition to hold

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$$(L_1) \qquad \sup_{s\le t} e^{\alpha|x|} \bullet \hat{\nu}_s < \infty,\ t\in\mathbb{R}_+,\ \alpha\in\mathbb{R}_+.$$

By Lemma 2.8.8 condition (L_1) implies condition (L_0) of Corollary 2.8.7. We recall that $G_t(\lambda)$ is differentiable in λ under (L_0). Let $x_0\in\mathbb{R}^d$. By Corollary 2.8.7 $\mathbf{\Pi}_{x_0}$ is a deviability on $\mathbb{D}$ and the canonical idempotent process X is a Luzin-continuous semimaxingale with independent increments on $(\mathbb{D},\mathbf{\Pi}_{x_0})$ starting at x_0 and having local characteristics $(b,c,\nu,\hat{\nu})$. As in Section 2.7, we denote as $(B,C,\nu,\hat{\nu})$ the characteristics of X associated with a limiter $h(x)$; B' denotes the nontruncated first characteristic and $\tilde{C}$ denotes the modified second characteristic, which are defined by (2.7.11), (2.7.57), and (2.7.58). To recall,

$$B'_t=\int_0^t b_s\,ds,\ B_t=B'_t+(h(x)-x)*\nu_t,\ C_t=\int_0^t c_s\,ds,$$

and for $\lambda\in\mathbb{R}^d$

$$\lambda\cdot\tilde{C}_t\lambda=\lambda\cdot C_t\lambda+(\lambda\cdot h(x))^2*\nu_t - \int_0^t(\lambda\cdot h(x)\bullet\hat{\nu}_s)^2ds.$$

Let U be a dense subset of $\mathbb{R}_+$ and let $\mathcal{C}_b$ denote the set of functions $f:\mathbb{R}^d\to\mathbb{R}^d$ that are bounded, continuous and equal to 0 in a neighbourhood of the origin. For $f\in\mathcal{C}_b$, we denote $f^\phi(x)=f(r_\phi x)/r_\phi$.

We consider the following conditions

$$(0) \qquad X_0^\phi \xrightarrow{P_\phi^{1/r_\phi}} x_0 \ \text{ as } \phi\in\Phi,$$

$$(A) \quad \lim_{A\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\nu^\phi([0,t],|x|>A)^{1/r_\phi}>\varepsilon\big)=0,\ t>0,\ \varepsilon>0,$$

$$(a) \quad \lim_{a\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}e^{\alpha r_\phi|x|}\,\mathbf{1}(r_\phi|x|>a)\,\mathbf{1}(|x|\le A)*\nu_t^\phi>\varepsilon\Big)=0,\quad t>0,\ \alpha>0,\ A>0,\ \varepsilon>0.$$

$$(\sup B) \qquad \sup_{t\le T}|B_t^\phi-B_t| \xrightarrow{P_\phi^{1/r_\phi}} 0 \ \text{ as } \phi\in\Phi,\ \ T>0,$$

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$$(C) \qquad \lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\,\|r_\phi\, C_t^{\phi,\delta} - C_t\| > \varepsilon\Big) = 0,\ t\in U,\ \varepsilon>0,$$

$$(\tilde C) \qquad \|r_\phi\,\tilde C_t^\phi - \tilde C_t\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ t\in U,$$

$$(\nu) \qquad f^\phi(x)*\nu_t^\phi - f(x)*\nu_t \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ t\in U,\ f\in\mathcal{C}_b,$$

$$(\hat\nu) \qquad \frac{1}{r_\phi}\sum_{0<s\le t}\big(f(r_\phi x)\bullet\nu_s^\phi\big)^k - \int_0^t \big(f(x)\bullet\hat\nu_s\big)^k\,ds \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,$$

$$k=2,3,\ldots,\ t\in U,\ f\in\mathcal{C}_b.$$

The following theorem is the main result of the section.

Theorem 4.2.1. *Let the limiter $h(x)$ be continuous. If conditions (0), $(A)+(a)$, $(\sup B)$, (C) (or $(\tilde C)$), (ν), and $(\hat\nu)$ hold, then $X^\phi \overset{ld}{\to} X$.*

Remark 4.2.2. *Conditions (C) and $(\tilde C)$ equivalently require convergence of the entries of the respective matrices $r_\phi C_t^{\phi,\delta}$ and $r_\phi \tilde C_t^\phi$ to the corresponding entries of C_t.*

Remark 4.2.3. *The assertion of the theorem is equivalent to the LD convergence $\mathcal{L}(X^\phi) \overset{ld}{\to} \mathbf{\Pi}_{x_0}$. We also recall that by Lemma 2.7.12 under the hypotheses*

$$\mathbf{\Pi}_{x_0}(\mathbf{x}) = \exp\Big(-\int_0^\infty \sup_{\lambda\in\mathbb{R}^d}\big(\lambda\cdot\dot{\mathbf{x}}_t - g_t(\lambda)\big)\,dt\Big)$$

if $\mathbf{x}$ is absolutely continuous and $\mathbf{x}_0 = x_0$, and $\mathbf{\Pi}_{x_0}(\mathbf{x}) = 0$ otherwise.

Remark 4.2.4. *The theorem also holds if condition (a) is replaced by the following weaker condition*

$$(a') \qquad \lim_{a\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi} j^\phi_{a,A,\alpha}(x)*\nu_t^{\phi,c} + \frac{1}{r_\phi}\sum_{0<s\le t}\ln\big(1+j^\phi_{a,A,\alpha}(x)\bullet\nu_s^\phi\big) > \varepsilon\Big) = 0,$$

$$t>0,\ \alpha>0,\ A>0,\ \varepsilon>0,$$

where

$$j^\phi_{a,A,\alpha}(x) = (e^{\alpha r_\phi |x|} - 1)\,\mathbf{1}(r_\phi|x|>a)\,\mathbf{1}(|x|\le A).$$

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Before proceeding with a proof we give a version for processes with independent increments (PII) that are not necessarily semimartingales. Let X^ϕ be PII. Then, given a limiter $h(x)$, the X^ϕ admit decomposition (4.1.2), see Jacod and Shiryaev [67, II.5], where

$B^\phi = (B_t^\phi,\, t \in \mathbb{R}_+), B_0^\phi = 0$, is an $\mathbb{R}^d$–valued right-continuous with left limits (deterministic) function;

$X^{\phi,c} = (X_t^{\phi,c}, t \in \mathbb{R}_+), X_0^{\phi,c} = 0$, is an $\mathbb{R}^d$–valued continuous local martingale with respect to $\mathbf{F}_\phi$ that is the continuous martingale part of X^ϕ;

μ^ϕ is the measure associated with jumps of X^ϕ;

ν^ϕ is a (deterministic) measure on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d))$ that is the $\mathbf{F}_\phi$–compensator of μ^ϕ.

Also, relations (4.1.3a) and (4.1.3c) hold.

Let $C^\phi = (C_t^\phi,\, t \in \mathbb{R}_+), C_0^\phi = 0$, be the $\mathbf{F}_\phi$–predictable quadratic-variation process of $X^{\phi,c}$. Then C^ϕ is a deterministic $\mathbb{R}^{d\times d}$–valued continuous function such that the matrices $C_t^\phi - C_s^\phi$ are symmetric and positive semi-definite for $s \le t$. As above, we denote by $C^{\phi,\delta}$ and $\tilde{C}^\phi$ the $\mathbf{F}_\phi$-predictable quadratic-variation processes of the respective local martingales $M^{\phi,\delta}$ and M^ϕ from the respective equalities (4.1.5) and (4.1.6). As with C^ϕ, the processes $C^{\phi,\delta}$ and $\tilde{C}^\phi$ are actually deterministic matrix-valued functions.

Since X^ϕ is not necessarily a semimartingale, the function B might no longer have bounded variation over bounded intervals and condition (4.1.3b) is not in general satisfied; so, one cannot specify $C^{\phi,\delta}$ and $\tilde{C}^\phi$ by the respective equalities (4.1.7) and (4.1.8). Instead, we have for $\lambda \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$

$$\lambda \cdot C_t^{\phi,\delta}\lambda = \lambda \cdot C_t^\phi \lambda + \bigl(\lambda \cdot x\, \mathbf{1}(r_\phi |x| \le \delta) - \lambda \cdot x\, \mathbf{1}(r_\phi |x| \le \delta) \bullet \nu_s^\phi\bigr)^2 * \nu_t^\phi \\ + \sum_{0<s\le t} \bigl(\lambda \cdot x\, \mathbf{1}(r_\phi |x| \le \delta) \bullet \nu_s^\phi\bigr)^2 \bigl(1 - \nu^\phi(\{s\}, \mathbb{R}^d)\bigr) \quad (4.2.5)$$

and

$$\lambda \cdot \tilde{C}_t^\phi \lambda = \lambda \cdot C_t^\phi \lambda + \bigl(\lambda \cdot h^\phi(x) - \lambda \cdot h^\phi(x) \bullet \nu_s^\phi\bigr)^2 * \nu_t^\phi \\ + \sum_{0<s\le t} \bigl(\lambda \cdot h^\phi(x) \bullet \nu_s^\phi\bigr)^2 \bigl(1 - \nu^\phi(\{s\}, \mathbb{R}^d)\bigr). \quad (4.2.6)$$

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The right-hand sides are well defined since by Jacod and Shiryaev [67, II.5.6]

$$\left|h^{\phi}(x)-\Delta B_s^{\phi}\right|^2 * \nu_t^{\phi} + \sum_{0<s\le t} |\Delta B_s^{\phi}|^2 \big(1-\nu^{\phi}(\{s\},\mathbb{R}^d)\big) < \infty. \tag{4.2.7}$$

Formulas (4.2.5) and (4.2.6) reduce to (4.1.7) and (4.1.8) if (4.1.3b) holds.

Since the characteristics of a PII are deterministic, conditions (A), (a), (a'), $(\sup B)$, (C), $(\tilde{C})$, (ν), and $(\hat{\nu})$ take the form

$(A)_I$
$$\lim_{A\to\infty} \limsup_{\phi\in\Phi} \nu^{\phi}([0,t],|x|>A)^{1/r_\phi} = 0,\ t>0,$$

$(a)_I$
$$\lim_{a\to\infty} \limsup_{\phi\in\Phi} \frac{1}{r_\phi} e^{\alpha r_\phi |x|}\, \mathbf{1}(r_\phi|x|>a)\, \mathbf{1}(|x|\le A) * \nu_t^{\phi} = 0,$$
$$t>0,\ \alpha>0,\ A>0,\ \varepsilon>0.$$

$(a')_I$
$$\lim_{a\to\infty} \limsup_{\phi\in\Phi} \Big(\frac{1}{r_\phi} j^{\phi}_{a,A,\alpha}(x) * \nu_t^{\phi,c} + \frac{1}{r_\phi} \sum_{0<s\le t} \ln\big(1 + j^{\phi}_{a,A,\alpha}(x) \bullet \nu_s^{\phi}\big)\Big) = 0,$$
$$t>0,\ \alpha>0,\ A>0,\ \varepsilon>0,$$

$(\sup B)_I$
$$\sup_{t\le T} |B_t^{\phi} - B_t| \to 0 \text{ as } \phi\in\Phi,\ \ T>0,$$

$(C)_I$
$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} \|r_\phi\, C_t^{\phi,\delta} - C_t\| = 0,\ t\in U,$$

$(\tilde{C})_I$
$$\lim_{\phi\in\Phi} \|r_\phi\, \tilde{C}_t^{\phi} - \tilde{C}_t\| = 0,\ t\in U,$$

$(\nu)_I$
$$f^{\phi}(x) * \nu_t^{\phi} - f(x) * \nu_t \to 0 \text{ as } \phi\in\Phi,\ t\in U,\ f\in\mathcal{C}_b,$$

$(\hat{\nu})_I$
$$\frac{1}{r_\phi} \sum_{0<s\le t} \big(f(r_\phi x) \bullet \nu_s^{\phi}\big)^k - \int_0^t \big(f(x) \bullet \hat{\nu}_s\big)^k\, ds \to 0 \text{ as } \phi\in\Phi,$$
$$k=2,3,\dots,\ t\in U,\ f\in\mathcal{C}_b.$$

Theorem 4.2.5. *Let X^{ϕ} be PII with predictable characteristics $(B^{\phi}, C^{\phi}, \nu^{\phi})$ corresponding to a continuous limiter $h(x)$. Let conditions* (0)*,* $(A)_I$*,* $(a)_I$ *(or* $(a')_I$*),* $(\sup B)_I$*,* $(C)_I$ *(or* $(\tilde{C}_I)$*),* $(\nu)_I$*, and* $(\hat{\nu})_I$ *hold. Then* $X^{\phi} \xrightarrow{ld} X$ *as* $\phi\in\Phi$*.*

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We prove Theorems 4.2.1 and 4.2.5 in parallel. The argument actually refers to the X^ϕ being semimartingales. The modifications needed when the X^ϕ are PII are either self-evident (e.g., replacing super-exponential convergence in probability by deterministic convergence) or explicitly mentioned. The proof proceeds through a number of steps: we first establish interconnections between the conditions of the theorem, then derive the assertions of the theorems for the case of jumps of order r_ϕ^{-1} and finally consider the general setting.

Lemma 4.2.6. *Let $Z^{\phi,\delta} = (Z_t^{\phi,\delta}, t \in \mathbb{R}_+), Z_0^{\phi,\delta} = 0, \phi \in \Phi, \delta > 0$, be $\mathbb{R}_+$-valued increasing processes on respective probability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ and $Z = (Z_t, t \in \mathbb{R}_+), Z_0 = 0$, be a (deterministic) $\mathbb{R}_+$-valued increasing continuous function. If, for all $t \in U$ and $\varepsilon > 0$,*

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(|Z_t^{\phi,\delta} - Z_t| > \varepsilon\Big) = 0,$$

then this convergence is uniform so that

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T}|Z_t^{\phi,\delta} - Z_t| > \varepsilon\Big) = 0, \quad T > 0, \varepsilon > 0.$$

Proof. The argument is standard. Let $w_t(\delta)$ denote the modulus of continuity of Z on $[0,t]$, i.e., $w_t(\delta) = \sup_{\substack{u,v\le t:\\ |u-v|\le\delta}} |Z_u - Z_v|$. For $N \in \mathbb{N}$, we choose $t_i^N \in U$, $i = 0, \ldots, k^N$, such that $0 = t_0^N < t_1^N < \ldots < t_{k^N-1}^N < T \le t_{k^N}^N < T+1$ and $|t_i^N - t_{i-1}^N| \le 1/N$, $i = 1, \ldots, k^N$. Then, since the $Z^{\phi,\delta}$ and Z are increasing, for $t \in [t_{i-1}^N, t_i^N], i = 1, \ldots, k^N$, we have that

$$\begin{aligned}|Z_t^{\phi,\delta} - Z_t| &\le |Z_{t_i^N}^{\phi,\delta} - Z_{t_{i-1}^N}| \vee |Z_{t_{i-1}^N}^{\phi,\delta} - Z_{t_i^N}| \\ &\le |Z_{t_i^N}^{\phi,\delta} - Z_{t_i^N}| \vee |Z_{t_{i-1}^N}^{\phi,\delta} - Z_{t_{i-1}^N}| + |Z_{t_i^N} - Z_{t_{i-1}^N}|,\end{aligned}$$

and hence using that $Z_0^{\phi,\delta} = Z_0 = 0$

$$\sup_{t\le T}|Z_t^{\phi,\delta} - Z_t| \le \max_{i=1,\ldots,k^N} |Z_{t_i^N}^{\phi,\delta} - Z_{t_i^N}| + w_{T+1}\Big(\frac{1}{N}\Big).$$

Since $w_{T+1}(1/N) \to 0$ as $N \to \infty$ by continuity of Z, for arbitrary $\varepsilon > 0$ we have for all N large enough

$$P_\phi^{1/r_\phi}\Big(\sup_{t\le T}|Z_t^{\phi,\delta} - Z_t| > \varepsilon\Big) \le \sum_{i=1}^{k^N} P_\phi^{1/r_\phi}\Big(|Z_{t_i^N}^{\phi,\delta} - Z_{t_i^N}| > \varepsilon/2\Big).$$

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The latter goes to 0 as $\phi \in \Phi$ and $\delta \to 0$ by hypotheses and the choice of the t_i^N. □

As a consequence, we have the following.

Corollary 4.2.7. *Conditions* (C), $(\tilde{C})$, *and* (ν) *are equivalent to the following respective conditions* $(\sup C)$, $(\sup \tilde{C})$, *and* $(\sup \nu)$.

$$(\sup C) \qquad \lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T}\|r_\phi C_t^{\phi,\delta} - C_t\| > \varepsilon\Big) = 0, \quad \varepsilon > 0,\ T > 0,$$

$$(\sup \tilde{C}) \qquad \lim_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T}\|r_\phi \tilde{C}_t^{\phi} - \tilde{C}_t\| > \varepsilon\Big) = 0,\ \varepsilon > 0,\ T > 0,$$

$$(\sup \nu) \qquad \sup_{t\le T}\Big|f^\phi(x) * \nu_t^\phi - f(x) * \nu_t\Big| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi, \quad T > 0,\ f \in \mathcal{C}_b.$$

The second preliminary lemma shows that if condition (ν) holds, then condition $(\sup B)$ is invariant with respect to the choice of a continuous limiter h.

Lemma 4.2.8. *If condition* (ν) *holds, then condition* $(\sup B)$ *does not depend on the choice of a continuous limiter* $h(x)$.

Proof. Let $B^\phi = (B_t^\phi,\, t \in \mathbb{R}_+)$ and $\bar{B}^\phi = (\bar{B}_t^\phi,\, t \in \mathbb{R}_+)$ be the first characteristics of X^ϕ corresponding to continuous limiters $h(x)$ and $\bar{h}(x)$, respectively. Let B_t and $\bar{B}_t$ be the first characteristics of X associated with $h(x)$ and $\bar{h}(x)$, respectively. By (4.1.4), up to a P_ϕ-null set,

$$\bar{B}_t^\phi = B_t^\phi + (\bar{h}^\phi(x) - h^\phi(x)) * \nu_t^\phi,$$

so that by the definitions of B_t and $\bar{B}_t$, $h^\phi(x)$ and $\bar{h}^\phi(x)$, up to a P_ϕ-null set,

$$\bar{B}_t^\phi - \bar{B}_t = (B_t^\phi - B_t) + (\bar{h}^\phi(x) - h^\phi(x)) * \nu_t^\phi - (\bar{h}(x) - h(x)) * \nu_t,$$

and the equivalence of $(\sup B)$ and $(\sup \bar{B})$ under (ν) follows by the equivalence of (ν) and $(\sup \nu)$, and the fact that $\bar{h} - h \in \mathcal{C}_b$. □

We now consider implications of conditions (ν) and $(\hat{\nu})$.

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Lemma 4.2.9. *Let* (ν) *and* $(\hat{\nu})$ *hold. Then for* $\varepsilon > 0$ *and* $t \in U$

1. $$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\Big|\frac{1}{r_\phi} f(r_\phi x)\,\mathbf{1}(r_\phi|x| > \delta) * \nu_t^\phi - f(x) * \nu_t\Big| > \varepsilon\Big) = 0,$$

   for all $\mathbb{R}_+$*-valued bounded and continuous functions* $f(x)$, $x \in \mathbb{R}^d$, *such that* $f(x) \le c|x|^2, c > 0$, *in a neighbourhood of the origin;*

2. $$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{\delta}{r_\phi}|g(r_\phi x)|\,\mathbf{1}(r_\phi|x| > \delta) * \nu_t^\phi > \varepsilon\Big) = 0$$

   and

3. $$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\Big|\frac{1}{r_\phi}\sum_{0<s\le t}\big(g(r_\phi x)\,\mathbf{1}(r_\phi|x| > \delta) \bullet \nu_s^\phi\big)^k - \int_0^t \big(g(x) \bullet \hat{\nu}_s\big)^k ds\Big| > \varepsilon\Big) = 0,$$
   $$k = 2, 3, \dots,$$

   for all $\mathbb{R}$*-valued bounded and continuous functions* $g(x), x \in \mathbb{R}^d$, *such that* $|g(x)| \le c|x|, c > 0$, *in a neighbourhood of the origin.*

Proof. Let

$$f_r(x) = \big(\frac{|x|}{r} - 1\big)^+ \wedge 1, \quad r > 0, x \in \mathbb{R}^d.$$

Then, since $f(x) \ge 0$ and $f_{\delta/2}(x) \ge \mathbf{1}(|x| > \delta) \ge f_\delta(x)$,

$$\Big|\frac{1}{r_\phi} f(r_\phi x)\,\mathbf{1}(r_\phi|x| > \delta) * \nu_t^\phi - f(x) * \nu_t\Big|$$
$$\le f(x)\,\mathbf{1}(|x| \le \delta) * \nu_t + f(x)(f_{\delta/2}(x) - f_\delta(x)) * \nu_t$$
$$+ \max_{i=1,2}\Big|\frac{1}{r_\phi} f(r_\phi x) f_{\delta/i}(r_\phi x) * \nu_t^\phi - f(x) f_{\delta/i}(x) * \nu_t\Big|.$$

The last term on the right goes to 0 super-exponentially in probability as $\phi \in \Phi$ by (ν) and the inclusion $f(x)f_r(x) \in \mathcal{C}_b$. The sum of the two other terms does not exceed, for δ small enough, $2c|x|^2\,\mathbf{1}(|x| \le 2\delta) * \nu_t$, which goes to 0 as $\delta \to 0$ by (4.2.3) and Lebesgue's dominated convergence theorem. Part 1 is proved.

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Now we prove part 2. By (ν), using that $\mathbf{1}(|x| > \delta) \leq f_{\delta/2}(x)$ and $g(x)f_{\delta/2}(x) \in \mathcal{C}_b$,

$$\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{\delta}{r_\phi}|g(r_\phi x)|\,\mathbf{1}(r_\phi|x| > \delta) * \nu_t^\phi > \varepsilon\Big)$$
$$\leq \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{\delta}{r_\phi}|g(r_\phi x)|f_{\delta/2}(r_\phi x) * \nu_t^\phi > \varepsilon\Big)$$
$$\leq \mathbf{1}\big(\delta|g(x)|f_{\delta/2}(x) * \nu_t > \varepsilon/2\big). \quad (4.2.8)$$

Now, for $\sigma > \delta/2$, since $f_{\delta/2}(x) \leq \mathbf{1}(|x| > \delta/2)$,

$$\delta|g(x)|f_{\delta/2}(x) * \nu_t \leq \delta|g(x)|\,\mathbf{1}(|x| > \sigma) * \nu_t + \delta|g(x)|\,\mathbf{1}(\delta/2 < |x| \leq \sigma) * \nu_t.$$

The first term on the right, obviously, goes to 0 as $\delta \to 0$. The second one, by the assumptions on $g(x)$ and with the use of Chebyshev's inequality, is not greater than (take σ small enough) $2|g(x)||x|\,\mathbf{1}(|x| \leq \sigma) * \nu_t \leq 2c|x|^2\,\mathbf{1}(|x| \leq \sigma) * \nu_t$, and goes to 0 as $\sigma \to 0$, as above. Thus, the right-hand side of (4.2.8) is zero for δ small enough and part 2 is proved.

We prove part 3. By $(\hat\nu)$ and the inclusion $g(x)f_\delta(x) \in \mathcal{C}_b$, the required would follow by

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}\sum_{0<s\leq t}\Big|\big(g(r_\phi x)\,\mathbf{1}(r_\phi|x| > \delta) \bullet \nu_s^\phi\big)^k - \big(g(r_\phi x)f_\delta(r_\phi x) \bullet \nu_s^\phi\big)^k\Big| > \frac{\varepsilon}{2}\Big) = 0 \quad (4.2.9)$$

and

$$\lim_{\delta\to 0}\int_0^t \Big|\big(g(x)\bullet\hat\nu_s\big)^k - \big(g(x)f_\delta(x)\bullet\hat\nu_s\big)^k\Big|\,ds = 0.$$

The validity of the latter limit is obvious since $g(x)$ and $f_\delta(x)$ are uniformly bounded, $f_\delta(x) \to 1$ as $\delta \to 0$ for $x \neq 0$, and (4.2.4) holds. For (4.2.9), we write, by the inequalities $|x^k - y^k| \leq k(x \vee y)^{k-1}|x -$

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$y|$, $x, y > 0$, and $\mathbf{1}(|x| > 2\delta) \le f_\delta(x) \le \mathbf{1}(|x| > \delta)$,

$$\frac{1}{r_\phi} \sum_{0<s\le t} \Big| \big(g(r_\phi x)\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi\big)^k - \big(g(r_\phi x) f_\delta(r_\phi x) \bullet \nu_s^\phi\big)^k \Big|$$
$$\le \frac{k}{r_\phi} \sum_{0<s\le t} \big(|g(r_\phi x)|\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi\big)^{k-1}$$
$$\big(|g(r_\phi x)|\, \mathbf{1}(r_\phi |x| \le 2\delta) \bullet \nu_s^\phi\big). \quad (4.2.10)$$

Applying to the first integral on the right of (4.2.10) Jensen's inequality and recalling that $\nu^\phi(\{s\}, \mathbb{R}^d) \le 1$, we conclude that, for δ small enough, the right-hand side of (4.2.10) is not greater than

$$2\delta c\, \frac{k}{r_\phi} \sum_{0<s\le t} |g(r_\phi x)|^{k-1}\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi,$$

and an application of the assertion of part 2 yields (4.2.9). Part 3 is proved. □

Lemma 4.2.10. *Under (ν) and ($\hat{\nu}$), conditions (C) and ($\tilde{C}$) are equivalent.*

Proof. By the definitions of $C_t^{\phi,\delta}$, $\tilde{C}_t^\phi$, $\tilde{C}_t$, and $h^\phi(x)$, it suffices to prove that for $t \in U, \varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi} \Big(\Big| \frac{1}{r_\phi} (\lambda \cdot h(r_\phi x))^2\, \mathbf{1}(r_\phi |x| > \delta) * \nu_t^\phi - (\lambda \cdot h(x))^2 * \nu_t \Big| > \varepsilon \Big) = 0,$$

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi} \Big(\frac{\delta}{r_\phi} \sum_{0<s\le t} |h(r_\phi x)|\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi > \varepsilon \Big) = 0,$$

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi} \Big(\Big| \frac{1}{r_\phi} \sum_{0<s\le t} \big(\lambda \cdot h(r_\phi x)\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi\big)^2 - \int_0^t \big(\lambda \cdot h(x) \bullet \hat{\nu}_s\big)^2 ds \Big| > \varepsilon \Big) = 0.$$

The limits follow by the respective parts 1, 2 and 3 of Lemma 4.2.9. □

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4.2.1 The case of small jumps

In this subsection we prove Theorems 4.2.1 and 4.2.5 for the case of jump size of order $1/r_\phi$.

Theorem 4.2.11. *Let the* X^ϕ *be semimartingales (respectively, PII). Let conditions* (0), $(\sup B)$, (C) *(or* $(\tilde{C})$*),* (ν)*, and* $(\hat{\nu})$ *(respectively,* (0), $(\sup B)_I$, $(C)_I$ *(or* $(\tilde{C})_I$*),* $(\nu)_I$*, and* $(\hat{\nu})_I$*) hold, and, in addition, for some* $a > 0$,

$$(\nu^F) \qquad \nu^\phi([0,t],\{r_\phi|x| > a\}) = 0, \quad t > 0, \phi \in \Phi.$$

Then $X^\phi \stackrel{ld}{\to} X$ *as* $\phi \in \Phi$.

Proof. We prove the theorem by checking the hypotheses of Theorem 4.1.2 in the semimartingale case, respectively, Theorem 4.1.8 in the PII case. By condition (ν^{F}) the Cramér condition (Cr) is met by the X^ϕ so that the associated stochastic exponentials $\mathcal{E}^\phi(\lambda)$ if the X^ϕ are semimartingales, respectively, the $E\exp\bigl(\lambda\cdot(X_t^\phi - X_0^\phi)\bigr)$ if the X^ϕ are PII, are well defined. For economy of notation we denote the latter expectation by $\mathcal{E}_t^\phi(\lambda)$ in the PII case as well. Since also the cumulant $G_t(\lambda)$ satisfies the conditions of Theorem 4.1.2, by Theorem 4.1.2 (respectively, Theorem 4.1.8) in order to prove Theorem 4.2.11 it is sufficient to check that as $\phi \in \Phi$

$$(\sup\ \mathcal{E}) \qquad \sup_{t\le T}\Bigl|\frac{1}{r_\phi}\ln\mathcal{E}_t^\phi(r_\phi\lambda) - G_t(\lambda)\Bigr| \stackrel{P_\phi^{1/r_\phi}}{\to} 0.$$

(In the PII case the convergence is deterministic.) Let us firstly note that by conditions (ν^F) and (ν)

$$\nu_t(|x| > a) = 0 \quad \text{(a.e.)} \tag{4.2.11}$$

We choose $h(x) = x$ for $|x| \le a$ so that by (ν^F) we have $B^\phi = B'^\phi$. In view of (4.1.14), (4.1.15) and (4.1.16) we can write

$$\mathcal{E}_t^\phi(\lambda) = \exp\Bigl(\lambda\cdot B_t^\phi + \frac{1}{2}\lambda\cdot C_t^\phi\lambda + (e^{\lambda\cdot x} - 1 - \lambda\cdot x) * \nu_t^{\phi,c}\Bigr) \prod_{s\le t} e^{-\lambda\cdot\Delta B_s^\phi}\bigl(1 + (e^{\lambda\cdot x} - 1)\bullet\nu_s^\phi\bigr). \tag{4.2.12}$$

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(In the PII case one needs to use the argument of the proofs of Jacod and Shiryaev [67, Theorems II.4.15 and II.5.2].) We show that the right-hand side of (4.2.12) is well defined. Since

$$\big(e^{\alpha|x|}-1-\alpha|x|\big)*\nu_t^{\phi,c}<\infty,\ \alpha\in\mathbb{R}_+, \tag{4.2.13}$$

all the individual terms are well defined. To show that the product is convergent, note that $\Delta B_s^\phi = x\bullet\nu_s^\phi$ so that

$$e^{-\lambda\cdot\Delta B_s^\phi}\big(1+(e^{\lambda\cdot x}-1)\bullet\nu_s^\phi\big)=1+\big(e^{\lambda\cdot(x-\Delta B_s^\phi)}-\lambda\cdot(x-\Delta B_s^\phi)-1\big)\bullet\nu_s^\phi$$
$$+\big(e^{-\lambda\cdot\Delta B_s^\phi}+\lambda\cdot\Delta B_s^\phi-1\big)\big(1-\nu^\phi(\{s\},\mathbb{R}^d)\big).$$

Thus, the expression on the left-hand side is not less than 1. Also, by the inequalities $0\le\exp(u)-1-u\le\exp(|u|)|u|^2/2$, $u\in\mathbb{R}$, and $|\Delta B_s^\phi|\le a/r_\phi$, condition (ν^F) and the choice of $h(x)$

$$\sum_{0<s\le t}\ln\Big(e^{-\lambda\cdot\Delta B_s^\phi}\big(1+(e^{\lambda\cdot x}-1)\bullet\nu_s^\phi\big)\Big)\le\frac{|\lambda|^2}{2}e^{2|\lambda|ar_\phi}$$
$$\sum_{0<s\le t}\Big(|h^\phi(x)-\Delta B_s^\phi|^2\bullet\nu_s^\phi+|\Delta B_s^\phi|^2\big(1-\nu^\phi(\{s\},\mathbb{R}^d)\big)\Big),$$

the latter sum being convergent by (4.1.3b) (respectively, by (4.2.7)).

Let us denote for $s\in\mathbb{R}_+$

$$a_s^\phi=\nu^\phi(\{s\},\mathbb{R}^d), \tag{4.2.14}$$

and for $\delta>0$, $\lambda\in\mathbb{R}^d$,

$$x_s^{\phi,\delta} = x\,\mathbf{1}(r_\phi|x|\le\delta)\bullet\nu_s^\phi, \tag{4.2.15a}$$

$$D_s^{\phi,\delta}(\lambda) = (e^{\lambda\cdot x}-1)\,\mathbf{1}(r_\phi|x|>\delta)\bullet\nu_s^\phi, \tag{4.2.15b}$$

(well defined by (ν^F)),

$$R_s^{\phi,\delta}(\lambda) = \big(\exp(\lambda\cdot(x\,\mathbf{1}(r_\phi|x|\le\delta)-x_s^{\phi,\delta}))-1 -\lambda\cdot(x\,\mathbf{1}(r_\phi|x|\le\delta)-x_s^{\phi,\delta})\big)\bullet\nu_s^\phi, \tag{4.2.15c}$$

$$Q_s^{\phi,\delta}(\lambda) = (\exp(-\lambda\cdot x_s^{\phi,\delta})-1+\lambda\cdot x_s^{\phi,\delta})(1-a_s^\phi), \tag{4.2.15d}$$

$$G_s^{\phi,\delta}(\lambda) = \exp(-\lambda\cdot x_s^{\phi,\delta})D_s^{\phi,\delta}(\lambda)+R_s^{\phi,\delta}(\lambda) \tag{4.2.15e}$$

$$+Q_s^{\phi,\delta}(\lambda), \tag{4.2.15f}$$

$$U_t^{\phi,\delta}(\lambda) = (e^{\lambda\cdot x}-1-\lambda\cdot x)\,\mathbf{1}(r_\phi|x|\le\delta)*\nu_t^{\phi,c} \tag{4.2.15g}$$

(well defined by (4.2.13)),

$$V_t^{\phi,\delta}(\lambda) = (e^{\lambda\cdot x}-1-\lambda\cdot x)\,\mathbf{1}(r_\phi|x|>\delta)*\nu_t^\phi \tag{4.2.15h}$$

(well defined by (ν^F)).

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For the sequel, we put down the following obvious relations:

$$0 \le a_s^\phi \le 1, \quad |x_s^{\phi,\delta}| \le \frac{\delta}{r_\phi} \tag{4.2.16}$$

and

$$\exp\Big(-\frac{|\lambda|a}{r_\phi}\Big)-1 \le D_s^{\phi,\delta}(\lambda) \le \exp\Big(\frac{|\lambda|a}{r_\phi}\Big)-1 \tag{4.2.17}$$

(use (ν^F) and (4.1.3a)).

Lemma 4.2.12. *The following representation holds*

$$(LS^\phi) \qquad \ln \mathcal{E}_t^\phi(\lambda) = \lambda \cdot B_t^\phi + V_t^{\phi,\delta}(\lambda) + Y_t^{\phi,\delta}(\lambda) + Z_t^{\phi,\delta}(\lambda),$$

where

$$Y_t^{\phi,\delta}(\lambda) = \sum_{0<s\le t} \big(\ln(1+D_s^{\phi,\delta}(\lambda)) - D_s^{\phi,\delta}(\lambda)\big),$$

$$Z_t^{\phi,\delta}(\lambda) = \sum_{0<s\le t} \ln(1 + G_s^{\phi,\delta}(\lambda)) + U_t^{\phi,\delta}(\lambda) + \frac{1}{2}\lambda \cdot C_t^\phi \lambda - \sum_{0<s\le t} \ln(1 + D_s^{\phi,\delta}(\lambda)).$$

Proof. The key is to observe that

$$1+(e^{\lambda\cdot x}-1)\bullet\nu_s^\phi = \exp(\lambda\cdot x_s^{\phi,\delta})(1+G_s^{\phi,\delta}(\lambda)), \tag{4.2.18}$$

which follows by routine calculations using (4.2.14)–(4.2.15e). Substituting the right-hand side into (4.2.12) and taking into account (4.1.10), (4.2.15g) and (4.2.15h) yields

$$\mathcal{E}_t^\phi(\lambda) = \exp\big(\lambda \cdot B_t^\phi + \frac{1}{2}\lambda \cdot C_t^\phi \lambda + U_t^{\phi,\delta}(\lambda) + V_t^{\phi,\delta}(\lambda) - \sum_{0<s\le t} D_s^{\phi,\delta}(\lambda)\big) \prod_{s\le t}(1 + G_s^{\phi,\delta}(\lambda)), \tag{4.2.19}$$

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which is equivalent to (LS^ϕ) provided the right-hand sides of (4.2.19) and (LS^ϕ) are well defined, i.e.,

$$\sum_{0<s\le t} |D_s^{\phi,\delta}(\lambda)| \ < \ \infty, \tag{4.2.20a}$$

$$1 + G_s^{\phi,\delta}(\lambda) > 0, \tag{4.2.20b}$$

$$\sum_{0<s\le t} |\ln(1 + G_s^{\phi,\delta}(\lambda))| \ < \ \infty, \tag{4.2.20c}$$

$$1 + D_s^{\phi,\delta}(\lambda) > 0, \tag{4.2.20d}$$

$$\sum_{0<s\le t} |\ln(1 + D_s^{\phi,\delta}(\lambda))| \ < \ \infty. \tag{4.2.20e}$$

Inequality (4.2.20a) follows by (4.2.15b), (ν^F) and the fact that by (4.1.3b) $\nu^\phi([0,t],\{|x| > \varepsilon\}) < \infty$, $\varepsilon > 0$. Inequality (4.2.20b) follows since by (4.2.18), (ν^F) and (4.2.16)

$$1 + G_s^{\phi,\delta}(\lambda) \ge \exp\bigl(-|\lambda|\delta/r_\phi\bigr)\Bigl(1 + \bigl(\exp(-|\lambda|a/r_\phi) - 1\bigr)a_s^\phi\Bigr) \\ \ge \exp\bigl(-|\lambda|(\delta + a)/r_\phi\bigr). \tag{4.2.21}$$

Next, by (4.2.15e) and (4.2.16)

$$\sum_{0<s\le t} |G_s^{\phi,\delta}(\lambda)| \le e^{|\lambda|\delta/r_\phi} \sum_{0<s\le t} |D_s^{\phi,\delta}(\lambda)| \\ + \sum_{0<s\le t} (R_s^{\phi,\delta}(\lambda) + Q_s^{\phi,\delta}(\lambda)) \tag{4.2.22}$$

(note that $R_s^{\phi,\delta}(\lambda) \ \ge \ 0$ and $Q_s^{\phi,\delta}(\lambda) \ \ge \ 0$). By (4.2.15c) and (4.2.15d), using (4.2.16) and the inequality $\exp(u) - 1 - u \le (|u|^2/2)\exp(|u|), u \in \mathbb{R}$, we have

$$\sum_{0<s\le t} (R_s^{\phi,\delta}(\lambda) + Q_s^{\phi,\delta}(\lambda)) \\ \le \frac{e^{2|\lambda|\delta/r_\phi}}{2} \sum_{0<s\le t} \Bigl(\bigl(\lambda\cdot(x\,\mathbf{1}(r_\phi|x| \le \delta) - x_s^{\phi,\delta})\bigr)^2 \bullet \nu_s^\phi \\ + \bigl(\lambda\cdot x_s^{\phi,\delta}\bigr)^2(1 - a_s^\phi)\Bigr),$$

which is finite by (4.1.3b). In view of (4.2.20a) we thus have that $\sum_{0<s\le t} |G_s^{\phi,\delta}(\lambda)| \ < \ \infty$, which implies (4.2.20c) by (4.2.21),

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(4.2.15e), (4.2.17), (4.2.16), and the fact that $R_s^{\phi,\delta}(\lambda)$ and $Q_s^{\phi,\delta}(\lambda)$ are non-negative.

Inequality (4.2.20d) follows by (4.2.15b). Finally, inequality (4.2.20e) follows by (4.2.20a) and the left-hand side of (4.2.17). □

Now we give a similar representation for $G_t(\lambda)$. Let

$$V_t(\lambda) = (e^{\lambda\cdot x} - 1 - \lambda \cdot x) * \nu_t, \quad (4.2.23)$$

$$Y_t(\lambda) = \int_0^t \Big(\ln\big(1 + (e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s\big) - (e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s\Big) ds, \quad (4.2.24)$$

$$Z_t(\lambda) = \frac{1}{2}\lambda \cdot C_t\lambda. \quad (4.2.25)$$

By (4.2.11) and the choice of $h(x)$ (recall we take $h(x) = x, |x| \le a$)

$$(LS) \qquad G_t(\lambda) = \lambda \cdot B_t + V_t(\lambda) + Y_t(\lambda) + Z_t(\lambda).$$

Decompositions (LS^ϕ) and (LS) show that (sup $\mathcal{E}$) would follow if for every $T > 0$ and $\varepsilon > 0$

$$\alpha) \qquad \sup_{t\le T} |B_t^\phi - B_t| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi,$$

$$\beta) \qquad \lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T} |\frac{1}{r_\phi} V_t^{\phi,\delta}(r_\phi\lambda) - V_t(\lambda)| > \varepsilon\Big) = 0,$$

$$\gamma) \qquad \lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T} |\frac{1}{r_\phi} Y_t^{\phi,\delta}(r_\phi\lambda) - Y_t(\lambda)| > \varepsilon\Big) = 0,$$

$$\delta) \qquad \lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T} |\frac{1}{r_\phi} Z_t^{\phi,\delta}(r_\phi\lambda) - Z_t(\lambda)| > \varepsilon\Big) = 0.$$

Part α) is just condition (sup B).

By (4.2.15h), (4.2.23), (ν^F), and (4.2.11), part 1 of Lemma 4.2.9 yields

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(|\frac{1}{r_\phi} V_t^{\phi,\delta}(r_\phi\lambda) - V_t(\lambda)| > \varepsilon\Big) = 0, \quad t \in U, \varepsilon > 0,$$

and an application of Lemma 4.2.6 proves part β).

We now prove part γ). Let

$$\psi(x) = x - \ln(1+x), \; x > -1. \quad (4.2.26)$$

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By the definitions of $Y_t^{\phi,\delta}(\lambda)$ and $Y_t(\lambda)$ (see Lemma 4.2.12 and (4.2.24))

$$Y_t^{\phi,\delta}(\lambda) = -\sum_{0<s\le t}\psi(D_s^{\phi,\delta}(\lambda)), \tag{4.2.27}$$

$$Y_t(\lambda) = -\int_0^t \psi\big((e^{\lambda\cdot x}-1)\bullet\hat{\nu}_s\big)ds. \tag{4.2.28}$$

Since $\psi(x)>0$, an application of Lemma 4.2.6 implies that γ) would follow by

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(|\frac{1}{r_\phi}Y_t^{\phi,\delta}(r_\phi\lambda)-Y_t(\lambda)|>\varepsilon\Big)=0,\quad t\in U,\varepsilon>0. \tag{4.2.29}$$

Let $u=\exp(-|\lambda|a)-1$ and $v=\exp(|\lambda|a)-1$. Since the function $\psi(x)/x^2$ is continuous on $[u,v]$, by Weierstrass' theorem it can uniformly be approximated on $[u,v]$ by polynomials, so, given arbitrary $\sigma>0$, there exists a polynomial $q_\sigma(x)$ with powers not less than 2 such that $|\psi(x)-q_\sigma(x)|<\sigma x^2$, $x\in[u,v]$.

Now, by (4.2.17) $D_s^{\phi,\delta}(r_\phi\lambda)\in[u,v]$, and by (4.2.11) and (4.2.4) $(e^{\lambda\cdot x}-1)\bullet\hat{\nu}_s\in[u,v]$ (a.e.) Thus, recalling (4.2.27) and (4.2.28),

$$\begin{aligned}
&P_\phi^{1/r_\phi}\Big(|\frac{1}{r_\phi}Y_t^{\phi,\delta}(r_\phi\lambda)-Y_t(\lambda)|>\varepsilon\Big)\\
&\le P_\phi^{1/r_\phi}\Big(\Big|\frac{1}{r_\phi}\sum_{0<s\le t}q_\sigma\big(D_s^{\phi,\delta}(r_\phi\lambda)\big)\\
&-\int_0^t q_\sigma\big((e^{\lambda\cdot x}-1)\bullet\hat{\nu}_s\big)\,ds\Big|>\frac{\varepsilon}{3}\Big)\\
&+P_\phi^{1/r_\phi}\Big(\sigma\frac{1}{r_\phi}\sum_{0<s\le t}D_s^{\phi,\delta}(r_\phi\lambda)^2>\frac{\varepsilon}{3}\Big)\\
&\qquad\qquad+\mathbf{1}\Big(\sigma\int_0^t\big((e^{\lambda\cdot x}-1)\bullet\hat{\nu}_s\big)^2ds>\frac{\varepsilon}{3}\Big),
\end{aligned}$$

and since σ can be taken arbitrarily small and the smallest power in

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q_σ is not less than 2, (4.2.29) is implied by

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\Big|\frac{1}{r_\phi}\sum_{0<s\le t} D_s^{\phi,\delta}(r_\phi\lambda)^k - \int_0^t \big((e^{\lambda\cdot x}-1)\bullet\hat\nu_s\big)^k\,ds\Big| > \eta\Big) = 0,$$

$$\eta>0, t\in U, k=2,3,\ldots; \quad (4.2.30a)$$

$$\lim_{A\to\infty}\limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}\sum_{0<s\le t} D_s^{\phi,\delta}(r_\phi\lambda)^2 > A\Big) = 0. \quad (4.2.30b)$$

Limit (4.2.30a) follows by part 3 of Lemma 4.2.9 in view of (4.2.15b), (ν^F) and (4.2.11). The $\limsup_{\phi\in\Phi}$ in (4.2.30b) being by $(\hat\nu)$ not greater than $\mathbf{1}\big(\int_0^t(|e^{\lambda\cdot x}-1|\bullet\hat\nu_s)^2\,ds > A/2\big)$ for all $\delta>0$, equals 0 for all large A. Limit (4.2.29) is proved. Part γ) is proved.

We prove part δ). Let us denote

$$L_s^{\phi,\delta}(\lambda) = \frac{1}{2}\big(\lambda\cdot(x\,\mathbf{1}(r_\phi|x|\le\delta) - x_s^{\phi,\delta})\big)^2\bullet\nu_s^\phi, \quad (4.2.31a)$$

$$K_s^{\phi,\delta}(\lambda) = \frac{1}{2}(\lambda\cdot x_s^{\phi,\delta})^2(1-a_s^\phi), \quad (4.2.31b)$$

$$H_s^{\phi,\delta}(\lambda) = L_s^{\phi,\delta}(\lambda) + K_s^{\phi,\delta}(\lambda), \quad (4.2.31c)$$

$$W_t^{\phi,\delta}(\lambda) = \frac{1}{2}(\lambda\cdot x)^2\ \mathbf{1}(r_\phi|x|\le\delta) * \nu_t^{\phi,c}. \quad (4.2.31d)$$

Then by (4.1.7), (4.2.15a) and (4.2.14)

$$\frac{1}{2}\lambda\cdot C_t^{\phi,\delta}\lambda = \frac{1}{2}\lambda\cdot C_t^\phi\lambda + W_t^{\phi,\delta}(\lambda) + \sum_{0<s\le t} H_s^{\phi,\delta}(\lambda). \quad (4.2.32)$$

Hence, in view of the definitions of $Z_t^{\phi,\delta}(\lambda)$ (see Lemma 4.2.12) and $Z_t(\lambda)$ (see (4.2.25)), and the fact that by Corollary 4.2.7 condition (sup C) holds, δ) would follow from

δ') $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T}\Big|\frac{1}{r_\phi}(U_t^{\phi,\delta}(r_\phi\lambda) - W_t^{\phi,\delta}(r_\phi\lambda))\Big| > \varepsilon\Big) = 0,$$

δ'') $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}\sum_{0<s\le t}\Big|\ln(1+G_s^{\phi,\delta}(r_\phi\lambda)) - \big(H_s^{\phi,\delta}(r_\phi\lambda) + \ln(1+D_s^{\phi,\delta}(r_\phi\lambda))\big)\Big| > \varepsilon\Big) = 0.$$

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Let us note that condition (C) implies that

$$\lim_{A\to\infty}\limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(r_\phi\lambda\cdot C_t^{\phi,\delta}\lambda > A\big) = 0. \tag{4.2.33}$$

For limit $\delta')$, we note that by the inequality $|e^u - 1 - u - u^2/2| \leq (e^{|u|}/6)|u|^3$, (4.2.15g) and (4.2.31d)

$$\sup_{t\leq T}|U_t^{\phi,\delta}(r_\phi\lambda) - W_t^{\phi,\delta}(r_\phi\lambda)| \leq \frac{e^{|\lambda|\delta}}{3}|\lambda|\delta\, W_T^{\phi,\delta}(r_\phi\lambda). \tag{4.2.34}$$

Next, since $H_s^{\phi,\delta}(\lambda)$ and $\lambda\cdot C_t^\phi\lambda$ are non-negative, by (4.2.32) $W_T^{\phi,\delta}(\lambda) \leq \lambda\cdot C_t^{\phi,\delta}\lambda/2$, so that by (4.2.33)

$$\lim_{A\to\infty}\limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}W_T^{\phi,\delta}(r_\phi\lambda) > A\Big) = 0,$$

which together with (4.2.34) implies $\delta')$.

We prove $\delta'')$. Let us first note that for $t\in\mathbb{R}_+, \varepsilon > 0$ and $\lambda\in\mathbb{R}^d$ by part 2 of Lemma 4.2.9, (ν^F) and (4.2.15b)

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{\delta}{r_\phi}\sum_{0<s\leq t}|D_s^{\phi,\delta}(r_\phi\lambda)| > \varepsilon\Big) = 0. \tag{4.2.35}$$

Next, (4.2.15e) implies by Taylor's formula

$$\ln(1+G_s^{\phi,\delta}(r_\phi\lambda)) = \ln(1+D_s^{\phi,\delta}(r_\phi\lambda)) + \frac{T_s^\phi}{F_s^\phi},$$

where

$$\begin{aligned} T_s^\phi &= (\exp(-r_\phi\lambda\cdot x_s^{\phi,\delta}) - 1)D_s^{\phi,\delta}(r_\phi\lambda)\\ &\quad + R_s^{\phi,\delta}(r_\phi\lambda) + Q_s^{\phi,\delta}(r_\phi\lambda), \qquad (4.2.36)\\ F_s^\phi &= 1 + D_s^{\phi,\delta}(r_\phi\lambda) + \theta T_s^\phi,\\ &\quad 0\leq\theta\leq 1, \qquad (4.2.37)\end{aligned}$$

and thus

$$\frac{1}{r_\phi}\sum_{0<s\leq t}\Big|\ln(1+G_s^{\phi,\delta}(r_\phi\lambda)) - \big(H_s^{\phi,\delta}(r_\phi\lambda) + \ln(1+D_s^{\phi,\delta}(r_\phi\lambda))\big)\Big| \leq A_1 + A_2 + A_3,$$

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where

$$\begin{aligned} A_1 &= \frac{1}{r_\phi}\sum_{0<s\le t}\frac{|D_s^{\phi,\delta}(r_\phi\lambda)|}{F_s^\phi}\big[|\exp(-r_\phi\lambda\cdot x_s^{\phi,\delta})-1| \\ &\quad +H_s^{\phi,\delta}(r_\phi\lambda)\big], \qquad (4.2.38a)\\ A_2 &= \frac{1}{r_\phi}\sum_{0<s\le t}\frac{|T_s^\phi|}{F_s^\phi}H_s^{\phi,\delta}(r_\phi\lambda), \qquad (4.2.38b)\\ A_3 &= \frac{1}{r_\phi}\sum_{0<s\le t}\frac{1}{F_s^\phi}(|R_s^{\phi,\delta}(r_\phi\lambda)-L_s^{\phi,\delta}(r_\phi\lambda)| \\ &\quad +|Q_s^{\phi,\delta}(r_\phi\lambda)-K_s^{\phi,\delta}(r_\phi\lambda)|) \qquad (4.2.38c)\end{aligned}$$

(for the latter equality recall (4.2.31c)). We thus prove $\delta'')$ by proving that

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(A_i>\varepsilon\Big)=0,\quad \varepsilon>0, i=1,2,3. \qquad (4.2.39)$$

We begin with some estimates. Since by (4.2.16)

$$|\exp(-r_\phi\lambda\cdot x_s^{\phi,\delta})-1|\le|\lambda|\delta e^{|\lambda|\delta}, \qquad (4.2.40)$$

we have, using (4.2.17),

$$|\exp(-r_\phi\lambda\cdot x_s^{\phi,\delta})-1||D_s^{\phi,\delta}(r_\phi\lambda)|\le|\lambda|\delta e^{|\lambda|\delta}e^{|\lambda|a}. \qquad (4.2.41)$$

From (4.2.15c) and (4.2.15d) using (4.2.16) and the inequality $\exp(u)-1-u\le(|u|^2/2)\exp(|u|)$, we have

$$|R_s^{\phi,\delta}(r_\phi\lambda)|\le 2|\lambda|^2\delta^2e^{2|\lambda|\delta},\quad |Q_s^{\phi,\delta}(r_\phi\lambda)|\le\frac{1}{2}|\lambda|^2\delta^2e^{|\lambda|\delta},$$

whereafter in view of (4.2.36) and (4.2.41)

$$|T_s^\phi|\le 4|\lambda|\delta e^{2|\lambda|a}\quad\text{provided } |\lambda|\delta\le 1, \delta\le a. \qquad (4.2.42)$$

Further, the left inequality in (4.2.17) and (4.2.37) yield, in view of (4.2.42),

$$F_s^\phi\ge\frac{1}{2}e^{-|\lambda|a}\quad\text{provided } \delta|\lambda|\le\frac{1}{8}e^{-3|\lambda|a}, \delta\le a. \qquad (4.2.43)$$

Besides, (4.2.31a)–(4.2.31c) and (4.2.16) imply

$$H_s^{\phi,\delta}(r_\phi\lambda)\le 3|\lambda|^2\delta^2. \qquad (4.2.44)$$

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Now, (4.2.39) for $i = 1$ follows by (4.2.38a), (4.2.35), (4.2.43), (4.2.40), and (4.2.44).

Next, since by (4.2.32)

$$\frac{1}{r_\phi}\sum_{0<s\le t} H_s^{\phi,\delta}(r_\phi\lambda) \le \frac{1}{2} r_\phi\lambda\cdot C_t^{\phi,\delta}\lambda, \tag{4.2.45}$$

by (4.2.42), (4.2.43) and (4.2.38b) for δ small enough

$$A_2 \le 4|\lambda|\delta e^{3|\lambda|a} r_\phi\lambda\cdot C_t^{\phi,\delta}\lambda,$$

and (4.2.39) for $i = 2$ follows by (4.2.33).

Finally, let $i = 3$. From (4.2.15c), (4.2.31a), (4.2.15d), and (4.2.31b) analogously to (4.2.34)

$$\begin{aligned} |R_s^{\phi,\delta}(r_\phi\lambda) - L_s^{\phi,\delta}(r_\phi\lambda)| &\le \frac{2\delta|\lambda|}{3} e^{2\delta|\lambda|} L_s^{\phi,\delta}(r_\phi\lambda), \\ |Q_s^{\phi,\delta}(r_\phi\lambda) - K_s^{\phi,\delta}(r_\phi\lambda)| &\le \frac{\delta|\lambda|}{3} e^{\delta|\lambda|} K_s^{\phi,\delta}(r_\phi\lambda), \end{aligned}$$

and hence in view of (4.2.38c), (4.2.43) and (4.2.31c) for δ small enough

$$A_3 \le 2\delta|\lambda| e^{2\delta|\lambda|} e^{|\lambda|a} \frac{1}{r_\phi}\sum_{0<s\le t} H_s^{\phi,\delta}(r_\phi\lambda),$$

so that (4.2.45) and (4.2.33) yield (4.2.39) for $i = 3$. Part δ'') is proved. Limit (sup $\mathcal{E}$) and with it Theorem 4.2.11 are proved. □

4.2.2 The general case

In this subsection we prove Theorems 4.2.1 and 4.2.5. The proof relies heavily on the theory of weak convergence for deviabilities. Since by Lemma 4.2.10 conditions (C) and $(\tilde{C})$ are equivalent under conditions (ν) and $(\hat{\nu})$, we assume that conditions (0), $(A)+(a)$, (sup B), (C), (ν), and $(\hat{\nu})$ hold. The proof below also applies to the case where condition (a') is assumed instead of condition (a). The X^ϕ are either semimartingales or PII.

For $a > 0$, we define the limiters

$$h_a(x) = \Big(\frac{a}{|x|}\wedge 1\Big)x,\ h_a^\phi(x) = \frac{1}{r_\phi} h_a(r_\phi x),\ \ x \in \mathbb{R}^d, \tag{4.2.46}$$

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and introduce processes $\breve{X}^{\phi,a} = (\breve{X}_t^{\phi,a}, t \in \mathbb{R}_+)$ and $\hat{X}^{\phi,a} = (\hat{X}_t^{\phi,a}, t \in \mathbb{R}_+)$ by

$$\breve{X}_t^{\phi,a} = \sum_{0<s\le t} (\Delta X_s^\phi - h_a^\phi(\Delta X_s^\phi)), \tag{4.2.47}$$

$$\hat{X}_t^{\phi,a} = X_t^\phi - \breve{X}_t^{\phi,a}. \tag{4.2.48}$$

Let P_ϕ^a denote the distribution of $\hat{X}^{\phi,a}$ and let $\Pi = \mathbf{\Pi}_{x_0}$, which is the idempotent distribution of X. Let $(B^{\phi,a}, C^\phi, \nu^\phi)$ be the triplet of X^ϕ corresponding to h_a. Since the jumps of $\hat{X}^{\phi,a}$ are $h_a^\phi(\Delta X_s^\phi)$, the triplet of $\hat{X}^{\phi,a}$ corresponding to h_a is $(B^{\phi,a}, C^\phi, \nu^{\phi,a})$, where

$$\nu^{\phi,a}([0,t],\Gamma) = \nu^\phi([0,t],(h_a^\phi)^{-1}(\Gamma)), \;\; t \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d). \tag{4.2.49}$$

The semimartingales (respectively, PII) $\hat{X}^{\phi,a}$ will LD converge in distribution to a certain semimaxingale (respectively, a semimaxingale with independent increments) X^a. We define the latter as having characteristics $(B^a, C, \nu^a, \hat{\nu}^a)$ relative to h_a, which are defined in analogy with the characteristics of $\hat{X}^{\phi,a}$ in that B^a is the first characteristic of X associated with $h_a(x)$,

$$\nu^a([0,t],\Gamma) = \nu([0,t],h_a^{-1}(\Gamma)), \; \hat{\nu}_t^a(\Gamma) = \hat{\nu}_t(h_a^{-1}(\Gamma)), \; \Gamma \in \mathcal{B}(\mathbb{R}^d). \tag{4.2.50}$$

Corollary 2.8.7 implies that X^a exists and is a Luzin-continuous idempotent process. We denote its idempotent distribution by Π^a and the associated cumulant by $G^a(\lambda) = (G_t^a(\lambda), t \in \mathbb{R}_+)$, $\lambda \in \mathbb{R}^d$.

The proof of Theorem 4.2.1 (respectively, Theorem 4.2.5) consists in proving that

(*i*) $P_\phi^a \stackrel{ld}{\to} \Pi^a$ as $\phi \in \Phi$, $a > 0$,

(*ii*) $\lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{s\le t} |\breve{X}_s^{\phi,a}| > \varepsilon\Big) = 0, \;\; t > 0, \varepsilon > 0,$

(*iii*) $\Pi^a \stackrel{iw}{\to} \Pi$ as $a \to \infty$.

Since part *(ii)* implies that

$$\lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\rho_S(X^\phi, \hat{X}^{\phi,a}) > \varepsilon\Big) = 0, \; \varepsilon > 0,$$

by Lemma 3.1.37 *(i)* through *(iii)* would yield $P_\phi \stackrel{ld}{\to} \Pi$ as required.

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Lemma 4.2.13. *Part* (i) *holds.*

Proof. We show that the $\hat{X}^{\phi,a}$ and X^a satisfy conditions (0), (sup B), (C), (ν), and $(\hat{\nu})$. Condition (0) holds by the hypotheses of Theorem 4.2.1. Since the first characteristics of the $\hat{X}^{\phi,a}$ and X^a associated with h_a coincide with the respective first characteristics of X^ϕ and X, condition (sup B) for $\hat{X}^{\phi,a}$ and X^a is identical to condition (sup B) for X^ϕ and X.

We now check that conditions (C) are identical. By (4.2.49) and since $(h_a^\phi)^{-1}(x) = \{x\}$ if $r_\phi|x| < a$, for $\delta < a$

$$\nu^{\phi,a}\big([0,t], \Gamma \cap \{r_\phi|x| \le \delta\}\big) = \nu^\phi\big([0,t], \Gamma \cap \{r_\phi|x| \le \delta\}\big), \qquad \Gamma \in \mathcal{B}(\mathbb{R}^d),$$

and hence (4.1.7) (respectively, (4.2.5)) yields $C_t^{\phi,a,\delta} = C_t^{\phi,\delta}$ for $\delta < a$ (with obvious notation). The claim follows.

The fact that (ν) for the $\hat{X}^{\phi,a}$ and X^a is implied by (ν) for X^ϕ and X follows by the equalities

$$\frac{1}{r_\phi} f(r_\phi x) * \nu_t^{\phi,a} = \frac{1}{r_\phi} f(h_a(r_\phi x)) * \nu_t^\phi, \qquad f(x) * \nu_t^a = f(h_a(x)) * \nu_t,$$

and the inclusion $f \circ h_a \in \mathcal{C}_b$ if $f \in \mathcal{C}_b$. Condition $(\hat{\nu})$ for the $\hat{X}^{\phi,a}$ and X^a is checked similarly. Since also $\hat{X}^{\phi,a}$ satisfies (ν^F) by (4.2.49) and (4.2.46), Theorem 4.2.11 yields the assertion of the lemma. □

Remark 4.2.14. *Note that it is here, while checking* (C), *that we used the property that the limiters in* (4.2.46), *by contrast with truncation functions, do not vanish at infinity.*

Now we proceed with a proof of *(ii).*

Lemma 4.2.15. *If* $f(x), x \in \mathbb{R}^d$, *is an* $\mathbb{R}_+$*-valued bounded Borel function equal to* 0 *in a neighbourhood of the origin, then for all* $\alpha > 0$ *and* $\beta > 0$

$$\begin{aligned} &P_\phi\big(f(x) * \mu_t^\phi > \alpha\big) \\ &\le e^{\beta-\alpha} + P_\phi\Big((e^{f(x)}-1) * \nu_t^{\phi,c} + \sum_{0<s\le t} \ln\big(1+(e^{f(x)}-1)\bullet\nu_s^\phi\big) > \beta\Big) \\ &\le e^{\beta-\alpha} + P_\phi\big((e^{f(x)}-1) * \nu_t^\phi > \beta\big). \end{aligned}$$

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Proof. Let $Y_t^\phi = f(x) * \mu_t^\phi$. Then $Y^\phi = (Y_t^\phi,\, t \in \mathbb{R}_+)$ has bounded variation over bounded intervals, and is, therefore, a semimartingale. The associated stochastic exponential $\mathcal{E}^{\phi,Y}(\mu) = (\mathcal{E}_t^{\phi,Y}(\mu), t \in \mathbb{R}_+), \mu \in \mathbb{R}$, is of the form

$$\mathcal{E}_t^{\phi,Y}(\mu) = e^{G_t^{\phi,Y}(\mu)} \prod_{s \le t} (1+\Delta G_s^{\phi,Y}(\mu)) e^{-\Delta G_s^{\phi,Y}(\mu)},$$

where $G_t^{\phi,Y}(\mu) = (e^{\mu f(x)} - 1) * \nu_t^\phi$. Lemma 4.1.1 implies that the process $\big(\exp(\mu Y_t^\phi)/\mathcal{E}_t^{\phi,Y}(\mu), t \in \mathbb{R}_+\big)$ is a supermartingale relative to $\mathbf{F}_\phi$; hence, $E_\phi\big(\exp(\mu Y_\tau^\phi)/\mathcal{E}_\tau^{\phi,Y}(\mu)\big) \le 1$ for every finite $\mathbf{F}_\phi$-stopping time τ. Since

$$\ln \mathcal{E}_t^{\phi,Y}(\mu) = (e^{\mu f(x)} - 1) * \nu_t^{\phi,c} + \sum_{0<s\le t} \ln\big(1 + (e^{\mu f(x)} - 1) \bullet \nu_s^\phi\big),$$

an application of Lemma 3.2.6 yields the left inequality. The right inequality follows since $\ln(1+x) \le x$. □

The next lemma proves part (*ii*).

Lemma 4.2.16. *Both under conditions* $(A) + (a)$ *and* $(A) + (a')$ *for every* $\varepsilon > 0$

$$\lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{s\le t} |\breve{X}_s^{\phi,a}| > \varepsilon\Big) = 0, \quad t \in \mathbb{R}_+.$$

Proof. Since by (4.2.46) and (4.2.47)

$$\sup_{s\le t} |\breve{X}_s^{\phi,a}| \le \sum_{0<s\le t} |\Delta X_s^\phi|\, \mathbf{1}(r_\phi |\Delta X_s^\phi| > a),$$

we have, for $A > 0, \varepsilon > 0$,

$$P_\phi\big(\sup_{s\le t} |\breve{X}_s^{\phi,a}| > \varepsilon\big) \le P_\phi\big(\sup_{s\le t} |\Delta X_s^\phi| > A\big) + P_\phi\Big(\sum_{0<s\le t} |\Delta X_s^\phi|\, \mathbf{1}(r_\phi |\Delta X_s^\phi| > a)\, \mathbf{1}(|\Delta X_s^\phi| \le A) > \varepsilon\Big). \tag{4.2.51}$$

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By the Lenglart-Rebolledo inequality, see, e.g., Liptser and Shiryaev [79, Theorem 1.9.3], for $\alpha > 0$

$$P_\phi\big(\sup_{s\le t}|\Delta X_s^\phi| > A\big) \le P_\phi\big(\sum_{0<s\le t}\mathbf{1}(|\Delta X_s^\phi| > A) \ge 1\big)$$
$$\le e^{-r_\phi \alpha} + P_\phi\big(\nu^\phi([0,t],\{|x|>A\}) > e^{-r_\phi\alpha}\big),$$

and, hence, by (A)

$$\limsup_{A\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{s\le t}|\Delta X_s^\phi| > A\Big) \le e^{-\alpha} \to 0 \text{ as } \alpha\to\infty. \tag{4.2.52}$$

For the second term on the right-hand side of (4.2.51), we have by Lemma 4.2.15 for $\alpha > 0$ and $\beta > 0$

$$P_\phi\Big(\sum_{0<s\le t}|\Delta X_s^\phi|\,\mathbf{1}(r_\phi|\Delta X_s^\phi| > a)\,\mathbf{1}(|\Delta X_s^\phi| \le A) > \varepsilon\Big)$$
$$\le \exp\big(r_\phi(\beta - \alpha\varepsilon)\big)$$
$$+ P_\phi\Big(\frac{1}{r_\phi} j^\phi_{a,A,\alpha}(x) * \nu_t^{\phi,c} + \frac{1}{r_\phi}\sum_{0<s\le t}\ln\big(1 + j^\phi_{a,A,\alpha}(x)\bullet\nu_s^\phi\big) > \beta\Big),$$

which yields under condition (a')

$$\lim_{a\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sum_{0<s\le t}|\Delta X_s^\phi|\,\mathbf{1}(r_\phi|\Delta X_s^\phi| > a)$$
$$\mathbf{1}(|\Delta X_s^\phi| \le A) > \varepsilon\Big) = 0. \tag{4.2.53}$$

Limits (4.2.52) and (4.2.53) in view of (4.2.51) prove the claim under conditions $(A) + (a')$. Since (a) is stronger than (a'), the required also holds under $(A) + (a)$. □

It is left to prove (iii). We use the method of finite-dimensional distributions for idempotent processes, so we prove that finite-dimensional idempotent distributions of the X^a converge to finite dimensional idempotent distributions of X as $a \to \infty$ and that the net $\{\mathcal{L}_i(X^a),\, a \in \mathbb{R}_+\}$ is tight. We begin the proof of the convergence of finite-dimensional distributions by checking that $G_t^a(\lambda) \to G_t(\lambda)$ as $a \to \infty$.

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Lemma 4.2.17. *For $t \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}^d$, as $a \to \infty$,*

$$\sup_{s \le t} |G_s^a(\lambda) - G_s(\lambda)| \to 0.$$

Proof. Let as in (4.2.26) $\psi(x) = x - \ln(1+x)$, $x > -1$. By (4.2.50) and (2.7.61)

$$G_t^a(\lambda) = \lambda \cdot B_t^a + \frac{1}{2}\lambda \cdot C_t\lambda + (e^{\lambda\cdot x} - 1 - \lambda \cdot h_a(x)) * \nu_t^a - \int_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s^a\big)\, ds = \lambda \cdot B_t^a + \frac{1}{2}\lambda \cdot C_t\lambda + (e^{\lambda\cdot h_a(x)} - 1 - \lambda \cdot h_a(x)) * \nu_t - \int_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s^a\big)\, ds.$$

Also by (2.7.61)

$$G_t(\lambda) = \lambda \cdot B_t^a + \frac{1}{2}\lambda \cdot C_t\lambda + (e^{\lambda\cdot x} - 1 - \lambda \cdot h_a(x)) * \nu_t - \int_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s\big)\, ds.$$

Thus,

$$|G_t^a(\lambda) - G_t(\lambda)| \le e^{|\lambda||x|}\, \mathbf{1}(|x| > a) * \nu_t + \Big| \int_0^t \psi\big((e^{\lambda\cdot h_a(x)} - 1) \bullet \hat{\nu}_s\big)\, ds - \int_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s\big)\, ds \Big|. \tag{4.2.54}$$

The right inequality in (4.2.3) implies that the first term on the right-hand side of (4.2.54) tends to 0 as $a \to \infty$. By the fact that $\psi(x)$ is positive, in order to prove that the second term tends to 0 uniformly over bounded intervals by Polya's theorem it is sufficient to check convergence to 0 for every $t \in \mathbb{R}_+$. Since $h_a(x) \to h(x)$ as $a \to \infty$, by Lebesgue's dominated convergence theorem, (4.2.3) and (4.2.4) $\big(e^{\lambda\cdot h_a(x)} - 1\big) \bullet \hat{\nu}_s \to \big(e^{\lambda\cdot x} - 1\big) \bullet \hat{\nu}_s$. Therefore, the required convergence

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would follow by Lebesgue's dominated convergence theorem, (4.2.3) and (4.2.4) provided

$$\lim_{a\to\infty} \sup_{s\le t} e^{\lambda\cdot x}\, \mathbf{1}(|x|>a) \bullet \hat{\nu}_s = 0,$$

$$\liminf_{a\to\infty} \Big(1 + \inf_{s\le t}\big(e^{\lambda\cdot h_a(x)} - 1\big) \bullet \hat{\nu}_s\Big) > 0,$$

which hold by condition (L_1). □

We now prove (iii).

Lemma 4.2.18. $\Pi^a \overset{iw}{\to} \Pi$ *as* $a \to \infty$.

Proof. Since the Π^a and Π are supported by $\mathbb{C}$ and the topology on $\mathbb{C}$ coincides with the one induced by the Skorohod topology, by Corollary 1.9.7 and Remark 1.9.8 we may apply the method of finite dimensional distributions of Theorem 2.2.27. Since both the X^a and X are Luzin idempotent processes with independent increments, by Lemma 1.10.8 weak convergence of finite-dimensional idempotent distributions would follow from weak convergence of one-dimensional idempotent distributions. The latter follows by Lemma 1.11.19 and Lemma 4.2.17 if we recall that $G_t(\lambda)$ is differentiable in λ.

We check tightness of the net $\{\Pi^a, a \in \mathbb{R}_+\}$ by verifying the conditions of Theorem 2.2.26. Condition 1° is obvious since $\Pi^a(\mathbf{x}) = 0$ if $\mathbf{x}_0 \ne x_0$. Let us consider condition 2°. We denote, for $\delta > 0, T > 0$ and $\eta > 0$,

$$A^T_{\delta,\eta} = \{\mathbf{x} \in \mathbb{C} : \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} |\mathbf{x}_t - \mathbf{x}_s| > \eta\}$$

Let $e_i, 1 \le i \le 2d$, denote the vector, whose $\lfloor (i+1)/2 \rfloor$th entry equals 1 if i is odd and -1 if i is even, the rest of the entries being equal to 0. Denoting by I^a the rate function associated with Π^a, we have for $\alpha > 0, s < t$, that if $e_i \cdot (\mathbf{x}_t - \mathbf{x}_s) > \gamma > 0$, then $I^a(\mathbf{x}) \ge \alpha\gamma - (G^a_t(\alpha e_i) - G^a_s(\alpha e_i))$. Therefore,

$$A^T_{\delta,\eta} \subset \big\{\mathbf{x} \in \mathbb{C} : \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} \max_{i=1,\dots,2d} e_i \cdot (\mathbf{x}_t - \mathbf{x}_s) > \frac{\eta}{d}\big\}$$

$$\subset \bigcup_{i=1}^{2d} \Big\{\mathbf{x} \in \mathbb{C} : I^a(\mathbf{x}) \ge \alpha\frac{\eta}{d} - \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} (G^a_t(\alpha e_i) - G^a_s(\alpha e_i))\Big\},$$

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and hence

$$
\begin{aligned}
\Pi^a(A^T_{\delta,\eta}) &= \sup_{\mathbf{x}\in A^T_{\delta,\eta}} \exp\bigl(-I^a(\mathbf{x})\bigr) \\
&\le \max_{i=1,\dots,2d} \exp\Bigl(-\alpha\frac{\eta}{d} + \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} \bigl|G^a_t(\alpha e_i) - G^a_s(\alpha e_i)\bigr|\Bigr) \\
&\le e^{-\alpha\eta/d} \max_{i=1,\dots,2d} \exp\bigl(\sup_{t\in[0,T]} \bigl|G^a_t(\alpha e_i) - G_t(\alpha e_i)\bigr| \\
&\qquad + \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} \bigl|G_t(\alpha e_i) - G_s(\alpha e_i)\bigr|\bigr).
\end{aligned}
$$

By Lemma 4.2.17 and continuity of $G(\lambda)$ we conclude that

$$
\limsup_{\delta\to 0}\limsup_{a\to\infty} \Pi^a(A^T_{\delta,\eta}) \le e^{-\alpha\eta/d} \to 0 \text{ as } \alpha\to\infty.
$$

Tightness of $\{\Pi^a\}$ is proved. Part (iii) is proved. □

Thus all the assertions *(i)*, *(ii)* and *(iii)* are proved and by Lemma 3.1.37 $P^\phi \overset{ld}{\to} \Pi$ as $\phi\in\Phi$. Theorems 4.2.1 and 4.2.5 have been proved.

4.3 Corollaries

In this section we discuss conditions and implications of Theorem 4.2.1. The PII case of Theorem 4.2.5 can be considered similarly. Thus, the X^ϕ are semimartingales in what follows.

We start with "integrable" versions when the convergence conditions can be checked for nontruncated characteristics. Let us recall that the nontruncated modified second characteristic $\tilde{C}'$ of X is defined by (see (2.7.59))

$$
\lambda\cdot\tilde{C}'_t\lambda = \lambda\cdot C_t\lambda + (\lambda\cdot x)^2 * \nu_t - \int_0^t (\lambda\cdot x\bullet\hat{\nu}_s)^2 ds.
$$

If the X^ϕ are special semimartingales, we introduce the conditions

$$
(\sup B') \qquad \sup_{t\le T} |B'^{\phi}_t - B'_t| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ T>0,
$$

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and

$$(I_1) \qquad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(|x|\,\mathbf{1}(r_\phi|x|>a)*\nu_t^\phi>\varepsilon\Big)=0, \quad t>0, \varepsilon>0.$$

If the X^ϕ are also locally square integrable semimartingales, we introduce the conditions

$$(\tilde{C}') \qquad \|r_\phi\,\tilde{C}_t'^\phi-\tilde{C}_t'\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ t\in U,$$

and

$$(I_2) \qquad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(r_\phi|x|^2\,\mathbf{1}(r_\phi|x|>a)*\nu_t^\phi>\varepsilon\Big)=0, \quad t>0,\ \varepsilon>0.$$

Note that (I_2) implies (I_1).

Lemma 4.3.1. *1. Let the X^ϕ be special semimartingales. If conditions (ν) and (I_1) hold, then conditions* (sup B) *and* (sup B') *are equivalent.*

2. Let the X^ϕ be locally square integrable semimartingales. If conditions (ν), $(\hat{\nu})$ and (I_2) hold, then conditions (C) and $(\tilde{C}')$ are equivalent.

Proof. The proofs are analogous to the proofs of Lemmas 4.2.8 and 4.2.10, respectively. For part 1 we write

$$\begin{aligned} B'^\phi - B_t' &= B_t^\phi - B_t \\ &+ \big((h_a^\phi(x)-h^\phi(x))*\nu_t^\phi-(h_a(x)-h(x))*\nu_t\big) \\ &+ (x-h_a^\phi(x))*\nu_t^\phi-(x-h_a(x))*\nu_t, \end{aligned}$$

where h_a is from (4.2.46). Since condition (ν) implies condition (sup ν), the expression in the first parentheses on the right converges as $\phi\in\Phi$ super-exponentially in probability to 0 locally uniformly in t. Since

$$\begin{aligned} |x-h_a^\phi(x)|*\nu_t^\phi &\le |x|\,\mathbf{1}(r_\phi|x|\ge a)*\nu_t^\phi, \\ |x-h_a(x)|*\nu_t &\le |x|\,\mathbf{1}(|x|\ge a)*\nu_t, \end{aligned}$$

$(x-h_a^\phi(x))*\nu_t^\phi$ converges super-exponentially in probability to 0 as $\phi\in\Phi$ and $a\to\infty$ locally uniformly in t by (I_1) and $(x-h_a^\phi(x))*\nu_t$

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converges to 0 as $a \to \infty$ locally uniformly in t by (4.2.3). Part 1 is proved.

In order to prove part 2, it suffices to show as in the proof of Lemma 4.2.10 that for $t \in U, \varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\big|r_\phi(\lambda\cdot x)^2\,\mathbf{1}(r_\phi|x|>\delta)*\nu_t^\phi - (\lambda\cdot x)^2 * \nu_t\big| > \varepsilon\Big) = 0,$$

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\delta\sum_{0<s\le t}|x|\,\mathbf{1}(r_\phi|x|>\delta)\bullet\nu_s^\phi > \varepsilon\Big) = 0,$$

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\Big|r_\phi\sum_{0<s\le t}\big(\lambda\cdot x\,\mathbf{1}(r_\phi|x|>\delta)\bullet\nu_s^\phi\big)^2 - \int_0^t\big(\lambda\cdot x\bullet\hat\nu_s\big)^2 ds\Big| > \varepsilon\Big) = 0.$$

The first convergence follows by the inequality, where $a > \delta$,

$$\begin{aligned}&\big|r_\phi(\lambda\cdot x)^2\,\mathbf{1}(r_\phi|x|>\delta)*\nu_t^\phi - (\lambda\cdot x)^2*\nu_t\big|\\ &\le \Big|\frac{1}{r_\phi}(\lambda\cdot h_a(r_\phi x))^2\,\mathbf{1}(r_\phi|x|>\delta)*\nu_t^\phi - (\lambda\cdot h_a(x))^2*\nu_t\Big|\\ &\qquad + |\lambda|^2 r_\phi|x|^2\,\mathbf{1}(r_\phi|x|>a)*\nu_t^\phi + |\lambda|^2|x|^2\,\mathbf{1}(|x|>a)*\nu_t,\end{aligned}$$

part 1 of Lemma 4.2.9, condition (I_2), and (4.2.3).

The second convergence follows by the inequality

$$\begin{aligned}\delta\sum_{0<s\le t}|x|\,\mathbf{1}(r_\phi|x|>\delta)\bullet\nu_s^\phi &\le \frac{\delta}{r_\phi}\sum_{0<s\le t}|h_a(r_\phi x)|\,\mathbf{1}(r_\phi|x|>\delta)\bullet\nu_s^\phi\\ &\quad + \delta\sum_{0<s\le t}|x|\,\mathbf{1}(r_\phi|x|>a)\bullet\nu_s^\phi,\end{aligned}$$

part 2 of Lemma 4.2.9 and condition (I_1).

For the third convergence, we write

$$\Big|r_\phi\sum_{0<s\le t}\big(\lambda\cdot x\,\mathbf{1}(r_\phi|x|>\delta)\bullet\nu_s^\phi\big)^2 - \int_0^t\big(\lambda\cdot x\bullet\hat\nu_s\big)^2 ds\Big| \le Q_1+Q_2+Q_3,$$

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where

$$Q_1 = \Big| \frac{1}{r_\phi} \sum_{0<s\le t} \big(\lambda \cdot h_a(r_\phi x)\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi\big)^2 - \int_0^t \big(\lambda \cdot h_a(x) \bullet \hat{\nu}_s\big)^2 ds \Big|,$$

$$Q_2 = \Big| r_\phi \sum_{0<s\le t} \big(\lambda \cdot x\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi\big)^2 - \frac{1}{r_\phi} \sum_{0<s\le t} \big(\lambda \cdot h_a(r_\phi x)\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi\big)^2 \Big|,$$

$$Q_3 = \Big| \int_0^t \big(\lambda \cdot h_a(x) \bullet \hat{\nu}_s\big)^2 ds - \int_0^t \big(\lambda \cdot x \bullet \hat{\nu}_s\big)^2 ds \Big|.$$

Quantity Q_1 converges super-exponentially in probability to 0 as $\phi \in \Phi$ by part 3 of Lemma 4.2.9. For Q_2 we have

$$\begin{aligned} Q_2 &\le 2|\lambda|^2 r_\phi \sum_{0<s\le t} \big(|x|\, \mathbf{1}(r_\phi |x| > a) \bullet \nu_s^\phi\big)\big(|x|\, \mathbf{1}(r_\phi |x| > \delta) \bullet \nu_s^\phi\big) \\ &\le 2|\lambda|^2 r_\phi \sum_{0<s\le t} \big(|x|\, \mathbf{1}(r_\phi |x| > a) \bullet \nu_s^\phi\big)\big(|x|\, \mathbf{1}(a \ge r_\phi |x| > \delta) \bullet \nu_s^\phi\big) \\ &+ 2|\lambda|^2 r_\phi \sum_{0<s\le t} \big(|x|\, \mathbf{1}(r_\phi |x| > a) \nu_s^\phi\big)^2 \\ &\le 4|\lambda|^2 r_\phi \sum_{0<s\le t} |x|^2\, \mathbf{1}(r_\phi |x| > a) \bullet \nu_s^\phi. \end{aligned}$$

The latter sum goes to 0 as $\phi \in \Phi$ super-exponentially in probability by (I_2). By a similar argument,

$$Q_3 \le 4|\lambda|^2 \int_0^t (\lambda \cdot x)^2\, \mathbf{1}(|x| > a) \bullet \hat{\nu}_s\, ds,$$

and converges to 0 as $a \to \infty$ by (4.2.3). Part 2 is proved. □

The next result is a direct consequence of Theorem 4.2.1 and Lemma 4.3.1.

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Theorem 4.3.2. *I. Let the X^ϕ be special semimartingales and condition (I_1) hold. If conditions (0), $(A)+(a)$, $(\sup B')$, (C) (or $(\tilde{C})$ associated with a continuous limiter), (ν), and $(\hat{\nu})$ hold, then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$.*

II. Let the X^ϕ be locally square integrable semimartingales and condition (I_2) hold. If conditions (0), $(A)+(a)$, $(\sup B')$, $(\tilde{C}')$, (ν), and $(\hat{\nu})$ hold, then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$.

Remark 4.3.3. *Similarly, in the statements below we can replace condition $(\sup B)$ by condition $(\sup B')$ each time condition (I_1) holds and replace condition $(\tilde{C})$ by condition $(\tilde{C}')$ each time condition (I_2) holds.*

We next consider the "quasi-continuous" case $\hat{\nu}_s(\mathbb{R}^d) = 0$. It is singled out by the condition

$$(QC) \qquad \frac{1}{r_\phi} \sum_{0<s\le t} \nu^\phi(\{s\}, \{r_\phi |x| > \epsilon\})^2 \overset{P_\phi^{1/r_\phi}}{\to} 0, \quad t > 0, \epsilon > 0.$$

Since (QC) implies $(\hat{\nu})$ with $\hat{\nu}_s(\mathbb{R}^d) = 0$, condition (L_1) trivially holds. We thus obtain the following corollary of Theorem 4.2.1.

Corollary 4.3.4. *Let condition (QC) hold and the limiter $h(x)$ be continuous. If conditions (0), $(A)+(a)$, $(\sup B)$, (C) (or $(\tilde{C})$), and (ν) hold, then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$.*

As a consequence, we derive a result on LD convergence to the Poisson idempotent process. Since the latter by Theorem 2.4.16 has characteristics $B'_t = t$, $C_t = 0$, $\nu_t(\Gamma) = \mathbf{1}(1 \in \Gamma)$, and $\hat{\nu}_t(\Gamma) = 0$, we have the following result.

Corollary 4.3.5. *Let the semimartingales X^ϕ be one-dimensional and the limiter $h(x)$ be continuous at $x = 1$. Let $\mathcal{N}$ be a Poisson idempotent process. Let conditions (0) for $x_0 = 0$, $(A)+(a)$ and (QC) hold. If*

$$\sup_{t\le T} |B_t^\phi - h(1)t| \overset{P_\phi^{1/r_\phi}}{\to} 0 \quad \text{as } \phi \in \Phi, \quad T > 0,$$

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\|r_\phi C_t^{\phi,\delta}\| > \varepsilon\Big) = 0, \quad t \in U, \varepsilon > 0,$$

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and for all $\varepsilon \in (0, 1/2)$*, as* $\phi \in \Phi$*,*

$$\frac{1}{r_\phi}\,\nu^\phi([0,t],\{|r_\phi x - 1| \le \varepsilon\}) \overset{P_\phi^{1/r_\phi}}{\to} t, \;\; t \in U,$$

$$\frac{1}{r_\phi}\,\nu^\phi([0,t],\{r_\phi|x| > \varepsilon\} \cap \{|r_\phi x - 1| > \varepsilon\}) \overset{P_\phi^{1/r_\phi}}{\to} 0, \;\; t > 0,$$

then $X^\phi \overset{ld}{\to} \mathcal{N}$ *as* $\phi \in \Phi$*.*

Proof. Let $h(x)$ be continuous. The first characteristic of X associated with $h(x)$ equals $h(1)t$. Therefore, the first two convergences in the statement check conditions $(\sup B)$ and (C). We check condition (ν). Let $f \in \mathcal{C}_b$ and $\eta > 0$ be arbitrary. Let $\epsilon > 0$ be such that $|f(x) - f(1)| \le \eta$ if $|x - 1| \le \epsilon$ and $f(x) = 0$ if $|x| \le \epsilon$. Then by the fact that $f * \nu_t = f(1)t$

$$\begin{aligned}|f^\phi * \nu_t^\phi - f * \nu| \le{}& \frac{\|f\|}{r_\phi}\nu^\phi\big([0,t],\{r_\phi|x| > \varepsilon\} \cap \{|r_\phi x - 1| > \varepsilon\}\big)\\ &+ \eta\frac{1}{r_\phi}\,\nu^\phi\big([0,t],\{|r_\phi x - 1| \le \varepsilon\}\big)\\ &+ |f(1)|\Big|\frac{1}{r_\phi}\,\nu^\phi\big([0,t],\{|r_\phi x - 1| \le \varepsilon\}\big) - t\Big|,\end{aligned}$$

which implies condition (ν) by hypotheses and arbitrariness of η. The stated LD convergence follows now by Corollary 4.3.4.

Now let $h(x)$ be continuous at $x = 1$ and $\overline{h}(x)$ be a continuous limiter such that $\overline{h}(1) = h(1)$. We denote by $\overline{B}^\phi$ the first characteristic of X^ϕ corresponding to $\overline{h}(x)$. Given $\eta > 0$, we choose $\epsilon > 0$ such that $\overline{h}(x) = h(x) = 0$ if $|x| \le \epsilon$ and $|\overline{h}(x) - h(x)| \le \eta$ if $|x - 1| \le \epsilon$. Then by (4.1.4) denoting $\|h\| = \sup_{x\in\mathbb{R}^d}|h(x)|$ and $\|\overline{h}\| = \sup_{x\in\mathbb{R}^d}|\overline{h}(x)|$

$$\begin{aligned}|\overline{B}_t^\phi - \overline{h}(1)t| &\le |B_t^\phi - h(1)t| + \big|\overline{h}^\phi(x) - h^\phi(x)\big| * \nu_t^\phi\\ &\le |B_t^\phi - h(1)t| + \eta\frac{1}{r_\phi}\,\nu^\phi\big([0,t],\{|r_\phi x - 1| \le \varepsilon\}\big)\\ &\quad + \frac{\|h\| + \|\overline{h}\|}{r_\phi}\nu^\phi\big([0,t],\{r_\phi|x| > \varepsilon\} \cap \{|r_\phi x - 1| > \varepsilon\}\big).\end{aligned}$$

Since η is arbitrary, the hypotheses imply that $\overline{B}_t^\phi - \overline{h}(1)t \overset{P_\phi^{1/r_\phi}}{\to} 0$. The claim follows by the part already proved. ☐

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Let us now consider the case $\nu_t(\Gamma) = 0$. It is implied by the condition

$$(MD) \qquad \frac{1}{r_\phi}\nu^\phi([0,t],\{r_\phi|x|>\epsilon\}) \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ t>0,\ \epsilon>0.$$

In the large deviation theory terminology this is the case of "moderate deviations". We recall that if $\nu = 0$, then the idempotent deviability distribution of X has density

$$\mathbf{\Pi}_{x_0}(\mathbf{x}) = \exp\Big(-\frac{1}{2}\int_0^\infty (\dot{\mathbf{x}}_t - b_t)\cdot c_t^\oplus(\dot{\mathbf{x}}_t - b_t)\,dt\Big)$$

if $\mathbf{x}$ is absolutely continuous, $\mathbf{x}_0 = x_0$ and $\dot{\mathbf{x}}_t - b_t$ is in the range of c_t (a.e.), and $\mathbf{\Pi}_{x_0}(\mathbf{x}) = 0$ otherwise. By Theorem 2.6.26 X is the Luzin-continuous idempotent Gaussian diffusion

$$X_t = x_0 + \int_0^t b_s\,ds + \int_0^t c_s^{1/2}\dot{W}_s\,ds, \tag{4.3.1}$$

where W is a Wiener idempotent process. Hence, this is a "central limit theorem" setting.

Lemma 4.3.6. *Let B^ϕ and $\tilde{C}^\phi$, $\overline{B}^\phi$ and $\overline{\tilde{C}}^\phi$ be the first and modified second characteristics of X^ϕ corresponding to respective limiters $h(x)$ and $\overline{h}(x)$, not necessarily continuous. If condition (MD) holds, then, as $\phi\in\Phi$,*

$$\sup_{t\le T}|B_t^\phi - \overline{B}_t^\phi| \overset{P_\phi^{1/r_\phi}}{\to} 0,\ \ T>0,$$

$$r_\phi\,\|\tilde{C}_t^\phi - \overline{\tilde{C}}_t^\phi\| \overset{P_\phi^{1/r_\phi}}{\to} 0,\ \ t>0.$$

Proof. For B^ϕ and $\overline{B}^\phi$ the claim is a direct consequence of (4.1.4), the definition of a limiter and (MD). For $\tilde{C}^\phi$ and $\overline{\tilde{C}}^\phi$, we have by the definition of modified second characteristics, choosing $\epsilon>0$ such that $h(x) = \overline{h}(x) = x, |x|\le\epsilon$, that

$$\begin{aligned} r_\phi\,\|\tilde{C}_t^\phi(\lambda) - \overline{\tilde{C}}_t^\phi(\lambda)\| \ &\le |\lambda|^2(\|h\|^2 + \|\overline{h}\|^2)\frac{1}{r_\phi}\nu^\phi([0,t],\{r_\phi|x|>\epsilon\}) \\ &+ |\lambda|^2(\|h\| + \|\overline{h}\|)^2\frac{1}{r_\phi}\sum_{0<s\le t}\nu^\phi(\{s\},\{r_\phi|x|>\epsilon\}). \end{aligned}$$

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Thus, the claim follows by (MD). □

We introduce the conditions

$$(\sup B_0') \qquad \sup_{t\le T} |B_t^\phi - B_t'| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi,\ T > 0,$$

$$(C_0) \qquad \|r_\phi \tilde{C}_t^\phi - C_t\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi,\ \ t \in U.$$

Theorem 4.2.1 and Lemma 4.3.6 yield the following.

Corollary 4.3.7. *("the LD central limit theorem") Let X be given by (4.3.1). Let conditions (0), $(A)+(a)$, $(\sup B_0')$, (C_0), and (MD) hold with some limiter $h(x)$. Then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$.*

If we would like to use nontruncated characteristics of the X^ϕ, we could require the following Lindeberg condition

$$(L_2) \qquad r_\phi |x|^2\, \mathbf{1}(r_\phi |x| > \epsilon) * \nu_t^\phi \overset{P_\phi^{1/r_\phi}}{\to} 0,\ t > 0, \epsilon > 0,$$

which implies both conditions (I_2) and (MD). Let us introduce the condition

$$(C_0') \qquad \|r_\phi \tilde{C}_t'^\phi - C_t\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi,\ \ t \in U.$$

We thus have the following version.

Corollary 4.3.8. *Let X be given by (4.3.1). Let the X^ϕ be locally square integrable semimartingales and condition (L_2) hold. If conditions (0), $(A)+(a)$, $(\sup B')$, and (C_0') hold, then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$.*

Now we consider simpler versions of conditions $(A)+(a)$ on the jumps of the X^ϕ. We first note that condition (A) can be checked by checking the condition

$$(A_0) \qquad \nu^\phi([0,t], |x| > A)^{1/r_\phi} \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi,\quad t > 0, \exists A > 0,$$

and condition (a) can be checked by checking the condition

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$$(a_0) \qquad \frac{1}{r_\phi}\, e^{\alpha r_\phi |x|}\, \mathbf{1}(r_\phi|x| > a)\, \mathbf{1}(|x| \le A) * \nu_t^\phi \overset{P_\phi^{1/r_\phi}}{\to} 0$$
$$\text{as } \phi \in \Phi,\ t > 0,$$
$$\alpha > 0, A > 0, \exists a > 0.$$

Let us also note that if condition (A_0) holds and the convergence in (a_0) holds for every $a > 0$, then condition (MD) holds. The following observation comes in useful below.

Lemma 4.3.9. *Condition* (a) *is implied by the conditions*

$$(a_1) \qquad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}\nu^\phi\big([0,t], \{r_\phi|x| > a\}\big) > \epsilon\Big) = 0,$$
$$t > 0,\ \epsilon > 0,$$

$$(a_2) \quad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}\int_a^{\alpha r_\phi A} e^u \nu^\phi\big([0,t], \{\alpha r_\phi|x| > u\}\big)du > \epsilon\Big) = 0,$$
$$t > 0,\ \epsilon > 0,\ \alpha > 0,\ A > 0,$$

Proof. The claim follows since

$$\begin{aligned}
&\frac{1}{r_\phi}\big(e^{\alpha r_\phi|x|} - 1\big)\, \mathbf{1}(r_\phi|x| > a)\, \mathbf{1}(|x| \le A) * \nu_t^\phi \\
&= \frac{1}{r_\phi}\int_0^\infty e^u\, \mathbf{1}(\alpha r_\phi|x| > u)du\, \mathbf{1}(r_\phi|x| > a)\, \mathbf{1}(|x| \le A) * \nu_t^\phi \\
&\le \frac{1}{r_\phi}\int_0^{\alpha r_\phi A} e^u \nu^\phi\big([0,t], \{\alpha r_\phi|x| > u\} \cap \{r_\phi|x| > a\}\big)du \\
&\le \frac{1}{r_\phi}\nu^\phi\big([0,t], \{r_\phi|x| > a\}\big)\int_0^R e^u du \\
&\qquad\qquad + \frac{1}{r_\phi}\int_R^{\alpha r_\phi A} e^u \nu^\phi\big([0,t], \{\alpha r_\phi|x| > u\}\big)du,
\end{aligned}$$

where $R > 0$ is arbitrary. □

The following conditions can also be used for checking conditions $(A) + (a)$.

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$$(VS) \qquad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\nu^\phi([0,t],\{r_\phi|x|>a\})^{1/r_\phi}>\varepsilon\Big)=0,$$
$$t>0,\varepsilon>0,$$

$$(VS_0) \qquad \nu^\phi([0,t],\{r_\phi|x|>\epsilon\})^{1/r_\phi} \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ t>0,\ \epsilon>0.$$

Clearly, $(VS_0) \Rightarrow (VS) \Rightarrow (A)+(a)$. Since also $(VS_0) \Rightarrow (MD)$, by Corollary 4.3.7 we have the following.

Corollary 4.3.10. *Let* X *be given by* (4.3.1)*. Let conditions* (0)*,* (VS_0)*,* $(\sup B_0')$*, and* (C_0) *hold with some limiter* $h(x)$*. Then* $X^\phi \overset{ld}{\to} X$ *as* $\phi\in\Phi$*.*

Now we consider the case of the "classical" large deviation setting when the Cramér condition holds:

$$e^{\alpha|x|}\ \mathbf{1}(|x|>1)*\nu_t^\phi<\infty,\ \phi\in\Phi, t>0,\alpha>0.$$

We introduce the conditions

$$(I_e) \qquad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}e^{\alpha r_\phi|x|}\ \mathbf{1}(r_\phi|x|>a)*\nu_t^\phi>\varepsilon\Big)=0,$$
$$t>0,\ \varepsilon>0,\ \alpha>0,$$

$$(L_e) \qquad \frac{1}{r_\phi}e^{\alpha r_\phi|x|}\ \mathbf{1}(r_\phi|x|>\epsilon)*\nu_t^\phi \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,$$
$$t>0,\ \alpha>0,\ \epsilon>0.$$

Condition (L_e) can be called an exponential Lindeberg condition. Obviously, $(L_e) \Rightarrow (I_e) \Rightarrow (A)+(a)$, $(I_e) \Rightarrow (I_2)$, and $(L_e) \Rightarrow (L_2) \Rightarrow (MD)$. By Theorem 4.3.2 the implication $(I_e) \Rightarrow (I_2)$ allows us to consider nontruncated characteristics under (I_e). We thus have the following result.

Corollary 4.3.11. *Let the Cramér condition and condition* (I_e) *hold. If conditions* (0)*,* $(\sup B')$*,* $(\tilde{C}')$*,* (ν)*, and* $(\hat{\nu})$ *hold, then* $X^\phi \overset{ld}{\to} X$ *as* $\phi\in\Phi$*.*

The following is an application to point processes.

Corollary 4.3.12. *Let* $X_t^\phi = N_t^\phi/r_\phi$*, where* $N^\phi=(N_t^\phi,\,t\in\mathbb{R}_+)$ *are one-dimensional point processes with compensators* $A^\phi=(A_t^\phi,\,t\in\mathbb{R}_+)$*.*

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a) If, as $\phi \in \Phi$, for some Lebesgue measurable function $\Delta_s \in [0,1]$,

$$\frac{1}{r_\phi} A_t^\phi \overset{P_\phi^{1/r_\phi}}{\to} t + \int_0^t \Delta_s \, ds \,, \quad t \in U,$$

$$\frac{1}{r_\phi} \sum_{0<s\le t} (\Delta A_s^\phi)^k \overset{P_\phi^{1/r_\phi}}{\to} \int_0^t \Delta_s^k \, ds \,, \ t \in U, k = 2, 3, \ldots,$$

then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$, where X is the idempotent process with independent increments with local characteristics $b_t = 1 + \Delta_t$, $c_t = 0$, $\nu_t(\Gamma) = (1 + \Delta_t)\, \mathbf{1}(1 \in \Gamma)$, and $\hat{\nu}_t(\Gamma) = \Delta_t \, \mathbf{1}(1 \in \Gamma)$.

b) In particular, if

$$\frac{1}{r_\phi} A_t^\phi \overset{P_\phi^{1/r_\phi}}{\to} t, \quad t \in U,$$

$$\frac{1}{r_\phi} \sum_{0<s\le t} (\Delta A_s^\phi)^2 \overset{P_\phi^{1/r_\phi}}{\to} 0, \quad t > 0,$$

then $X^\phi \overset{ld}{\to} \mathcal{N}$ as $\phi \in \Phi$, where $\mathcal{N}$ is an idempotent Poisson process.

Proof. In part a) the nontruncated characteristics of X^ϕ are of the form $B'^\phi_t = A_t^\phi / r_\phi$, $C_t^\phi = 0$, $\nu^\phi([0,t], \Gamma) = \mathbf{1}(r_\phi^{-1} \in \Gamma) A_t^\phi$ so that conditions (sup B'), $(\tilde{C}')$, (ν), and $(\hat{\nu})$ hold with $B_t = t + \int_0^t \Delta_s \, ds$, $C_t = 0$, and ν_t and $\hat{\nu}_t$ as indicated in the statement. Part b) is a consequence of part a). □

The implications $(L_e) \Rightarrow (L_2)$ and $(L_e) \Rightarrow (A) + (a)$ give the following version of Corollary 4.3.8.

Corollary 4.3.13. *Let X be given by* (4.3.1)*. Let the Cramér condition and condition (L_e) hold. If conditions* (0)*, (*sup B'*), and (C_0') hold, then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$.*

4.4 Applications to partial-sum processes

In this section we consider applications of the above results to the setting of the processes of partial sums of random variables. Let

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$\{\xi_i^n,\ i\in\mathbb{N}\}$, $n\in\mathbb{N}$, be sequences of $\mathbb{R}^d$-valued random variables defined on respective probability spaces $(\Omega_n,\mathcal{F}_n,P_n)$ and adapted to discrete-time filtrations $\mathbf{F}_n=\{\mathcal{F}_i^n,\ i\in\mathbb{N}\}$. Let $r_n\to\infty$ as $n\to\infty$ and

$$X_t^n=\sum_{i=1}^{\lfloor nt\rfloor}\xi_i^n,\ t\in\mathbb{R}_+, \tag{4.4.1}$$

where $\sum_{i=1}^0=0$. The predictable triplet (B^n,C^n,ν^n) of $X^n=(X_t^n,\ t\in\mathbb{R}_+)$ corresponding to a limiter $h(x)$ is given by

$$B_t^n=\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}E_n\big(h(r_n\xi_i^n)|\mathcal{F}_{i-1}^n\big),\qquad C_t^n=0,$$

$$\nu^n\big([0,t],\Gamma\big)=\sum_{i=1}^{\lfloor nt\rfloor}P_n(\xi_i^n\in\Gamma\setminus\{0\}|\mathcal{F}_{i-1}^n),\ \Gamma\in\mathcal{B}(\mathbb{R}^d),$$

where E_n denotes expectation with respect to P_n. We consider large deviation convergence of the X^n with rate r_n. Let X be a semimaxingale with characteristics $(B,C,\nu,\hat\nu)$ and modified second characteristic $\tilde C$ associated with $h(x)$ as defined in Section 4.2. The conditions of Theorem 4.2.1 assume the form.

$(A)_\Sigma$
$$\lim_{A\to\infty}\limsup_{n\to\infty}P_n^{1/r_n}\Big(\Big(\sum_{i=1}^{\lfloor nt\rfloor}P_n(|\xi_i^n|>A|\mathcal{F}_{i-1}^n)\Big)^{1/r_n}>\epsilon\Big)=0,\quad t>0,\ \varepsilon>0,$$

$(a)_\Sigma$
$$\lim_{a\to\infty}\limsup_{n\to\infty}P_n^{1/r_n}\Big(\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}E_n\big(e^{\alpha r_n|\xi_i^n|}\mathbf{1}(r_n|\xi_i^n|>a)\,\mathbf{1}(|\xi_i^n|\le A)|\mathcal{F}_{i-1}^n\big)>\varepsilon\Big)=0,$$
$$t>0,\ \alpha>0,\ A>0,\ \varepsilon>0,$$

$(a')_\Sigma$
$$\lim_{a\to\infty}\limsup_{n\to\infty}P_n^{1/r_n}\Big(\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}\ln\big(1+E_n\big((e^{\alpha r_n|\xi_i^n|}-1)\,\mathbf{1}(r_n|\xi_i^n|>a)\,\mathbf{1}(|\xi_i^n|\le A)|\mathcal{F}_{i-1}^n\big)\big)>\varepsilon\Big)=0,$$
$$t>0,\ \alpha>0,\ A>0,\ \varepsilon>0,$$

$(\sup B)_\Sigma$
$$\sup_{t\le T}\Big|\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}E_n\big(h(r_n\xi_i^n)|\mathcal{F}_{i-1}^n\big)-B_t\Big|\overset{P_n^{1/r_n}}{\to}0\text{ as }n\to\infty,\quad T>0,$$

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$(C)_\Sigma$
$$\lim_{\delta\to 0}\limsup_{n\to\infty} P_n^{1/r_n}\Big(\Big|r_n\sum_{i=1}^{\lfloor nt\rfloor}\big[E_n\big((\lambda\cdot\xi_i^n)^2\,\mathbf{1}(r_n|\xi_i^n|\le\delta)|\mathcal{F}_{i-1}^n\big) - \big(E_n\big(\lambda\cdot\xi_i^n\,\mathbf{1}(r_n|\xi_i^n|\le\delta)|\mathcal{F}_{i-1}^n\big)\big)^2\big] - \lambda\cdot C_t\lambda\Big|>\epsilon\Big)=0,$$
$$t\in U,\ \varepsilon>0,\ \lambda\in\mathbb{R}^d,$$

$(\tilde{C})_\Sigma$
$$\lim_{n\to\infty} P_n^{1/r_n}\Big(\Big|\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}\big[E_n\big((\lambda\cdot h(r_n\xi_i^n))^2|\mathcal{F}_{i-1}^n\big) - \big(E_n\big(\lambda\cdot h(r_n\xi_i^n)|\mathcal{F}_{i-1}^n\big)\big)^2\big] - \lambda\cdot\tilde{C}_t\lambda\Big|>\epsilon\Big)=0,$$
$$t\in U,\ \varepsilon>0,\ \lambda\in\mathbb{R}^d,$$

$(\nu)_\Sigma$
$$\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}E_n\big(f(r_n\xi_i^n)|\mathcal{F}_{i-1}^n\big)\overset{P_n^{1/r_n}}{\to} f(x)*\nu_t \text{ as } n\to\infty,$$
$$t\in U,\ f\in\mathcal{C}_b,$$

$(\hat{\nu})_\Sigma$
$$\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}\big(E_n(f(r_n\xi_i^n)|\mathcal{F}_{i-1}^n)\big)^k\overset{P_n^{1/r_n}}{\to}\int_0^t\big(f(x)\bullet\hat{\nu}_s\big)^k ds \text{ as } n\to\infty,$$
$$k=2,3,\ldots,\ t\in U,\ f\in\mathcal{C}_b.$$

The following theorem is a triangular array version of Theorem 4.2.1.

Theorem 4.4.1. *Let X^n be defined by* (4.4.1) *and $h(x)$ be a continuous limiter. If conditions $(A)_\Sigma+(a)_\Sigma$, $(\sup B)_\Sigma$, $(C)_\Sigma$ (or $(\tilde{C}_\Sigma)$), $(\nu)_\Sigma$, and $(\hat{\nu})_\Sigma$ hold, then $X^n \overset{ld}{\to} X$.*

The integrable and square integrable versions look as follows. As above, B' and $\tilde{C}'$ denote the nontruncated first and modified second characteristics of X, respectively.

Theorem 4.4.2. *I. Let $E_n|\xi_i^n|<\infty$, $i\in\mathbb{N}$. Let*

$(I_1)_\Sigma$
$$\lim_{a\to\infty}\limsup_{n\to\infty}P_n^{1/r_n}\Big(\sum_{i=1}^{\lfloor nt\rfloor}E_n\big(|\xi_i^n|\,\mathbf{1}(r_n|\xi_i^n|>a)|\mathcal{F}_{i-1}^n\big)>\varepsilon\Big)=0,\quad t>0,\varepsilon>0$$

and

$(\sup B')_\Sigma$
$$\sup_{t\le T}\Big|\sum_{i=1}^{\lfloor nt\rfloor}E_n(\xi_i^n|\mathcal{F}_{i-1}^n)-B'_t\Big|\overset{P_n^{1/r_n}}{\to} 0 \quad \textit{as } n\to\infty,$$
$$T>0.$$

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If, in addition, conditions $(A)_\Sigma + (a)_\Sigma$, $(C)_\Sigma$ *(or* $(\tilde{C})_\Sigma$ *with a continuous limiter),* $(\nu)_\Sigma$, *and* $(\hat{\nu})_\Sigma$ *hold, then* $X^n \stackrel{ld}{\to} X$ *as* $n \to \infty$.

II. Let $E_n|\xi_i^n|^2 < \infty$, $i \in \mathbb{N}$. *Let*

$$(I_2)_\Sigma \qquad \lim_{a\to\infty} \limsup_{n\to\infty} P_n^{1/r_n}\Big(r_n \sum_{i=1}^{\lfloor nt \rfloor} E_n\big(|\xi_i^n|^2\,\mathbf{1}(r_n|\xi_i^n| > a)|\mathcal{F}_{i-1}^n\big) > \varepsilon\Big) = 0, \quad t > 0, \varepsilon > 0$$

and

$$(\tilde{C}')_\Sigma \qquad r_n \sum_{i=1}^{\lfloor nt \rfloor}\big(E_n((\lambda\cdot\xi_i^n)^2|\mathcal{F}_{i-1}^n) - \big(E_n(\lambda\cdot\xi_i^n|\mathcal{F}_{i-1}^n)\big)^2\big) \stackrel{P_n^{1/r_n}}{\to} \lambda\cdot\tilde{C}_t'\lambda \quad as\ n\to\infty, \quad t\in U,\ \lambda\in\mathbb{R}^d.$$

If, in addition, conditions $(A)_\Sigma + (a)_\Sigma$, $(\sup B')_\Sigma$, $(\nu)_\Sigma$, *and* $(\hat{\nu})_\Sigma$ *hold, then* $X^n \stackrel{ld}{\to} X$ *as* $n\to\infty$.

Conditions (QC) and (MD) look in the triangular array setting as follows.

$$(QC)_\Sigma \qquad \frac{1}{r_n}\sum_{i=1}^{\lfloor nt \rfloor}\big(P_n(r_n|\xi_i^n| > \epsilon|\mathcal{F}_{i-1}^n)\big)^2 \stackrel{P_n^{1/r_n}}{\to} 0 \text{ as } n\to\infty, \quad t>0,\ \epsilon>0,$$

$$(MD)_\Sigma \qquad \frac{1}{r_n}\sum_{i=1}^{\lfloor nt \rfloor} P_n(r_n|\xi_i^n| > \epsilon|\mathcal{F}_{i-1}^n) \stackrel{P_n^{1/r_n}}{\to} 0 \text{ as } n\to\infty, \quad t>0,\ \epsilon>0.$$

The other conditions take the form

$$(L_2)_\Sigma \qquad r_n\sum_{i=1}^{\lfloor nt \rfloor} E_n\big(|\xi_i^n|^2\,\mathbf{1}(r_n|\xi_i^n| > \epsilon)|\mathcal{F}_{i-1}^n\big) \stackrel{P_n^{1/r_n}}{\to} 0 \text{ as } n\to\infty, \quad t>0,\ \epsilon>0,$$

$$(A_0)_\Sigma \qquad \Big(\sum_{i=1}^{\lfloor nt \rfloor} P_n(|\xi_i^n| > A|\mathcal{F}_{i-1}^n)\Big)^{1/r_n} \stackrel{P_n^{1/r_n}}{\to} 0 \text{ as } n\to\infty, \quad t>0,\ \varepsilon>0, \exists A>0,$$

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$$(a_0)_\Sigma \qquad \frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor} E_n\big(e^{\alpha r_n|\xi_i^n|}\,\mathbf{1}(r_n|\xi_i^n|>a)\,\mathbf{1}(|\xi_i^n|\le A)\big|\mathcal{F}_{i-1}^n\big) \stackrel{P_n^{1/r_n}}{\to} 0$$
$$\text{as } n\to\infty,\ \ t>0,\ \alpha>0,\ A>0,\ \exists a>0.$$

$$(a_1)_\Sigma \qquad \lim_{a\to\infty}\limsup_{n\to\infty} P_n^{1/r_n}\Big(\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor} P_n(r_n|\xi_i^n|>a|\mathcal{F}_{i-1}^n)>\epsilon\Big)=0,$$
$$t>0,\ \epsilon>0,$$

$$(a_2)_\Sigma \quad \lim_{a\to\infty}\limsup_{n\to\infty} P_n^{1/r_n}\Big(\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}\int_a^{\alpha r_n A} e^u P_n(\alpha r_n|\xi_i^n|>u|\mathcal{F}_{i-1}^n)du>\epsilon\Big)$$
$$=0, \quad t>0,\ \epsilon>0,\ \alpha>0,\ A>0,$$

$$(VS)_\Sigma \quad \lim_{a\to\infty}\limsup_{n\to\infty} P_n^{1/r_n}\Big(\Big(\sum_{i=1}^{\lfloor nt\rfloor} P_n(r_n|\xi_i^n|>a|\mathcal{F}_{i-1}^n)\Big)^{1/r_n}>\epsilon\Big)=0,$$
$$t>0,\ \epsilon>0,$$

$$(VS_0)_\Sigma \qquad \Big(\sum_{i=1}^{\lfloor nt\rfloor} P_n(r_n|\xi_i^n|>\epsilon|\mathcal{F}_{i-1}^n)\Big)^{1/r_n} \stackrel{P_n^{1/r_n}}{\to} 0 \ \text{ as } n\to\infty,$$
$$t>0,\ \epsilon>0,$$

$$(I_e)_\Sigma \quad \lim_{a\to\infty}\limsup_{n\to\infty} P_n^{1/r_n}\Big(\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor} E_n\big(e^{\alpha r_n|\xi_i^n|}\,\mathbf{1}(r_n|\xi_i^n|>a)|\mathcal{F}_{i-1}^n\big)>\epsilon\Big)$$
$$=0, \quad t>0,$$

$$(L_e)_\Sigma \qquad \frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor} E_n\big(e^{\alpha r_n|\xi_i^n|}\,\mathbf{1}(r_n|\xi_i^n|>\epsilon)|\mathcal{F}_{i-1}^n\big) \stackrel{P_n^{1/r_n}}{\to} 0 \text{ as } n\to\infty,$$
$$t>0,\ \epsilon>0,$$

$$(\sup\ B_0')_\Sigma \qquad \sup_{t\le T}\Big|\frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor} E_n\big(h(r_n\xi_i^n)|\mathcal{F}_{i-1}^n\big)-B_t'\Big| \stackrel{P_n^{1/r_n}}{\to} 0$$
$$\text{as } n\to\infty,\ \ T>0,$$

$$(C_0)_\Sigma \quad \frac{1}{r_n}\sum_{i=1}^{\lfloor nt\rfloor}\Big(E_n((\lambda\cdot h(r_n\xi_i^n))^2|\mathcal{F}_{i-1}^n)-\big(E_n(\lambda\cdot h(r_n\xi_i^n)|\mathcal{F}_{i-1}^n)\big)^2\Big)$$
$$\stackrel{P_n^{1/r_n}}{\to} \lambda\cdot C_t\lambda \ \text{ as } n\to\infty,\ \ t>0,\ \lambda\in\mathbb{R}^d,$$

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$$(C_0')_\Sigma \quad r_n \sum_{i=1}^{\lfloor nt \rfloor} \Big(E_n((\lambda\cdot\xi_i^n)^2|\mathcal{F}_{i-1}^n) - \big(E_n(\lambda\cdot\xi_i^n|\mathcal{F}_{i-1}^n)\big)^2\Big) \overset{P_n^{1/r_n}}{\to} \lambda\cdot C_t\lambda$$
$$\text{as } n\to\infty,\ \ t>0,\ \lambda\in\mathbb{R}^d.$$

We now consider versions of the results of the preceding section on LD convergence in distribution to idempotent diffusions. Let X be the idempotent diffusion given by (4.3.1), where $x_0 = 0$.

Corollary 4.4.3. *Let $E_n|\xi_i^n|^2 < \infty$, $i \in \mathbb{N}$, and condition $(L_2)_\Sigma$ hold. If conditions $(A)_\Sigma + (a)_\Sigma$, $(\sup B')_\Sigma$, and $(C_0')_\Sigma$ hold, then $X^n \overset{ld}{\to} X$ as $n\to\infty$.*

Corollary 4.4.4. *Let conditions $(VS_0)_\Sigma$, $(\sup B_0')_\Sigma$ and $(C_0)_\Sigma$ hold with some limiter $h(x)$. Then $X^n \overset{ld}{\to} X$ as $n\to\infty$.*

Corollary 4.4.5. *Let $E\exp(\alpha|\xi_i^n|) < \infty$, $\alpha\in\mathbb{R}_+$, and condition $(L_e)_\Sigma$ hold. If conditions $(\sup B')_\Sigma$ and $(C_0')_\Sigma$ hold, then $X^n \overset{ld}{\to} X$ as $n\to\infty$.*

We next consider an application to a typical moderate deviation setting.

Theorem 4.4.6. *Let ξ_i, $i\in\mathbb{N}$, be i.i.d. $\mathbb{R}^d$-valued random variables on a probability space $(\Omega,\mathcal{F},P)$ such that $E|\xi_1|^2<\infty$ and $E\xi_1 = 0$, and let*

$$X_t^n = \frac{1}{b_n}\sum_{i=1}^{\lfloor nt \rfloor}\xi_i,$$

where $b_n/n\to 0$ and $b_n^2/n\to\infty$ as $n\to\infty$. If, for some $v>0$,

$$\lim_{n\to\infty}\big(nP(|\xi_1|>vb_n)\big)^{n/b_n^2} = 0, \tag{4.4.2}$$

then the X^n LD converge in distribution at rate b_n^2/n to the Luzin-continuous idempotent diffusion $X = (E\xi_1\xi_1^T)^{1/2}W$, where W is an $\mathbb{R}^d$-valued Wiener idempotent process. The deviability distribution of X is idempotent Gaussian and given by

$$\Pi^X(\mathbf{x}) = \exp\Big(-\frac{1}{2}\int_0^\infty \dot{\mathbf{x}}_t\cdot(E\xi_1\xi_1^T)^{\oplus}\dot{\mathbf{x}}_t\,dt\Big),$$

if $\mathbf{x}$ is absolutely continuous, $\mathbf{x}_0 = 0$ and $\dot{\mathbf{x}}_t$ belongs to the range of $E\xi_1\xi_1^T$ (a.e.), and $\Pi^X(\mathbf{x}) = 0$ otherwise.

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Proof. We take $r_n = b_n^2/n$. It is easy to check that condition $(L_2)_\Sigma$ holds so we can apply Corollary 4.4.3. Since $E\xi_1 = 0$, condition $(\sup B')_\Sigma$ holds with $B'_t = 0$. Condition $(C'_0)_\Sigma$ holds with $C_t = (E\xi_1\xi_1^T)t$. We thus need to check conditions $(A)_\Sigma + (a)_\Sigma$. In view of Lemma 4.3.9 it is sufficient to check conditions $(A_0)_\Sigma$, $(a_1)_\Sigma$ and $(a_2)_\Sigma$. We assume with no loss of generality that b_n, n/b_n and b_n^2/n are monotonically increasing.

Condition $(A_0)_\Sigma$ has the form

$$\limsup_{n\to\infty}\Bigl(\lfloor nt\rfloor P(|\xi_1| > Ab_n)\Bigr)^{n/b_n^2} = 0$$

for some $A \in \mathbb{R}_+$ and follows from (4.4.2) with $A = v$.

Condition $(a_1)_\Sigma$ assumes the form

$$\lim_{a\to\infty}\limsup_{n\to\infty}\frac{n\lfloor nt\rfloor}{b_n^2}P\Bigl(|\xi_1| > a\frac{n}{b_n}\Bigr) = 0.$$

We actually check that

$$\lim_{n\to\infty}\frac{n\lfloor nt\rfloor}{b_n^2}P\Bigl(|\xi_1| > v\frac{n}{b_n}\Bigr) = 0. \tag{4.4.3}$$

Let integer $N = N(n)$ be such that

$$b_N \le \frac{n}{b_n} < b_{N+1}.$$

Then, noting that $b_{N+1} \le b_N(N+1)/N \le 2b_N$ by monotonicity of b_n/n , we have

$$\frac{n\lfloor nt\rfloor}{b_n^2}P\Bigl(|\xi_1| > v\frac{n}{b_n}\Bigr) \le tb_{N+1}^2P(|\xi_1| > vb_N) \le 4tb_N^2P(|\xi_1| > vb_N).$$

Since $N \to \infty$ as $n \to \infty$, condition (4.4.2) implies that

$$\lim_{n\to\infty}\Bigl(4tb_N^2P(|\xi_1| > vb_N)\Bigr)^{N/b_N^2} = 0$$

so that

$$\lim_{n\to\infty}4tb_N^2P(|\xi_1| > vb_N) = 0.$$

Limit (4.4.3) follows.

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Condition $(a_2)_\Sigma$ assumes the form

$$\lim_{a\to\infty}\limsup_{n\to\infty}\frac{n\lfloor nt\rfloor}{b_n^2}\int\limits_a^{\alpha A b_n^2/n} e^u P\Big(|\xi_1|>\frac{u}{\alpha}\frac{n}{b_n}\Big)du=0.$$

We actually prove that for all A and a large enough

$$\lim_{n\to\infty}\frac{n\lfloor nt\rfloor}{b_n^2}\int\limits_a^{\alpha A b_n^2/n} e^u P\Big(|\xi_1|>\frac{u}{\alpha}\frac{n}{b_n}\Big)du=0. \tag{4.4.4}$$

Let $A\ge v$ and define integer $L=L(n,u)$ by

$$b_L\le\frac{u}{A\alpha}\frac{n}{b_n}<b_{L+1}. \tag{4.4.5}$$

In the integral in (4.4.4) $u\ge a$, so assuming that $a>\alpha A$, we have that $u\ge\alpha A$ and, hence, $b_{L+1}>n/b_n$. Therefore, by the fact that as above $b_{L+1}\le 2b_L$, for $u\in[a,\alpha A b_n^2/n]$

$$\begin{aligned}\frac{n\lfloor nt\rfloor}{b_n^2}e^u P\Big(|\xi_1|>\frac{u}{\alpha}\frac{n}{b_n}\Big)&\le te^u b_{L+1}^2 P(|\xi_1|>Ab_L)\\&\le 4te^u b_L^2 P(|\xi_1|>Ab_L).\end{aligned} \tag{4.4.6}$$

Since $b_{L+1}\ge n/b_n$ for $u\ge a$, we have that $L\to\infty$ as $n\to\infty$ uniformly over $u\ge a$, so by (4.4.2) and the fact that $A\ge v$

$$\lim_{n\to\infty}\sup_{u\ge a}\Big(LP(|\xi_1|>Ab_L)\Big)^{L/b_L^2}=0.$$

Therefore, given arbitrary $\beta>0$, for all n large enough and $u\ge a$

$$LP(|\xi_1|>Ab_L)\le e^{-\beta b_L^2/L}. \tag{4.4.7}$$

Since in the integral in (4.4.4) $u\le\alpha A b_n^2/n$, it follows by (4.4.5) that $b_L\le b_n$, so by monotonicity $L\le n$ and $b_L/L\ge b_n/n$, which implies by (4.4.5) that

$$\frac{b_L^2}{L}\ge b_L\frac{b_n}{n}\ge\frac{b_{L+1}}{2}\frac{b_n}{n}\ge\frac{u}{2\alpha A}. \tag{4.4.8}$$

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Using (4.4.7) and (4.4.8) we obtain by (4.4.6) that for n large enough and $u \in [a, \alpha A b_n^2/n]$

$$\frac{n\lfloor nt\rfloor}{b_n^2} e^u P\Big(|\xi_1| > \frac{u}{\alpha}\frac{n}{b_n}\Big) \le 4te^u \frac{b_L^2}{L} e^{-\beta b_L^2/L} \le e^u e^{-\beta b_L^2/(2L)} \le e^{-\big(\beta/(4\alpha A)-1\big)u},$$

hence,

$$\limsup_{n\to\infty} \frac{n\lfloor nt\rfloor}{b_n^2} \int_a^{\alpha A b_n^2/n} e^u P\Big(|\xi_1| > \frac{u}{\alpha}\frac{n}{b_n}\Big) du \le \int_a^\infty e^{-\big(\beta/(4\alpha A)-1\big)u} du.$$

Since the latter integral converges to 0 as $\beta \to \infty$, limit (4.4.4) is proved. □

Remark 4.4.7. *As the proof shows, under the hypotheses condition $(a_0)_\Sigma$ holds.*

The next result is in the same theme but considers triangular arrays of row-wise i.i.d.r.v.

Theorem 4.4.8. *Let ζ_i^n, $i \in \mathbb{N}$, $n \in \mathbb{N}$, be a triangular array of row-wise i.i.d. $\mathbb{R}^d$-valued random variables on respective probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$ such that $E_n\zeta_1^n = 0$ and $E_n|\zeta_1^n|^2 < \infty$, and let*

$$X_t^n = \frac{1}{b_n}\sum_{i=1}^{\lfloor nt\rfloor} \zeta_i^n,$$

where $b_n/n \to 0$ and $b_n^2/n \to \infty$ as $n \to \infty$. If $E_n\zeta_1^n\zeta_1^{nT} \to \Sigma$ as $n \to \infty$, where Σ is a positive semi-definite symmetric matrix and either

$$\sup_n E_n|\zeta_1^n|^{2+\delta} < \infty \text{ for some } \delta > 0 \text{ and } \frac{b_n^2}{n\ln n} \to 0 \quad \text{as } n \to \infty, \tag{4.4.9}$$

or

$$\sup_n E_n \exp(\gamma|\zeta_1^n|^\beta) < \infty \text{ and } \frac{b_n^{2-\beta}}{n} \to 0 \quad \text{as } n \to \infty \text{ for some } \gamma > 0 \text{ and } \beta \in (0,1], \tag{4.4.10}$$

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then the X^n LD converge in distribution at rate b_n^2/n to the Luzin-continuous idempotent Gaussian diffusion $X = (\Sigma\Sigma^T)^{1/2}W$, where W is an $\mathbb{R}^d$-valued Wiener idempotent process.

Proof. We take $r_n = b_n^2/n$ and apply Corollary 4.4.3. Since the moment conditions imply $(L_2)_\Sigma$, and conditions $(\sup B')_\Sigma$ and $(C_0')_\Sigma$ hold, we have to check conditions $(A) + (a)$.

Under (4.4.9) this is done by checking condition $(VS_0)_\Sigma$, which takes the form

$$n^{n/b_n^2}\, P_n\Big(|\zeta_1^n| > \frac{n}{b_n}\epsilon\Big)^{n/b_n^2} \to 0 \ \text{ as } n \to \infty$$

and follows by (4.4.9).

Under condition (4.4.10) we check that conditions $(A_0)_\Sigma$ and $(a_0)_\Sigma$ hold. Condition $(A_0)_\Sigma$ holds since by (4.4.10)

$$\Big(nP_n\Big(\frac{|\zeta_1^n|}{b_n} > A\Big)\Big)^{n/b_n^2} \le n^{n/b_n^2}\big(E_n \exp(\gamma|\zeta_1^n|^\beta)\big)^{n/b_n^2} \exp\big(-\gamma A^\beta n/b_n^{2-\beta}\big),$$

which converges to 0 as $n \to \infty$.

Verification of condition $(a_0)_\Sigma$ is a bit more intricate. Let us assume that $\beta < 1$. We have for $\epsilon > 0$, $\eta > 0$, $A > 0$, and $\alpha > 0$

$$\begin{aligned}
&\frac{n^2}{b_n^2}E_n\Big(\exp\Big(\alpha\frac{b_n}{n}|\zeta_1^n|\Big)\,\mathbf{1}\Big(\frac{b_n}{n}|\zeta_1^n| > \epsilon\Big)\,\mathbf{1}\Big(\frac{|\zeta_1^n|}{b_n} \le A\Big)\Big)\\
&\le \frac{n^2}{b_n^2}E_n\Big(\exp\Big(\alpha\frac{b_n}{n}\,|\zeta_1^n|\Big)\,\mathbf{1}\Big(\frac{b_n}{n}|\zeta_1^n|^{1-\beta} < \eta\Big)\,\mathbf{1}\Big(\frac{b_n}{n}|\zeta_1^n| > \epsilon\Big)\\
&\quad + \frac{n^2}{b_n^2}E_n\Big(\exp\Big(\alpha\frac{b_n}{n}\,|\zeta_1^n|\Big)\,\mathbf{1}\Big(\frac{b_n}{n}|\zeta_1^n|^{1-\beta} \ge \eta\Big)\,\mathbf{1}\Big(\frac{|\zeta_1^n|}{b_n} \le A\Big)\Big).
\end{aligned} \tag{4.4.11}$$

The first term on the right of (4.4.11) is not greater than

$$\begin{aligned}
&\frac{n^2}{b_n^2}E_n\Big(\exp\big(\alpha\eta|\zeta_1^n|^\beta\big)\,\mathbf{1}\big(\frac{b_n}{n}|\zeta_1^n| > \epsilon\big)\Big)\\
&\le \frac{n^2}{b_n^2}E_n\Big(\exp\big(\alpha\eta|\zeta_1^n|^\beta + \frac{1}{2}\gamma|\zeta_1^n|^\beta\big)\Big)\exp\Big(-\frac{1}{2}\gamma\epsilon^\beta\Big(\frac{n}{b_n}\Big)^\beta\Big)
\end{aligned} \tag{4.4.12}$$

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and converges to 0 as $n \to \infty$ if $\alpha\eta < \gamma/2$ by the moment condition in (4.4.10) and the assumption $n/b_n \to \infty$.

We estimate the second term on the right of (4.4.11) as

$$\frac{n^2}{b_n^2} E_n\Big(\exp\Big(\alpha \frac{b_n}{n}|\zeta_1^n|\Big)\, \mathbf{1}\Big(\frac{b_n}{n}|\zeta_1^n|^{1-\beta} \ge \eta\Big)\, \mathbf{1}\Big(\frac{|\zeta_1^n|}{b_n} \le A\Big)\Big)$$
$$\le \frac{n^2}{b_n^2} \exp\Big(\alpha A \frac{b_n^2}{n} - \gamma\eta^{\beta/(1-\beta)}\Big(\frac{n}{b_n}\Big)^{\beta/(1-\beta)}\Big) E_n\big(\exp(\gamma|\zeta_1^n|^\beta)\big),$$

which goes to 0 by (4.4.10).

Thus, the right-hand side of (4.4.11) goes to 0 as $n \to \infty$ so that condition $(a_0)_\Sigma$ is checked for $\beta < 1$. If $\beta = 1$, the required follows by (4.4.12). □

Remark 4.4.9. *If the distributions of the ζ_1^n do not depend on n, the moment conditions above imply condition* (4.4.2)*.*

Remark 4.4.10. *We have actually checked that under* (4.4.10) *condition $(a_0)_\Sigma$ holds for every $a > 0$.*

We now consider examples on LD convergence to different kinds of idempotent processes.

Example 4.4.11. *"Very large deviations"*

Let X^n be given by (4.4.1). Let $r_n = n$, which specifies the set-up of "very large deviations". We assume that $\xi_i^n = g(i/n, \zeta_i)/n$, where $\zeta_i, i \in \mathbb{N}$, are $\mathbb{R}^d$-valued i.i.d.r.v. on a probability space $(\Omega, \mathcal{F}, P)$, and $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous in the first variable and such that $E\exp(\alpha|g(t,\zeta_1)|) < \infty$ for all $\alpha > 0, t > 0$.

It is easy to see that all the conditions of Corollary 4.3.11 hold with

$$B_t' = \int_0^t Eg(s,\zeta_1)\, ds,\ C_t = 0,\ \nu_t(\Gamma) = \hat{\nu}_t(\Gamma) = P\big(g(t,\zeta_1) \in \Gamma\setminus\{0\}\big).$$

It is instructive to note that the Cramér condition is not indispensable in this sort of result. Indeed, let $\xi_i^n = \zeta_i^n/n$, where ζ_i^n, $i \in \mathbb{N}$, are $\mathbb{R}_+$-valued random variables, i.i.d. for each n with the distribution function

$$P(\zeta_1^n \le x) = 1 - \exp(-x^2)\,\mathbf{1}(x \le n^2) - n^2\exp(-n^4)\frac{\mathbf{1}(x > n^2)}{x}.$$

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Then conditions $(A)_\Sigma + (a)_\Sigma$ are easily seen to hold while neither condition $(VS)_\Sigma$ nor $(I_e)_\Sigma$ is satisfied, and even $E\zeta_1^n = \infty$.

The other conditions of Theorem 4.4.1 are satisfied as well with

$$B_t = t\int_0^\infty h(x)d(1-\exp(-x^2)),\ C_t = 0,$$

$$\nu_t(\Gamma) = \hat{\nu}_t(\Gamma) = \int_\Gamma \mathbf{1}(x>0)d(1-\exp(-x^2)),\ \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

Example 4.4.12. *LD convergence to Poisson idempotent processes.*

Let X^n be given by (4.4.1). Let $\xi_i^n = \zeta_i^n/r_n$, where $r_n \to \infty$, $r_n/n \to 0$ as $n \to \infty$, and $\{\zeta_i^n,\ i \in \mathbb{N}\}$ are independent r.v. assuming values 1 and 0 with respective probabilities r_n/n and $(1 - r_n/n)$. Then part b) of Corollary 4.3.12 implies that $X^n \overset{ld}{\to} \mathcal{N}$ as $n \to \infty$ at rate r_n, where $\mathcal{N}$ is a Poisson idempotent process. Note that here $\nu^n([0,t],\{r_n|x| > \epsilon\})/r_n \to t$ for $\epsilon < 1$, so condition (MD) does not hold, while condition (QC) does.

Example 4.4.13. *LD convergence of empirical processes.*

Let

$$X_t^n = \frac{1}{r_n}\sum_{i=1}^n \mathbf{1}\Big(\xi_i \le \frac{r_n}{n}\,t\Big),$$

where ξ_i are i.i.d.r.v. with values in $\mathbb{R}_+$, whose distribution admits density $g(x)$, which is continuous and positive at 0. Also $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$.

We denote by $G(x)$ the distribution function of ξ_1 and introduce the point process $N_t^n = r_n X_t^n$. Then the compensator of $N^n = (N_t^n, t \in \mathbb{R}_+)$ relative to the natural filtration is, Jacod and Shiryaev [67, II.3.32],

$$A_t^n = \int_0^t \Big(r_n - \frac{r_n}{n}\sum_{i=1}^n \mathbf{1}\Big(\xi_i \le \frac{r_n}{n}\,s\Big)\Big)\frac{g\Big(\dfrac{r_n}{n}s\Big)}{1-G\Big(\dfrac{r_n}{n}s\Big)}ds.$$

It is not difficult to check that

$$\frac{1}{n}\sum_{i=1}^n \mathbf{1}\Big(\xi_i \le \frac{r_n}{n}\,s\Big) \overset{P_n^{1/r_n}}{\to} 0$$

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as $n \to \infty$ and hence $A_t^n / r_n \overset{P_n^{1/r_n}}{\to} tg(0)$. Part b) of Corollary 4.3.12 implies that the $X^n \overset{ld}{\to} X$ as $n \to \infty$ at rate r_n, where $X_t = \mathcal{N}_{g(0)t}$, $\mathcal{N}$ being an idempotent Poisson process.

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Chapter 5

The method of the maxingale problem

The method of finite-dimensional distributions considered in Chapter 4 does not allow us to prove LD convergence in distribution to idempotent processes other than idempotent processes with independent increments. In this chapter we consider a different approach, which is an analogue of the martingale problem approach in weak convergence theory and consists in identifying the limit deviability as a solution to a maxingale problem. As in Chapter 4, we consider a net of semimartingales $\{X^\phi,\ \phi \in \Phi\}$ defined on respective stochastic bases $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$ with paths in $\mathbb{D} = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. We assume as fixed a net $\{r_\phi,\ \phi \in \Phi\}$ of real numbers greater than 1 converging to ∞ as $\phi \in \Phi$. It is used as a rate for LD convergences below, which refer to the Skorohod topology. The limit semimaxingale X is assumed to be "canonical" in that it is defined on $\mathbb{D}$ by $X_t(\mathbf{x}) = \mathbf{x}_t$, $\mathbf{x} \in \mathbb{D}$, $t \in \mathbb{R}_+$. It will actually be Luzin-continuous so that we can equivalently consider it as the canonical idempotent process on $\mathbb{C} = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$. The next two sections are concerned with identifying maxingale problems whose solutions are LD accumulution points of $\{\mathcal{L}(X^\phi),\ \phi \in \Phi\}$: Section 5.1 specifies the maxingale problem in terms of convergence of stochastic exponentials and assumes the Cramér condition for the X^ϕ, while Section 5.2 considers convergence of the characteristics of the semimaxingales and does without the Cramér condition. Section 5.3 is devoted to specific LD convergence results. Section 5.4 considers applications to large deviation convergence of Markov processes.

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5.1 Convergence of stochastic exponentials

This section contains results on LD convergence of semimartingales stated in terms of convergence of the associated stochastic exponentials.

Let $G(\lambda) = \bigl(G_t(\lambda;\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D}\bigr)$, $\lambda \in \mathbb{R}^d$, be an $\mathbb{R}$-valued function such that $G_0(\lambda;\mathbf{x}) = G_t(0;\mathbf{x}) = 0$, which is continuous in t and $\mathbf{D}$-adapted in $\mathbf{x}$. As above, we refer to $G(\lambda)$ as a cumulant. We introduce a number of conditions.

Definition 5.1.1. *The function $G(\lambda)$ is said to satisfy the uniform continuity condition if the map $\mathbf{x} \to (G_t(\lambda;\mathbf{x}),\, t \in \mathbb{R}_+)$ is a $\mathbb{C}$–continuous map from $\mathbb{D}$ into $\mathbb{C}(\mathbb{R}_+,\mathbb{R})$.*

Definition 5.1.2. *Let $F = (F_t(\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D}), F_0(\mathbf{x}) = 0$, be an $\mathbb{R}$-valued continuous function. We say that F satisfies the majoration condition if there exists an $\mathbb{R}$-valued, increasing and continuous function $\overline{F} = (\overline{F}_t, t \in \mathbb{R}_+), \overline{F}_0 = 0$, such that for all $0 \le s < t$,*

$$\sup_{\mathbf{x}\in\mathbb{D}} (F_t(\mathbf{x}) - F_s(\mathbf{x})) \le \overline{F}_t - \overline{F}_s. \tag{5.1.1}$$

The function F is said to satisfy the local majoration condition if, for each $b > 0$, there exists an $\mathbb{R}$-valued, increasing and continuous in t function $\overline{F}^b = (\overline{F}^b_t, t \in \mathbb{R}_+), \overline{F}^b_0 = 0$, such that, for all $0 \le s < t$,

$$\sup_{\substack{\mathbf{x}\in\mathbb{D}:\\ \mathbf{x}^*_\infty \le b}} (F_t(\mathbf{x}) - F_s(\mathbf{x})) \le \overline{F}^b_t - \overline{F}^b_s. \tag{5.1.2}$$

Remark 5.1.3. *If F is $\mathbf{D}$-adapted, then, being continuous, it is $\mathbf{D}$-predictable, so the preceding supremum may be taken over $\mathbf{x} \in \mathbb{D}$ such that $\mathbf{x}^*_{t-} \le b$. More generally, if τ is a finite $\mathbf{D}$–stopping time on $\mathbb{D}$, then, for every finite $\mathbf{D}$–stopping time $\sigma \le \tau$,*

$$\sup_{\substack{\mathbf{x}\in\mathbb{D}:\\ \mathbf{x}^*_\infty \le b}} (F_\tau(\mathbf{x}) - F_\sigma(\mathbf{x})) = \sup_{\substack{\mathbf{x}\in\mathbb{D}:\\ \mathbf{x}^*_{\tau-} \le b}} (F_\tau(\mathbf{x}) - F_\sigma(\mathbf{x}))$$

(See, e.g., Jacod and Shiryaev [67, III.2.43].)

At times we require that the restriction of $G(\lambda)$ to $\mathbb{C}$ satisfy the linear-growth condition of Definition 2.8.11, which we recall here.

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Definition 5.1.4. *We say that $G(\lambda)$ satisfies the linear-growth condition if there exist $\mathbb{R}_+$–valued, increasing and continuous in t functions $F^l(\lambda) = (F^l_t(\lambda), t \in \mathbb{R}_+)$, $\lambda \in \mathbb{R}^d$, such that $F^l_0(\lambda) = F^l_t(0) = 0$ and for some $\mathbb{R}_+$-valued increasing function k_t we have for all $0 \le s < t$, $\mathbf{x} \in \mathbb{C}$ and $\lambda \in \mathbb{R}^d$*

$$G_t(\lambda; \mathbf{x}) - G_s(\lambda; \mathbf{x}) \le F^l_t(\lambda(1 + k_t \mathbf{x}^*_t)) - F^l_s(\lambda(1 + k_t \mathbf{x}^*_t)).$$

We also recall that a deviability Π on $\mathbb{C}$ is a solution to the maxingale problem (x_0, G), where $x_0 \in \mathbb{R}^d$, if the canonical process X on $\mathbb{C}$ is a semimaxingale with cumulant $G(\lambda)$ on $(\mathbb{C}, \mathbf{C}, \Pi)$ such that $X_0 = x_0$ Π-a.e. (Definition 2.8.1).

Let the X^ϕ satisfy the Cramér condition (Cr) and $\mathcal{E}^\phi(\lambda) = \big(\mathcal{E}^\phi_t(\lambda),\, t \in \mathbb{R}_+\big)$, $\lambda \in \mathbb{R}^d$, be the associated stochastic exponentials. The following conditions on the X^ϕ are similar to those used in Section 4.1:

(0) $$X^\phi_0 \overset{P_\phi^{1/r_\phi}}{\to} x_0 \quad \text{as } \phi \in \Phi,$$

(sup $\mathcal{E}$) $$\sup_{t \le T} \Big| \frac{1}{r_\phi} \ln \mathcal{E}^\phi_t(r_\phi \lambda) - G_t(\lambda; X^\phi) \Big| \overset{P_\phi^{1/r_\phi}}{\to} 0 \quad \text{as } \phi \in \Phi, \quad \lambda \in \mathbb{R}^d, T > 0.$$

Theorem 5.1.5. *Let the X^ϕ satisfy (Cr), and $G(\lambda)$, for each $\lambda \in \mathbb{R}^d$, satisfy the uniform continuity and majoration conditions. If conditions* (0) *and* (sup $\mathcal{E}$) *hold, then the net $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight and its every LD accumulation point solves the maxingale problem (x_0, G).*

Remark 5.1.6. *By the fact that a cumulant $G(\lambda)$ that does not depend on* $\mathbf{x}$ *satisfies the uniform continuity and majoration conditions, Theorems 5.1.5 and 2.8.5 imply Theorem 4.1.2.*

The majoration condition on $G(\lambda)$ is too restrictive in applications. We replace it next by the local majoration condition and another condition. Recall that $\mathbf{\Pi}_{x_0}$ is defined by (2.7.6) and $\mathbf{\Pi}_{x,t}$ by (2.8.6). The following is the condition we will require.

(NE) The function $\mathbf{\Pi}_x(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}$, is upper compact and the sets $\cup_{s \in [0,t]} \{\mathbf{x}^*_s : \mathbf{\Pi}_{x,s}(\mathbf{x}) \ge a\}$ are bounded for $a \in (0, 1]$ and $t \in \mathbb{R}_+$.

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Remark 5.1.7. *Condition (NE) implies that $\mathbf{\Pi}_{x_0}$ is a tight τ-smooth idempotent measure on $\mathbb{C}$.*

Remark 5.1.8. *By Lemma 2.8.12 and Remark 2.8.13 condition (NE) is met when $G(\lambda)$ satisfies the linear-growth condition.*

Let us define for $\mathbf{x} \in \mathbb{D}$

$$\tau_N(\mathbf{x}) = \inf\{t \in \mathbb{R}_+ : \mathbf{x}_t^* + t \geq N\},\ N \in \mathbb{N}. \tag{5.1.3}$$

The next version of Lemma 2.7.5, which is proved by a similar argument, implies that τ_N is a $\mathbf{D}$–stopping time and is $\mathbb{C}$–continuous.

Lemma 5.1.9. *Let $(H_t(\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ be an $\mathbb{R}_+$-valued $\mathbf{D}$–adapted function, which is continuous and increasing in t and $\mathbb{C}$–continuous in $\mathbf{x}$. Let for $c \in \mathbb{R}_+$*

$$\tau(\mathbf{x}) = \inf\{t \in \mathbb{R}_+ : H_t(\mathbf{x}) + t \geq c\}.$$

Then $\tau(\mathbf{x}), \mathbf{x} \in \mathbb{D}$, is a $\mathbf{D}$–stopping time and is $\mathbb{C}$-continuous.

The following condition is a localised version of $(\sup \mathcal{E})$.

$$(\sup \mathcal{E})_{loc} \qquad \sup_{t \leq T} \Big|\frac{1}{r_\phi} \ln \mathcal{E}^\phi_{t\wedge\tau_N(X^\phi)}(r_\phi\lambda) - G_{t\wedge\tau_N(X^\phi)}(\lambda; X^\phi)\Big| \overset{P_\phi^{1/r_\phi}}{\to} 0$$
$$\text{as } \phi \in \Phi, \quad \lambda \in \mathbb{R}^d, T > 0, N \in \mathbb{N}.$$

Theorem 5.1.10. *Let the X^ϕ satisfy (Cr), $G(\lambda)$, for each $\lambda \in \mathbb{R}^d$, satisfy the uniform continuity and local majoration conditions, and (NE) hold. If conditions (0) and $(\sup \mathcal{E})_{loc}$ hold, then the net $\{\mathcal{L}(X^\phi),\ \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight and its every LD accumulation point solves the maxingale problem (x_0, G).*

Remark 5.1.11. *The uniform continuity and majoration conditions used above can be somewhat modified. Let us say that $G(\lambda)$ satisfies the continuity condition for a given $\lambda \in \mathbb{R}^d$ if $G_t(\lambda; \mathbf{x})$ is $\mathbb{C}$–continuous in $\mathbf{x}$ for all t from a dense subset of $\mathbb{R}_+$. Let us say that an $\mathbb{R}$-valued function $F = (F_t(\mathbf{x}),\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ obeys the strict majoration condition if* (5.1.1) *holds with the increments on the left-hand side replaced by their absolute values. Similarly, we can define the local strict majoration condition by taking absolute values on the left of* (5.1.2). *Since the local strict majoration condition*

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and the continuity condition imply the uniform continuity condition for $G(\lambda)$, we could in Theorem 5.1.5 (respectively, Theorem 5.1.10) require only the continuity condition if we strengthened the majoration condition (respectively, local majoration condition) to the strict majoration condition (respectively, local strict majoration condition).

Since the linear-growth condition on $G(\lambda)$ implies both the local majoration condition and (NE), we obtain the following important consequence of Theorem 5.1.10.

Theorem 5.1.12. *Let the X^ϕ satisfy (Cr) and the cumulant $G(\lambda)$, for each $\lambda \in \mathbb{R}^d$, satisfy the uniform continuity and linear-growth conditions. If conditions* (0) *and* $(\sup \mathcal{E})_{loc}$ *hold, then the net $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight and its every LD accumulation point solves the maxingale problem (x_0, G).*

The proofs of Theorems 5.1.5, 5.1.10, and 5.1.12 below show that the only property of the processes $\mathcal{E}^\phi$ that matters, besides being positive and predictable, is that they satisfy the assertion of Lemma 4.1.1. This observation allows us, as in Chapter 4, to extend the theorems to the case when the processes X^ϕ are not necessarily semimartingales if we postulate the property stated in Lemma 4.1.1. More specifically, let us consider the following condition on processes X^ϕ with paths in $\mathbb{D}$ defined on stochastic bases $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$.

(E) For each $\phi \in \Phi$, there exist $\mathbf{F}_\phi$–predictable positive processes $\mathcal{E}^\phi(\lambda) = (\mathcal{E}^\phi_t(\lambda),\, t \in \mathbb{R}_+),\ \lambda \in \mathbb{R}^d$, such that $\mathcal{E}^\phi_0(\lambda) = 1$ and the processes $\Big(\exp(\lambda \cdot (X^\phi_t - X^\phi_0))\mathcal{E}^\phi_t(\lambda)^{-1}, t \in \mathbb{R}_+\Big)$ are $\mathbf{F}_\phi$–local martingales.

Then we have the following extension of Theorem 5.1.12.

Theorem 5.1.13. *Let X^ϕ be stochastic processes on $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$, which satisfy condition (E), and let $G(\lambda)$, for each $\lambda \in \mathbb{R}^d$, satisfy the uniform continuity and linear-growth conditions. If conditions* (0) *and* $(\sup \mathcal{E})_{loc}$ *hold, then the net $\{\mathcal{L}(X^\phi),\, \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight and its every LD accumulation point solves the maxingale problem (x_0, G).*

Theorems 5.1.5 and 5.1.10 admit similar versions.

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Remark 5.1.14. *In Theorems 5.1.5, 5.1.10, 5.1.12, and 5.1.13 we can equivalently describe the accumulation points by saying that if* Π *is an accumulation point of* $\{\mathcal{L}(X^\phi),\ \phi \in \Phi\}$*, then the canonical process* X *on* $(\mathbb{C}, \mathbf{C}, \Pi)$ *is a Luzin-continuous semimaxingale with cumulant* $G(\lambda)$ *starting at* x_0*. Also if* $G(\lambda)$ *and* x_0 *uniquely specify* Π*, then* $X^\phi \xrightarrow{ld} X$*.*

5.1.1 Proofs

In the proofs below we assume with no loss of generality that $x_0 = 0$. We start with two preliminary lemmas. We assume the conditions imposed on the X^ϕ at the beginning of the section. As above we denote by e_i, $i = 1, \ldots, 2d$, the d-vector, whose $\lfloor (i+1)/2 \rfloor$th entry equals 1 if i is odd and -1 if i is even, the rest of the entries being equal to 0.

Lemma 5.1.15. *For every finite* $\mathbf{F}_\phi$*–stopping time* τ*,* $a > 0$*,* $b > 0$*,* $c > 0$*, and* $u \in \mathbb{R}_+$ *the following inequalities hold*

$$\begin{aligned}
P_\phi\bigl(\sup_{t \le u} |X^\phi_{t+\tau} - X^\phi_\tau| \ge a\bigr) &\le 2d \exp\Bigl(\frac{c}{d}(b-a)\Bigr) \\
&+ 2d \max_{i=1,\ldots,2d} P_\phi\Bigl(\sup_{t \le u} \frac{1}{c}(\ln \mathcal{E}^\phi_{t+\tau}(ce_i) - \ln \mathcal{E}^\phi_\tau(ce_i)) > \frac{b}{d}\Bigr) \\
&\le 2d \exp\Bigl(\frac{c}{d}(b-a)\Bigr) \\
&\qquad + 2d \max_{i=1,\ldots,2d} P_\phi\Bigl(\sup_{t \le u} \frac{1}{c}(G^\phi_{t+\tau}(ce_i) - G^\phi_\tau(ce_i)) > \frac{b}{d}\Bigr).
\end{aligned}$$

Proof. The second inequality is implied by the first since by (4.1.15)

$$\ln \mathcal{E}^\phi_{t+\tau}(ce_i) - \ln \mathcal{E}^\phi_\tau(ce_i) \le G^\phi_{t+\tau}(ce_i) - G^\phi_\tau(ce_i).$$

The first inequality results from Lemma 3.2.6. Specifically, let

$$Z^{\phi,\tau}_t(\lambda) = Y^\phi_{t+\tau}(\lambda)/Y^\phi_\tau(\lambda),\ t \in \mathbb{R}_+,$$

where $Y^\phi_t(\lambda)$ is defined by (4.1.18). By Lemma 4.1.1 and Doob's stopping theorem $Z^{\phi,\tau}(\lambda) = (Z^{\phi,\tau}_t(\lambda),\ t \in \mathbb{R}_+)$ is an $\mathbb{R}_+$-valued local martingale with respect to the filtration $\mathbf{F}_{\phi,\tau} = (\mathcal{F}^\phi_{t+\tau}, t \in \mathbb{R}_+)$;

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hence, $EZ^{\phi,\tau}_\sigma(\lambda) \le 1$ for every $\mathbf{F}_{\phi,\tau}$–stopping time σ. Lemma 3.2.6 and the definition of $Y^\phi(\lambda)$ then yield for $i = 1, \dots, 2d$

$$P_\phi\Big(\sup_{t\le u} e_i \cdot (X^\phi_{t+\tau} - X^\phi_\tau) \ge \frac{a}{d}\Big) \le \exp\Big(\frac{c}{d}(b-a)\Big) + P_\phi\Big(\sup_{t\le u} \frac{1}{c}(\ln \mathcal{E}^\phi_{t+\tau}(ce_i) - \ln \mathcal{E}^\phi_\tau(ce_i)) > \frac{b}{d}\Big),$$

hence,

$$\begin{aligned} &P_\phi\big(\sup_{t\le u} |X^\phi_{t+\tau} - X^\phi_\tau| \ge a\big) \\ &\le 2d \max_{i=1,\dots,2d} P_\phi\Big(\sup_{t\le u} e_i \cdot (X^\phi_{t+\tau} - X^\phi_\tau) \ge \frac{a}{d}\Big) \\ &\le 2d \exp\Big(\frac{c}{d}(b-a)\Big) \\ &\qquad + 2d \max_{i=1,\dots,2d} P_\phi\Big(\sup_{t\le u} \frac{1}{c}(\ln \mathcal{E}^\phi_{t+\tau}(ce_i) - \ln \mathcal{E}^\phi_\tau(ce_i)) > \frac{b}{d}\Big). \end{aligned}$$

□

Next comes one of the most technically important results of the chapter. It is more general than is required at the moment for the proofs of Theorems 5.1.5, 5.1.10 and 5.1.12, but this generality will be exploited while proving Theorems 5.2.9, 5.2.12, and 5.2.15 below.

Let $\{X'^\phi, \phi \in \Phi\}$, where $X'^\phi = (X'^\phi_t,\, t \in \mathbb{R}_+)$, be along with X^ϕ a net of $\mathbb{R}^d$–valued semimartingales defined on $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$. We consider the pair (X^ϕ, X'^ϕ) as a process with paths in the Skorohod space $\mathbb{D}' = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^d)$. The space $\mathbb{D}'$ is equipped with the natural flow of σ-algebras $\mathbf{D}' = (\mathcal{D}'_t,\, t \in \mathbb{R}_+)$, defined in analogy with $\mathbf{D}$, and elements of $\mathbb{D}'$ are denoted by $(\mathbf{x}, \mathbf{x}')$. We denote by $\mathbb{C}' = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^d)$ the subspace of $\mathbb{D}'$ of continuous functions equipped with the τ-flow $\mathbf{C}' = (\mathcal{C}'_t, t \in \mathbb{R}_+)$ as defined in Section 3.2. For $(\mathbf{x}, \mathbf{x}') \in \mathbb{D}'$ and $\lambda \in \mathbb{R}^d$, we introduce

$$Y'_t(\lambda; (\mathbf{x}, \mathbf{x}')) = \exp(\lambda \cdot \mathbf{x}_t - G_t(\lambda; \mathbf{x}')),\ t \in \mathbb{R}_+, \tag{5.1.4}$$

and let

$$Y'(\lambda) = (Y'_t(\lambda; (\mathbf{x}, \mathbf{x}')), t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \mathbb{C}'). \tag{5.1.5}$$

Deviability Π' on $\mathbb{C}'$ is said to be a solution of maxingale problem (M') if

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$$(M') \qquad \begin{array}{ll} \mathbf{x}_0 = 0 & \Pi' \text{ – a.e.,} \\ Y'(\lambda), \lambda \in \mathbb{R}^d, & \text{is a } \mathbf{C}' \text{ –local exponential maxingale on} \\ & (\mathbb{C}', \Pi'). \end{array}$$

Theorem 5.1.16. *Let $G(\lambda)$ satisfy the uniform continuity condition. If the net $\{\mathcal{L}((X^\phi, X'^\phi)), \phi \in \Phi\}$ is $\mathbb{C}'$–exponentially tight, and conditions* (0) *and*

$$(\sup\ \mathcal{E})' \qquad \sup_{t \le T} \Big|\frac{1}{r_\phi} \ln \mathcal{E}_t^\phi(r_\phi \lambda) - G_t(\lambda; X'^\phi)\Big| \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \ as\ \phi \in \Phi, \quad T > 0, \lambda \in \mathbb{R}^d,$$

hold, then every LD accumulation point of $\{\mathcal{L}((X^\phi, X'^\phi)), \phi \in \Phi\}$ (when restricted to $\mathbb{C}'$) is a solution to (M').

Proof. Let Π' be an LD accumulation point of $\{\mathcal{L}((X^\phi, X'^\phi)), \phi \in \Phi\}$. To simplify notation, we assume that $\mathcal{L}((X^\phi, X'^\phi)) \overset{ld}{\to} \Pi'$ as $\phi \in \Phi$. By $\mathbb{C}'$–exponential tightness of $\{\mathcal{L}((X^\phi, X'^\phi)), \phi \in \Phi\}$ the deviability Π' is supported by $\mathbb{C}'$, so it can be considered as a deviability on $\mathbb{C}'$.

We show that $\Pi'((\mathbf{x}, \mathbf{x}') : \mathbf{x}_0 \neq 0) = 0$. Since the map $\pi_0 : (\mathbf{x}, \mathbf{x}') \to \mathbf{x}_0$ from $\mathbb{D}'$ into $\mathbb{R}^d$ is continuous, by the contraction principle

$$\mathcal{L}(X_0^\phi) \overset{ld}{\to} \Pi' \circ \pi_0^{-1} \ \text{ as } \phi \in \Phi,$$

and then by (0) and the definition of LD convergence

$$\Pi' \circ \pi_0^{-1}(x) = \mathbf{1}(x = 0),\ x \in \mathbb{R}^d,$$

which is equivalent to the required.

Now we prove that the $Y'(\lambda), \lambda \in \mathbb{R}^d$, are $\mathbf{C}'$ –local exponential maxingales on $(\mathbb{C}', \Pi')$. We do that by reduction to Theorem 3.2.9. As above we denote $G_t^*(\lambda, \mathbf{x}') = \sup_{s \le t} |G_s(\lambda, \mathbf{x}')|$. By the uniform continuity condition on $G(\lambda)$ the function $G_t^*(\lambda, \mathbf{x}')$ is $\mathbb{C}$–continuous in $\mathbf{x}' \in \mathbb{D}$ for each $t \in \mathbb{R}_+$. For $N \in \mathbb{N}$ and $\mathbf{x}' \in \mathbb{D}$ we introduce

$$\alpha_N^\phi(\mathbf{x}') = \inf\{t \in \mathbb{R}_+ : G_t^*(\lambda, \mathbf{x}') \vee G_t^*(2\lambda, \mathbf{x}') + t \ge N\}. \qquad (5.1.6)$$

By Lemma 5.1.9 $\alpha_N^\phi(\mathbf{x}')$, $\mathbf{x}' \in \mathbb{D}$, is a finite $\mathbf{D}$–stopping time and is $\mathbb{C}$–continuous. Therefore, α_N^ϕ, as a function on $\mathbb{D}'$, is a $\mathbf{D}'$–stopping time and is $\mathbb{C}'$–continuous.

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Let also, for $N \in \mathbb{N}$ and $\phi \in \Phi$,

$$\beta_N^\phi = \inf\big\{t \in \mathbb{R}_+ : \mathcal{E}_t^\phi(r_\phi\lambda)^{1/r_\phi} \vee \mathcal{E}_t^\phi(r_\phi\lambda)^{-1/r_\phi} \vee \mathcal{E}_t^\phi(2r_\phi\lambda)^{1/r_\phi} \vee \mathcal{E}_t^\phi(2r_\phi\lambda)^{-1/r_\phi} \geq 2e^N\big\}. \quad (5.1.7)$$

Then by $\mathbf{F}_\phi$–predictability and right continuity of $\mathcal{E}_t^\phi(\lambda)$, β_N^ϕ is an $\mathbf{F}_\phi$–predictable stopping time (see Dellacherie [34, IV.T.16]), and by (5.1.6) and $(\sup \mathcal{E})'$

$$\lim_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\beta_N^\phi \leq \alpha_N^\phi(X'^\phi)\big) = 0.$$

The facts that β_N^ϕ is $\mathbf{F}_\phi$–predictable and $\beta_N^\phi > 0$ P_ϕ–a.s. (since $\mathcal{E}_0^\phi(\lambda) = 1$) imply as in the proof of Theorem 4.1.2 that there exist finite $\mathbf{F}_\phi$–stopping times σ_N^ϕ such that

$$\sigma_N^\phi < \beta_N^\phi \quad P_\phi\text{–a.s.} \quad (5.1.8)$$

and

$$\lim_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\sigma_N^\phi \leq \alpha_N^\phi(X'^\phi)\big) = 0. \quad (5.1.9)$$

Note that by (5.1.7) and (5.1.8) for $t \in \mathbb{R}_+$ P_ϕ-a.s.

$$\mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda) \vee \mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{-1} \vee \mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(2r_\phi\lambda) \vee \mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(2r_\phi\lambda)^{-1} < 2^{r_\phi}e^{r_\phi N}. \quad (5.1.10)$$

Now, since Lemma 4.1.1 implies that $Y^\phi(\lambda)$ is a supermartingale so $EY_\tau^\phi(\lambda) \leq 1$ for every finite $\mathbf{F}_\phi$–stopping time τ, we have by (5.1.10) and the definition of $Y^\phi(\lambda)$ in (4.1.18) that

$$E_\phi\Big(Y^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^2\Big) = E_\phi\Big(Y^\phi_{t\wedge\sigma_N^\phi}(2r_\phi\lambda)\mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(2r_\phi\lambda)\mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{-2}\Big) \leq 2^{3r_\phi}e^{3Nr_\phi}.$$

Thus, in view of Doob's stopping theorem, $(Y^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda), t \in \mathbb{R}_+)$ is a square-integrable martingale and for every $\mathbf{F}_\phi$–stopping time τ

$$E_\phi^{1/r_\phi}\big(Y^\phi_{\tau\wedge\sigma_N^\phi}(r_\phi\lambda)^2\big) \leq 8e^{3N}. \quad (5.1.11)$$

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Next, by the respective definitions (4.1.18) and (5.1.4) of $Y_t^\phi(\lambda)$ and $Y_t'(\lambda;(\mathbf{x},\mathbf{x}'))$, and the inequality $|e^u-1|\le |u|e^{|u|}$, $u\in\mathbb{R}$, we have that for $A>0,\varepsilon>0,\eta>0$, and $T>0$

$$\begin{aligned}
&P_\phi\Big(\sup_{t\le T}\big|Y'_{t\wedge\sigma_N^\phi}(\lambda;(X^\phi,X'^\phi))-Y^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{1/r_\phi}\big|>\varepsilon\Big)\\
&\le P_\phi\big(\sup_{t\le T}|\lambda||X_t^\phi|>A\big)+P_\phi(|X_0^\phi|>\eta)\\
&+P_\phi\Big(\sup_{t\le T}\mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{-1/r_\phi}>\frac{\varepsilon}{2}e^{-A}(|\lambda|\eta)^{-1}e^{-|\lambda|\eta}\Big)\\
&+P_\phi\Big(\sup_{t\le T}\big|\exp(-G_{t\wedge\sigma_N^\phi}(\lambda;X'^\phi))-\mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{-1/r_\phi}\big|>\frac{\varepsilon}{2}e^{-A}\Big).
\end{aligned}\tag{5.1.12}$$

We prove that the right-hand side converges super-exponentially to 0 as $\phi\in\Phi$. By $\mathbb{C}'$–exponential tightness of $\{\mathcal{L}((X^\phi,X'^\phi)),\phi\in\Phi\}$ and Theorem 3.2.3

$$\lim_{A\to\infty}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}(\sup_{t\le T}|\lambda||X_t^\phi|>A)=0,\tag{5.1.13}$$

by (0)

$$\lim_{\phi\in\Phi}P_\phi^{1/r_\phi}(|X_0^\phi|>\eta)=0,\tag{5.1.14}$$

and by (5.1.10)

$$\lim_{\eta\to 0}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(\sup_{t\le T}\mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{-1/r_\phi}>\frac{\varepsilon}{2}e^{-A}(|\lambda|\eta)^{-1}e^{|\lambda|\eta}\Big)=0.\tag{5.1.15}$$

Finally, (5.1.10) and (sup $\mathcal{E}$)′ are easily seen to imply that

$$\lim_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(\sup_{t\le T}\big|\exp(-G_{t\wedge\sigma_N^\phi}(\lambda;X'^\phi))-\mathcal{E}^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{-1/r_\phi}\big|>\frac{\varepsilon}{2}e^{-A}\Big)=0.\tag{5.1.16}$$

By (5.1.13)–(5.1.16) the right-hand side of (5.1.12) raised to the power of $1/r_\phi$ goes to 0 in the limit $\lim_{A\to\infty}\limsup_{\eta\to 0}\limsup_{\phi\in\Phi}$.

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Thus, we have proved that as $\phi \in \Phi$

$$\sup_{t \le T} \left| Y'_{t\wedge\sigma_N^\phi}(\lambda;(X^\phi, X'^\phi)) - Y^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda)^{1/r_\phi} \right| \overset{P_\phi^{1/r_\phi}}{\to} 0, \ T > 0.$$

As a consequence, introducing

$$\gamma_N^\phi = \sigma_N^\phi \wedge \alpha_N^\phi(X'^\phi), \tag{5.1.17}$$

we have that for $t \in \mathbb{R}_+$

$$Y'_{t\wedge\gamma_N^\phi}(\lambda;(X^\phi, X'^\phi)) - Y^\phi_{t\wedge\gamma_N^\phi}(r_\phi\lambda)^{1/r_\phi} \overset{P_\phi^{1/r_\phi}}{\to} 0 \quad \text{as } \phi \in \Phi;$$

hence, since by (5.1.9) and (5.1.17)

$$\lim_{\phi\in\Phi} P_\phi^{1/r_\phi}\left(\alpha_N^\phi(X'^\phi) \neq \gamma_N^\phi\right) = 0,$$

we arrive at the convergence

$$Y'_{t\wedge\alpha_N^\phi(X'^\phi)}(\lambda;(X^\phi, X'^\phi)) - Y^\phi_{t\wedge\gamma_N^\phi}(r_\phi\lambda)^{1/r_\phi} \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi. \tag{5.1.18}$$

Now we check the conditions of Theorem 3.2.9 with $\mathbb{D}'$ as $\mathbb{D}$, (X^ϕ, X'^ϕ) as X^ϕ, $Y_N^\phi(\lambda) = (Y^\phi_{t\wedge\gamma_N^\phi}(r_\phi\lambda), t \in \mathbb{R}_+)$ as M^ϕ, $(\mathbf{x}, \mathbf{x}')$ as $\mathbf{x}$, and $Y'_{t\wedge\alpha_N^\phi(\mathbf{x}')}(\lambda;(\mathbf{x}, \mathbf{x}'))$ as $M_t(\mathbf{x})$.

The net $\{\mathcal{L}((X^\phi, X'^\phi)), \phi \in \Phi\}$ is $\mathbb{C}'$–exponentially tight by hypotheses. Next, since $\alpha_N^\phi(\mathbf{x}')$ is a $\mathbf{D}$–stopping time and X'^ϕ is $\mathbf{F}_\phi$–adapted, $\alpha_N^\phi(X'^\phi)$ is an $\mathbf{F}_\phi$–stopping time; since σ_N^ϕ also is an $\mathbf{F}_\phi$–stopping time, we conclude in view of (5.1.17) that γ_N^ϕ is an $\mathbf{F}_\phi$–stopping time. Therefore, recalling that $(Y^\phi_{t\wedge\sigma_N^\phi}(r_\phi\lambda), t \in \mathbb{R}_+)$ is a square-integrable martingale with respect to $\mathbf{F}_\phi$ and $\gamma_N^\phi \le \sigma_N^\phi$, we have that $Y_N^\phi(\lambda)$ is a square-integrable martingale too. Moreover, in view of (5.1.11), the net $\{Y^\phi_{t\wedge\gamma_N^\phi}(r_\phi\lambda)^{1/r_\phi}, \phi \in \Phi\}$ is uniformly exponentially integrable relative to $\{P_\phi\}$ for all $t \in \mathbb{R}_+$. Also $Y'_{t\wedge\alpha^\phi(\mathbf{x}')}(\lambda;(\mathbf{x}, \mathbf{x}'))$ is $\mathbb{C}'$–continuous by the fact that, as we remarked earlier, $\alpha_N^\phi(\mathbf{x}')$ is $\mathbb{C}'$–continuous, and $Y'_t(\lambda;(\mathbf{x}, \mathbf{x}'))$ is continuous in

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$(t, (\mathbf{x}, \mathbf{x}'))$ at $(\mathbf{x}, \mathbf{x}') \in \mathbb{C}'$ by the uniform continuity condition on $G(\lambda)$ and (5.1.4).

Finally, since $\alpha_N^\phi(\mathbf{x}')$ is a $\mathbf{D}'$–stopping time, it is a $\mathbf{C}'$–stopping time if restricted to $\mathbb{C}'$. Therefore, $Y'_{t\wedge\alpha^\phi(\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))$ is $\mathcal{C}'_t$–measurable by Lemma 2.2.19. Since also (5.1.18) holds, we conclude that the conditions of Theorem 3.2.9 are met with the above choice of M^ϕ, X^ϕ and $M_t(\mathbf{x})$. The theorem implies that the function $(Y'_{t\wedge\alpha_N^\phi(\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}')), t \in \mathbb{R}_+)$ is a $\mathbf{C}'$–exponential maxingale on $(\mathbb{C}', \Pi')$.

Π'–uniform maximability of $(Y'_{t\wedge\alpha_N^\phi(\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}')), t \in \mathbb{R}_+)$ is proved in analogy with (5.1.11). Since by (5.1.6) and continuity of $G_t(\lambda; \mathbf{x})$ in t we have that $|G_{t\wedge\alpha_N^\phi(\mathbf{x}')}(\lambda; \mathbf{x}')| \le N$ and $|G_{t\wedge\alpha_N^\phi(\mathbf{x}')}(2\lambda; \mathbf{x}')| \le N$, it follows by (5.1.4) that

$$\sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} Y'_{t\wedge\alpha_N^\phi(\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))^2 \Pi'((\mathbf{x}, \mathbf{x}'))$$
$$\le e^{3N} \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} Y'_{t\wedge\alpha_N^\phi(\mathbf{x}')}(2\lambda; (\mathbf{x}, \mathbf{x}'))\Pi'((\mathbf{x}, \mathbf{x}')) = e^{3N},$$

where the latter equality holds by the maxingale property of $(Y'_{t\wedge\alpha_N^\phi(\mathbf{x}')}(2\lambda; (\mathbf{x}, \mathbf{x}')), t \in \mathbb{R}_+)$. Thus, $(Y'_{t\wedge\alpha_N^\phi(\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}')), t \in \mathbb{R}_+)$ is Π'–uniformly maximable by Corollary 1.4.15. ☐

We are now in a position to prove Theorem 5.1.5.

Proof of Theorem 5.1.5. We apply Theorem 5.1.16 with $X'^\phi = X^\phi$. All we need to prove is that under the conditions of Theorem 5.1.5 the net $\{\mathcal{L}((X^\phi, X^\phi)), \phi \in \Phi\}$ is $\mathbb{C}'$–exponentially tight in $\mathbb{D}'$ or, equivalently, the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight in $\mathbb{D}$. We check the $\mathbb{C}$–exponential tightness by verifying the conditions of part II of Theorem 3.2.3. This is carried out similarly to the argument used in the proof of Theorem 4.2.11 with the use of Lemma 5.1.15. We consider only condition II(ii) of Theorem 3.2.3, because II(i) is checked in an analogous manner.

By Lemma 5.1.15 for $T > 0, \eta > 0, c > 0, 0 < \delta < 1$, and $\tau \in$

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$\mathbf{S}_T(\mathbf{F}_\phi)$

$$P_\phi\big(\sup_{t\le\delta}|X^\phi_{t+\tau}-X^\phi_\tau|>\eta\big)\le 2d\exp\Big(-\frac{c\eta r_\phi}{2d}\Big)$$
$$+\,2d\max_{i=1,\ldots,2d}P_\phi\Big(\sup_{t\le\delta}\frac{1}{cr_\phi}\big(\ln\mathcal{E}^\phi_{t+\tau}(r_\phi ce_i)-\ln\mathcal{E}^\phi_\tau(r_\phi ce_i)\big)\ge\frac{\eta}{2d}\Big)$$
$$\le 2d\max_{i=1,\ldots,2d}P_\phi\Big(\sup_{\substack{s,t\le T+1\\ s\le t\le s+\delta}}\frac{1}{r_\phi}\big(\ln\mathcal{E}^\phi_t(r_\phi ce_i)-\ln\mathcal{E}^\phi_s(r_\phi ce_i)\big)\ge\frac{\eta c}{2d}\Big)$$
$$+\,2d\exp\Big(-\frac{c\eta r_\phi}{2d}\Big).\quad(5.1.19)$$

Applying successively $(\sup\mathcal{E})$ and the majoration condition on $G(\lambda)$, we have for $i=1,\ldots,2d$

$$\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(\sup_{\substack{s,t\le T+1\\ s\le t\le s+\delta}}\frac{1}{r_\phi}\big(\ln\mathcal{E}^\phi_t(r_\phi ce_i)-\ln\mathcal{E}^\phi_s(r_\phi ce_i)\big)\ge\frac{\eta c}{2d}\Big)$$
$$\le\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(\sup_{\substack{s,t\le T+1\\ s\le t\le s+\delta}}\big(G_t(ce_i;X^\phi)-G_s(ce_i;X^\phi)\big)\ge\frac{\eta c}{3d}\Big)$$
$$\le\mathbf{1}\Big(\sup_{\substack{s,t\le T+1\\ s\le t\le s+\delta}}\big(\overline{G}^i_t-\overline{G}^i_s\big)\ge\frac{\eta c}{3d}\Big),$$

where $\overline{G}^i=(\overline{G}^i_t,t\in\mathbb{R}_+)$ is a function majorising $G(ce_i)$. By continuity of $\overline{G}^i_t$ in t the latter indicator is 0 for all small $\delta>0$. Hence, by (5.1.19)

$$\limsup_{\delta\to0}\limsup_{\phi\in\Phi}\sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)}P_\phi^{1/r_\phi}\big(\sup_{t\le\delta}|X^\phi_{t+\tau}-X^\phi_\tau|>\eta\big)\le\exp\Big(-\frac{c\eta}{2d}\Big).$$

Since c is arbitrarily large, condition II(ii) of Theorem 3.2.3 has been checked. □

For a proof of Theorem 5.1.10, we need another auxiliary result which will also be used in the proof of Theorem 5.2.12. Let the maps $\tilde{p}_N:\mathbb{D}\to\mathbb{D}$, $N\in\mathbb{N}$, be defined by

$$(\tilde{p}_N\mathbf{x})_t=\mathbf{x}_{t\wedge\tau_N(\mathbf{x})},\ \mathbf{x}\in\mathbb{D},t\in\mathbb{R}_+,\qquad(5.1.20)$$

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where the τ_N are from (5.1.3). The maps $\tilde{p}_N$ are $\mathbb{C}$–continuous since the τ_N are $\mathbb{C}$–continuous and Skorohod convergence to continuous functions is equivalent to locally uniform convergence.

Let also

$$X^{\phi,N} = \tilde{p}_N X^{\phi} \tag{5.1.21}$$

and $Y^N(\lambda) = (Y_t^N(\lambda;\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ be defined by

$$Y_t^N(\lambda;\mathbf{x}) = Y_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}). \tag{5.1.22}$$

Let for $N \in \mathbb{N}$ maxingale problems (M^N) on $\mathbb{C}$ be defined by

$$(M^N) \qquad \begin{array}{ll} \mathbf{x}_0 = 0 & \Pi^N\text{–a.e.,} \\ Y^N(\lambda),\ \lambda \in \mathbb{R}^d, & \text{is a } \mathbf{C}\text{-local exponential maxingale on} \\ & (\mathbb{C}, \Pi^N). \end{array}$$

Lemma 5.1.17. *Let the nets* $\{\mathcal{L}(X^{\phi,N}), \phi \in \Phi\}$, $N \in \mathbb{N}$, *be* $\mathbb{C}$*–exponentially tight and every LD accumulation point of* $\{\mathcal{L}(X^{\phi,N}), \phi \in \Phi\}$ *solve* (M^N)*. If, in addition,* (NE) *holds, then*

$$\lim_{N\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\tau_N(X^\phi) \le t\big) = 0,\ t \in \mathbb{R}_+.$$

Proof. We first note that since Π^N solves (M^N), an argument similar to the one used in the proof of Lemma 2.7.11 shows that

$$\Pi^N(\mathbf{x}) \le \mathbf{\Pi}_{0,\tau_N(\mathbf{x})}(\mathbf{x}). \tag{5.1.23}$$

Let $\big\{P_{\phi\circ\psi}^{1/r_{\phi\circ\psi}}\big(\tau_{N\circ\psi}(X^{\phi\circ\psi}) \le t\big), \psi \in \Psi\big\}$ be a subnet of $\big\{P_\phi^{1/r_\phi}\big(\tau_N(X^\phi) \le t\big), (\phi, N) \in \Phi \times \mathbb{N}\big\}$ such that

$$\lim_{\psi\in\Psi} P_{\phi\circ\psi}^{1/r_{\phi\circ\psi}}\big(\tau_{N\circ\psi}(X^{\phi\circ\psi}) \le t\big) = \limsup_{N\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\tau_N(X^\phi) \le t). \tag{5.1.24}$$

By Corollary 3.1.20 there exists a subnet $\{(\mathcal{L}(X^{\phi\circ\psi\circ\upsilon,N}), N \in \mathbb{N}), \upsilon \in \Upsilon\}$ of $\{(\mathcal{L}(X^{\phi\circ\psi,N}), N \in \mathbb{N}), \psi \in \Psi\}$ such that $\mathcal{L}(X^{\phi\circ\psi\circ\upsilon,N}) \stackrel{ld}{\to} \Pi^N$, $N \in \mathbb{N}$, as $\upsilon \in \Upsilon$ at rate $r_{\phi\circ\psi\circ\upsilon}$, where Π^N are deviabilities on $\mathbb{D}$ with support in $\mathbb{C}$. Since $\Pi^N(\mathbb{D} \setminus \mathbb{C}) = 0$, we

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identify Π^N with its restriction to $\mathbb{C}$. By (5.1.3) and Lemma 5.1.9 $\tau_N(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}$, is a finite $\mathbf{C}$–stopping time so that the τ-algebra $\mathcal{C}_{\tau_N}$ is well defined. We prove that there exists a deviability Π on $\mathbb{C}$ such that

$$\Pi(A) = \Pi^N(A), \; A \in \mathcal{C}_{\tau_N}, \; N \in \mathbb{N}. \tag{5.1.25}$$

This is done by applying Theorem 1.8.1. Let us check that $\{\Pi^N, \, N \in \mathbb{N}\}$ is a projective system of deviabilities on the same space $\mathbb{C}$ with the $\tilde{p}_N$ as "bonding maps". In other words, we have to check that $\Pi^N = \Pi^{N'} \circ \tilde{p}_N^{-1}$ for $N' > N$. Since $\mathcal{L}(X^{\phi \circ \psi \circ v, N'}) \stackrel{ld}{\to} \Pi^{N'}$, $X^{\phi, N} = \tilde{p}_N X^{\phi, N'}$ (see (5.1.20) and (5.1.21)), and $\tilde{p}_N$ is $\mathbb{C}$–continuous, by the contraction principle $\mathcal{L}(X^{\phi \circ \psi \circ v, N}) \stackrel{ld}{\to} \Pi^{N'} \circ \tilde{p}_N^{-1}$. Since also $\mathcal{L}(X^{\phi \circ \psi \circ v, N}) \stackrel{ld}{\to} \Pi^N$, by uniqueness of an LD limit $\Pi^N = \Pi^{N'} \circ \tilde{p}_N^{-1}$.

In order to apply Theorem 1.8.1 we have to check the (ϵ, K)-condition. By the first part of condition (NE) it is sufficient to check that $K_{\Pi^{N'}}(\epsilon) \subset \tilde{p}_{N'} K_{\mathbf{\Pi}_0}(\epsilon)$ for arbitrary $\epsilon \in (0,1]$ and $N' \in \mathbb{N}$. Let $\mathbf{x}^{N'} \in K_{\Pi^{N'}}(\epsilon)$. The fact that $\{\Pi^N, \, N \in \mathbb{N}\}$ is a projective system of deviabilities allows us to construct functions $\mathbf{x}^N \in \mathbb{C}$, $N = N', N'+1, \ldots$, such that $\tilde{p}_N \mathbf{x}^{N+1} = \mathbf{x}^N$ and $\Pi^N(\mathbf{x}^N) = \Pi^{N'}(\mathbf{x}^{N'})$. Since the sequence $\{\tau_N(\mathbf{x}^N)\}$ is increasing, it converges to a limit L. Since by (5.1.23) $\mathbf{\Pi}_{0,\tau_N(\mathbf{x}^N)}(\mathbf{x}^N) \geq \Pi^N(\mathbf{x}^N) \geq \epsilon$ and the sequence $\{\mathbf{x}^N_{\tau_N(\mathbf{x}^N)} + \tau_N(\mathbf{x}^N), \, N = N', N'+1, \ldots\}$ is unbounded, the second part of condition (NE) implies that $L = \infty$. Hence, there exists a function $\hat{\mathbf{x}} \in \mathbb{C}$ that coincides with $\mathbf{x}^{N'}$ on $[0, \tau_{N'}(\mathbf{x}^{N'})]$ and coincides with the $\mathbf{x}^N$ on $[\tau_{N-1}(\mathbf{x}^N), \tau_N(\mathbf{x}^N)]$ for $N = N'+1, N'+2, \ldots$. Since $\mathbf{\Pi}_{0,\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}}) = \mathbf{\Pi}_{0,\tau_N(\mathbf{x}^N)}(\mathbf{x}^N) \geq \epsilon$ and $\mathbf{\Pi}_{0,\tau_N(\hat{\mathbf{x}})}(\hat{\mathbf{x}}) \to \mathbf{\Pi}_0(\hat{\mathbf{x}})$ as $N \to \infty$, we conclude that $\mathbf{\Pi}_0(\hat{\mathbf{x}}) \geq \epsilon$ as required. Hence, by Theorem 1.8.1 there exists a deviability Π on $\mathbb{C}$ such that $\Pi^N = \Pi \circ \tilde{p}_N^{-1}$, which is equivalent to (5.1.25) by Lemma 2.2.21.

Since $\tau_N(X^\phi) = \tau_N(X^{\phi,N})$, $N \circ \psi \circ v \to \infty$ as $v \in \Upsilon$, the set $\{\mathbf{x} \in \mathbb{D} : \; \tau_N(\mathbf{x}) \leq t\}$ is $\mathbb{C}$–closed, $\Pi^N(\mathbb{D} \setminus \mathbb{C}) = 0$, and $\{\mathbf{x} \in \mathbb{C} : \tau_N(\mathbf{x}) \leq t\} \in \mathcal{C}_{\tau_N}$, we have by (5.1.24), Corollary 3.1.9, and (5.1.25) that for arbitrary $N' \in \mathbb{N}$

$$\begin{aligned} &\limsup_{N \to \infty} \limsup_{\phi \in \Phi} P_\phi^{1/r_\phi}(\tau_N(X^\phi) \leq t) \\ &\qquad\qquad = \lim_{v \in \Upsilon} P_{\phi \circ \psi \circ v}^{1/r_{\phi \circ \psi \circ v}} \big(\tau_{N \circ \psi \circ v}(X^{\phi \circ \psi \circ v}) \leq t\big) \end{aligned}$$

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$$\leq \limsup_{\upsilon \in \Upsilon} P_{\phi\circ\psi\circ\upsilon}^{1/r_{\phi\circ\psi\circ\upsilon}}(\tau_{N'}(X^{\phi\circ\psi\circ\upsilon,N'}) \leq t) \leq \Pi^{N'}(\tau_{N'}(\mathbf{x}) \leq t)$$

$$= \Pi(\tau_{N'}(\mathbf{x}) \leq t). \quad (5.1.26)$$

By the τ-smoothness property of deviability

$$\lim_{N\to\infty} \Pi(\tau_N(\mathbf{x}) \leq t) = \Pi\Big(\bigcap_{N\in\mathbb{N}} \{\mathbf{x} \in \mathbb{C} :\ \tau_N(\mathbf{x}) \leq t\}\Big) = 0,$$

which together with (5.1.26) proves the lemma. □

Proof of Theorem 5.1.10. We begin by showing that the nets $\{\mathcal{L}(X^{\phi,N}), \phi \in \Phi\}$, $N \in \mathbb{N}$, defined in (5.1.21), are $\mathbb{C}$–exponentially tight and their respective LD accumulation points solve (M^N). We first check that $\{X^{\phi,N}, \phi \in \Phi\}$ satisfies the conditions of Theorem 5.1.5 with $G^N(\lambda) = (G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ as $G(\lambda)$. Condition (0) is obvious. Next, $X^{\phi,N}$ has as its stochastic exponential the process $\mathcal{E}^{\phi,N}(\lambda) = (\mathcal{E}^{\phi}_{t\wedge\tau_N(X^\phi)}(\lambda), t \in \mathbb{R}_+)$. Hence, (sup $\mathcal{E})_{loc}$ implies (sup $\mathcal{E}$) for $X^{\phi,N}$ with $G^N(\lambda)$ as $G(\lambda)$.

Now we check that $G^N(\lambda)$ satisfies the conditions imposed in Theorem 5.1.5 on $G(\lambda)$. Let us consider the majoration condition. Since by the definition of τ_N, $\mathbf{x}^*_{(t\wedge\tau_N(\mathbf{x}))-} \leq N$, $\mathbf{x} \in \mathbb{D}$, we have that, for $0 \leq s \leq t$,

$$\sup_{\mathbf{x}\in\mathbb{D}}(G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}) - G_{s\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}))$$

$$= \sup_{\substack{\mathbf{x}\in\mathbb{D}:\\ \mathbf{x}^*_{(t\wedge\tau_N(\mathbf{x}))-}\leq N}} (G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}) - G_{s\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})). \quad (5.1.27)$$

By Remark 5.1.3 and the facts that $G(\lambda)$ is $\mathbf{D}$–adapted and $t \wedge \tau_N(\mathbf{x})$ is a $\mathbf{D}$–stopping time, the right-hand side of (5.1.27) equals $\sup(G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}) - G_{s\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}))$ over $\mathbf{x} \in \mathbb{D}$ such that $\mathbf{x}^*_\infty \leq N$; so by the local majoration condition on $G(\lambda)$ (say, with $\overline{G}^N$ for given λ) if $\mathbf{x}^*_\infty \leq N$, then

$$G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}) - G_{s\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}) \leq \overline{G}^N_{t\wedge\tau_N(\mathbf{x})} - \overline{G}^N_{s\wedge\tau_N(\mathbf{x})}$$

$$\leq \overline{G}^N_t - \overline{G}^N_s,$$

where for the last inequality we used that $\overline{G}^N_t$ is increasing in t. Hence,

$$\sup_{\mathbf{x}\in\mathbb{D}}(G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}) - G_{s\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})) \leq \overline{G}^N_t - \overline{G}^N_s, \quad (5.1.28)$$

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proving the majoration condition for $G^N(\lambda)$.

Next, obviously, $G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})$ is $\mathcal{D}_t$–measurable in $\mathbf{x}\in\mathbb{D}$ and continuous in t. We check that it is $\mathbb{C}$–continuous in $\mathbf{x}$ uniformly over $t\in[0,T]$ for arbitrary $T>0$. Let $\mathbf{x}_n\to\mathbf{x}\in\mathbb{C}$. We again fix λ and denote by $\overline{G}^N$ the associated local majorant for $G(\lambda)$. For arbitrary $\varepsilon>0$, by continuity of $\overline{G}_t^N$ and $G_t(\lambda;\mathbf{x})$ in t we can choose $\delta>0$, $\delta<\tau_N(\mathbf{x})\wedge 1$, such that

$$\sup_{\substack{u,v\le T\\ |u-v|\le\delta}}|\overline{G}_u^N-\overline{G}_v^N|\le\varepsilon,\ \sup_{t\le T}|G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x})-G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})|\le\varepsilon.$$

Since $\tau_N(\mathbf{x}_n)\to\tau_N(\mathbf{x})$ as $n\to\infty$ by Lemma 2.7.5, we can take n large enough to have

$$|\tau_N(\mathbf{x}_n)-\tau_N(\mathbf{x})|\le\delta, \tag{5.1.29}$$

and then, for $t\le T$,

$$\begin{aligned}
&G_{t\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n)-G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})\\
&\le(G_{t\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n)-G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x}_n))\\
&+|G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x}_n)-G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x})|\\
&+|G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x})-G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})|\\
&\le(G_{t\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n)-G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x}_n))\\
&\qquad+|G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x}_n)-G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x})|+\varepsilon.
\end{aligned}\tag{5.1.30}$$

By (5.1.29), (5.1.28) and the choice of δ

$$\begin{aligned}
&G_{t\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n)-G_{t\wedge(\tau_N(\mathbf{x})-\delta)}(\lambda;\mathbf{x}_n)\\
&=G_{t\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n)-G_{t\wedge(\tau_N(\mathbf{x})-\delta)\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n)\\
&\qquad\qquad\le\overline{G}^N_{t\wedge\tau_N(\mathbf{x}_n)}-\overline{G}^N_{t\wedge(\tau_N(\mathbf{x})-\delta)}\le\varepsilon.
\end{aligned}$$

Therefore, (5.1.30) yields by $\mathbb{C}$–continuity of the mapping $\mathbf{x}\to(G_t(\lambda;\mathbf{x}),\ t\in\mathbb{R}_+)$

$$\limsup_{n\to\infty}\sup_{t\le T}(G_{t\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n)-G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}))\le 2\varepsilon.$$

The complementary inequality

$$\limsup_{n\to\infty}\sup_{t\le T}(G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})-G_{t\wedge\tau_N(\mathbf{x}_n)}(\lambda;\mathbf{x}_n))\le 2\varepsilon$$

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is proved similarly if we choose $\delta > 0$ such that $2\delta \le \tau_{N+1}(\mathbf{x}) - \tau_N(\mathbf{x})$,

$$\sup_{\substack{u,v\le T\\|u-v|\le\delta}} |\overline{G}_u^{N+1} - \overline{G}_v^{N+1}| \le \varepsilon,$$

$$\sup_{t\le T} |G_{t\wedge(\tau_N(\mathbf{x})+\delta)}(\lambda;\mathbf{x}) - G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x})| \le \varepsilon,$$

and consider n for which, in addition to (5.1.29), $\tau_N(\mathbf{x}) + \delta \le \tau_{N+1}(\mathbf{x}_n)$.

Thus, $\{X^{\phi,N}, \phi \in \Phi\}$ and $G^N(\lambda), \lambda \in \mathbb{R}^d$, satisfy all the conditions of Theorem 5.1.5. Hence, the net $\{\mathcal{L}(X^{\phi,N}), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight, and if Π^N is an LD accumulation point, then $\mathbf{x}_0 = 0$ Π^N– a.e. and the function $\overline{Y}^N(\lambda) = (\overline{Y}_t^N(\lambda;\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C})$ defined by $\overline{Y}_t^N(\lambda;\mathbf{x}) = \exp(\lambda\cdot\mathbf{x}_t - G_{t\wedge\tau_N(\mathbf{x})}(\lambda;\mathbf{x}))$ is a $\mathbf{C}$–local exponential maxingale on $(\mathbb{C}, \Pi^N)$.

Therefore, to prove that Π^N solves (M^N), it is left to show that $\overline{Y}_t^N(\lambda;\mathbf{x}) = Y_t^N(\lambda;\mathbf{x})$ Π^N–a.e., which in view of (5.1.22) and the definition of Y follows by the equality

$$\mathbf{x}_{t\wedge\tau_N(\mathbf{x})} = \mathbf{x}_t \quad \Pi^N\text{–a.e.} \tag{5.1.31}$$

To see the latter, let $\{X^{\phi',N},\ \phi' \in \Phi'\}$ be a subnet of $\{X^{\phi,N},\ \phi \in \Phi\}$ that LD converges to Π. Since $X_t^{\phi,N} = X^{\phi,N}_{t\wedge\tau_N(X^{\phi,N})}$ by (5.1.20) and (5.1.21), Π^N is supported by $\mathbb{C}$, $\tau_N(\mathbf{x})$ is $\mathbb{C}$–continuous, and the set $\{\mathbf{x} \in \mathbb{D}:\ \mathbf{x}_{t\wedge\tau_N(\mathbf{x})} \ne \mathbf{x}_t\}$ is $\mathbb{C}$–open, by Corollary 3.1.9

$$0 = \limsup_{\phi'\in\Phi'} P_{\phi'}^{1/r_{\phi'}}\big(X_t^{\phi',N} \ne X^{\phi',N}_{t\wedge\tau_N(X^{\phi',N})}\big) \ge \Pi^N(\mathbf{x}_t \ne \mathbf{x}_{t\wedge\tau_N(\mathbf{x})}),$$

which proves (5.1.31).

Thus, the nets $\{X^{\phi,N}, \phi \in \Phi\}$, $N \in \mathbb{N}$, satisfy the conditions of Lemma 5.1.17. By the lemma for $T > 0$

$$\lim_{N\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\tau_N(X^\phi) \le T) = 0, \tag{5.1.32}$$

and using (5.1.20) and (5.1.21) we have that

$$\lim_{N\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\sup_{t\le T}|X_t^\phi - X_t^{\phi,N}| > 0\big) = 0,$$

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which implies by $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(X^{\phi,N}), \phi \in \Phi\}$ for every $N \in \mathbb{N}$ and Theorem 3.2.3 that $\{\mathcal{L}(X^{\phi}), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight. Also (5.1.32) and (sup $\mathcal{E})_{loc}$ imply (sup $\mathcal{E}$). Thus, all the conditions of Theorem 5.1.16 with $X'^{\phi} = X^{\phi}$ hold. An application of that theorem ends the proof. □

Theorem 5.1.12 follows by Theorem 5.1.10 and Remark 5.1.8.

5.2 Convergence of characteristics

This section formulates conditions on convergence of the characteristics of the X^{ϕ} in order for the net $\{\mathcal{L}(X^{\phi}), \phi \in \Phi\}$ to be exponentially tight with all the LD accumulation points being solutions of a maxingale problem. We retain the above notation.

As in Section 4.2 the cumulant in the limiting maxingale problem will have the semimaxingale representation (2.7.7) and (2.7.55), however, the characteristics can depend on $\mathbf{x}$, on the one hand, and are defined for $\mathbf{x} \in \mathbb{D}$, on the other hand.

Definition 5.2.1. *Let us say that a function $f : \mathbb{R}_+ \times \mathbb{D} \to \mathbb{R}^k$ is* $\mathbf{D}$*–progressively measurable if its restriction to $[0,t] \times \mathbb{D}$ is $\overline{\mathcal{B}}([0,t]) \otimes \mathcal{D}_t/\mathcal{B}(\mathbb{R}^k)$-measurable.*

We assume as given the following objects:
$(b_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ is an $\mathbb{R}^d$-valued $\mathbf{D}$-progressively measurable function such that $\int_0^t |b_s(\mathbf{x})| ds < \infty$ for $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{D}$,
$\big(c_s(\mathbf{x}),\ s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D}\big)$ is a $\mathbf{D}$-progressively measurable function with values in the space of symmetric, positive semi-definite $d \times d$-matrices such that $\int_0^t \|c_s(\mathbf{x})\|\, ds < \infty$ for $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{D}$,
$\big(\nu_s(\Gamma; \mathbf{x}),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d), \mathbf{x} \in \mathbb{D}\big)$ is a transition kernel from $([0,t] \times \mathbb{D}, \overline{\mathcal{B}}([0,t]) \otimes \mathcal{D}_t)$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for every $t \in \mathbb{R}_+$ such that for $t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D}$ and $\alpha \in \mathbb{R}_+$

$$\int_{\mathbb{R}^d} |x|^2 \wedge 1\, \nu_t(dx; \mathbf{x}) < \infty, \int_{\mathbb{R}^d} e^{\alpha|x|}\, \mathbf{1}(|x| > 1)\, \nu_t(dx; \mathbf{x}) < \infty,$$

$$\nu_t(\{0\}; \mathbf{x}) = 0,\ |x|^2 \wedge 1 * \nu_t(\mathbf{x}) < \infty,\ e^{\alpha|x|}\, \mathbf{1}(|x| > 1) * \nu_t(\mathbf{x}) < \infty,$$

$\big(\hat{\nu}_s(\Gamma; \mathbf{x}),\ s \in \mathbb{R}_+, \Gamma \in \mathcal{B}(\mathbb{R}^d), \mathbf{x} \in \mathbb{D}\big)$ is a transition kernel from $([0,t] \times \mathbb{D}, \overline{\mathcal{B}}(\mathbb{R}_+) \otimes \mathcal{D}_t)$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for every $t \in \mathbb{R}_+$ such that

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for $s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D}$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$\hat{\nu}_s(\Gamma; \mathbf{x}) \leq \nu_s(\Gamma; \mathbf{x}),\ \hat{\nu}_s(\mathbb{R}^d; \mathbf{x}) \leq 1. \tag{5.2.2}$$

Since **D**-progressively measurable functions are **C**-progressively measurable, the restrictions of $(b_s(\mathbf{x}))$, $(c_s(\mathbf{x})), (\nu_s(\Gamma; \mathbf{x}))$, and $(\hat{\nu}_s(\Gamma; \mathbf{x}))$ to $\mathbb{C}$ satisfy the conditions on the local characteristics of a semimaxingale as defined in Section 4.2. Also, given a limiter $h : \mathbb{R}^d \to \mathbb{R}^d$ we define extensions of the characteristics of a semimaxingale to $\mathbb{D}$ by

$$B'_t(\mathbf{x}) = \int_0^t b_s(\mathbf{x})\, ds, \tag{5.2.3}$$

$$B_t(\mathbf{x}) = B'_t(\mathbf{x}) + (h(x) - x) * \nu_t\,(\mathbf{x}), \tag{5.2.4}$$

$$C_t(\mathbf{x}) = \int_0^t c_s(\mathbf{x})\, ds, \tag{5.2.5}$$

and refer to $B' = (B'_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ as the first characteristic "without truncation" of the limiting semimaxingale, to $B = (B_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ as the first characteristic associated with limiter $h(x)$, to $C = (C_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ as the second characteristic, to $\nu_s(\Gamma; \mathbf{x})$ as the density of the measure of jumps, and to $\hat{\nu}_s(\Gamma; \mathbf{x})$ as the density of the discontinuous measure of jumps. If $\bar{B} = (\bar{B}_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ is the first characteristic associated with a limiter $\bar{h}(x)$, then

$$\bar{B}_t(\mathbf{x}) = B_t(\mathbf{x}) + (\bar{h}(x) - h(x)) * \nu_t\,(\mathbf{x}). \tag{5.2.6}$$

The modified second characteristic $\tilde{C} = (\tilde{C}_t(\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ is specified by the equality

$$\lambda \cdot \tilde{C}_t(\mathbf{x})\lambda = \lambda \cdot C_t(\mathbf{x})\lambda + (\lambda \cdot h(x))^2 * \nu_t\,(\mathbf{x}) - \int_0^t (\lambda \cdot h(x) \bullet \hat{\nu}_s(\mathbf{x}))^2 ds,\ \lambda \in \mathbb{R}^d. \tag{5.2.7}$$

We introduce a number of conditions on the characteristics, which are analogues of the continuity and majoration conditions on $G(\lambda)$ in Section 5.1. As above, we denote by U a dense subset of $\mathbb{R}_+$.

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Definition 5.2.2. *We say that B (respectively, C; $\tilde{C}$; ν; or $\hat{\nu}$) satisfies the continuity condition if $B_t(\mathbf{x})$ (respectively, $C_t(\mathbf{x})$; $\tilde{C}_t(\mathbf{x})$; $f(x) * \nu_t(\mathbf{x})$ for $f : \mathbb{R}^d \to \mathbb{R}$ Borel measurable and such that $|f(x)| \leq 1 \wedge |x|^2$; $\int_0^t (g(x) \bullet \hat{\nu}_s(\mathbf{x}))^k \, ds$ for $g : \mathbb{R}^d \to \mathbb{R}$ Borel measurable and bounded, and $k = 2, 3, \ldots$) is $\mathbb{C}$–continuous in $\mathbf{x}$ for all $t \in U$.*

Remark 5.2.3. *If the continuity condition on ν holds, then by* (5.2.6) *the continuity condition on B does not depend on a limiter. If, in addition, the continuity condition on $\hat{\nu}$ holds, then by* (5.2.7) *the continuity conditions on C and $\tilde{C}$ are equivalent.*

Occasionally, we will need a stronger form of the continuity condition on B which is an analogue of the uniform continuity condition for $G(\lambda)$.

Definition 5.2.4. *We say that B satisfies the uniform continuity condition if the map $\mathbf{x} \to (B_t(\mathbf{x}), t \in \mathbb{R}_+)$ is $\mathbb{C}$–continuous as a map from $\mathbb{D}$ into $\mathbb{C}$.*

Remark 5.2.5. *Since $\lambda \cdot C_t(\mathbf{x})\lambda$, $\lambda \cdot \tilde{C}_t(\mathbf{x})\lambda$ and $f(x) * \nu_t(\mathbf{x})$, if $f \geq 0$, are increasing, continuous in t and equal to 0 at 0, the continuity conditions on C, $\tilde{C}$ and ν are equivalent to $\mathbb{C}$–continuity of the respective maps $\mathbf{x} \to (C_t(\mathbf{x}), t \in \mathbb{R}_+)$ from $\mathbb{D}$ into $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^{d \times d})$, $\mathbf{x} \to (\tilde{C}_t(\mathbf{x}), t \in \mathbb{R}_+)$ from $\mathbb{D}$ into $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^{d \times d})$ and $\mathbf{x} \to (f(x) * \nu_t(\mathbf{x}), t \in \mathbb{R}_+)$, where $|f(x)| \leq 1 \wedge |x|^2$, from $\mathbb{D}$ into $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$. Thus, the continuity conditions on C, $\tilde{C}$ and ν imply the associated uniform continuity conditions. Therefore, we will sometimes also be referring to the continuity conditions for C, $\tilde{C}$ and ν as uniform continuity conditions.*

Definition 5.2.6. *We say that B (respectively, C, $\tilde{C}$, or ν) satisfies the majoration condition (respectively, the local majoration condition) if the functions $(\lambda \cdot B_t(\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ for all $\lambda \in \mathbb{R}^d$; $(\lambda \cdot C_t(\mathbf{x})\lambda, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ for all $\lambda \in \mathbb{R}^d$; $(\lambda \cdot \tilde{C}_t(\mathbf{x})\lambda, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ for all $\lambda \in \mathbb{R}^d$; $(f(x) * \nu_t(\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ for all $f \in \mathcal{C}_b$ satisfy the majoration condition (respectively, the local majoration condition).*

Remark 5.2.7. *The majoration condition (respectively, the local majoration condition) on B equivalently requires that the function $(Var_t B(\mathbf{x}), t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ of total variation of B obey the majoration condition (respectively, the local majoration condition). The*

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majoration conditions (respectively, the local majoration conditions) on C and $\tilde{C}$ are equivalent to the majoration conditions (respectively, local majoration conditions) on the respective functions of the sums of the diagonal entries of C_t and $\tilde{C}_t$.

Definition 5.2.8. *We say that ν satisfies the $\mathbb{C}$–local boundedness condition if for every compact $K \subset \mathbb{C}$ and every $\alpha > 0, t > 0$,*

$$\sup_{\mathbf{x}\in K} e^{\alpha|x|}\,\mathbf{1}(|x|>1)*\nu_t(\mathbf{x}) < \infty. \tag{5.2.8}$$

We say that $\hat{\nu}$ satisfies the $\mathbb{C}$–local boundedness condition if for every compact $K \subset \mathbb{C}$ and every $\alpha > 0, t > 0$,

$$\sup_{\mathbf{x}\in K}\sup_{s\le t} e^{\alpha|x|}\bullet\hat{\nu}_s(\mathbf{x}) < \infty. \tag{5.2.9}$$

We now state conditions on the triplets of the X^ϕ. They are similar to the conditions of Section 4.2. Actually, the conditions on the X_0^ϕ and big jumps are the same. We repeat them here for completeness. Let $(B^\phi, C^\phi, \nu^\phi)$ be the predictable characteristics of X^ϕ corresponding to a limiter $h(x)$. As above, $x_0 \in \mathbb{R}^d$.

(0) $$X_0^\phi \overset{P_\phi^{1/r_\phi}}{\to} x_0 \text{ as } \phi\in\Phi,$$

(A) $$\lim_{A\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\big(\nu^\phi([0,t],|x|>A)\big)^{1/r_\phi} > \varepsilon\Big) = 0, \qquad t>0, \varepsilon>0,$$

(a) $$\lim_{a\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}e^{\alpha r_\phi|x|}\,\mathbf{1}(r_\phi|x|>a)\,\mathbf{1}(|x|\le A)*\nu_t^\phi > \varepsilon\Big) = 0,$$
$$t>0,\ \alpha>0,\ A>0,\ \varepsilon>0,$$

(sup B) $$\sup_{t\le T}|B_t^\phi - B_t(X^\phi)| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ T>0,$$

(C) $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\,\|r_\phi\, C_t^{\phi,\delta} - C_t(X^\phi)\| > \varepsilon) = 0,\ t\in U, \varepsilon>0,$$

$(\tilde{C})$ $$\|r_\phi \tilde{C}_t^\phi - \tilde{C}_t(X^\phi)\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ t\in U,$$

(ν) $$f^\phi(x)*\nu_t^\phi - f(x)*\nu_t(X^\phi) \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ t\in U,\ f\in\mathcal{C}_b,$$

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$$(\hat{\nu}) \quad \frac{1}{r_\phi} \sum_{0<s\le t} \bigl(f(r_\phi x) \bullet \nu_s^\phi\bigr)^k - \int_0^t \bigl(f(x) \bullet \hat{\nu}_s(X^\phi)\bigr)^k ds \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi,$$
$$t \in U, k = 2, 3, \dots, f \in \mathcal{C}_b.$$

(We recall that $f^\phi(x) = f(r_\phi x)/r_\phi$.)

Theorem 5.2.9. *Let $h(x)$ be continuous, B, C (respectively, $\tilde{C}$), ν, and $\hat{\nu}$ satisfy the continuity conditions, and ν and $\hat{\nu}$ satisfy the $\mathbb{C}$–local boundedness conditions. Let also the majoration conditions on B, C (respectively, $\tilde{C}$) and ν hold.*

If conditions $(0), (A) + (a), (\sup B), (C)$ (respectively, $(\tilde{C})$), (ν), and $(\hat{\nu})$ hold, then the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight and its every LD accumulation point is a solution to the maxingale problem (x_0, G).

Remark 5.2.10. *Condition (a) can be replaced with the condition*

$$(a') \quad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Bigl(\frac{1}{r_\phi} j^\phi_{a,A,\alpha}(x) * \nu_t^{\phi,c} + \frac{1}{r_\phi} \sum_{0<s\le t} \ln\bigl(1 + j^\phi_{a,A,\alpha}(x) \bullet \nu_s^\phi\bigr) > \varepsilon\Bigr) = 0,$$
$$t > 0,\ \alpha > 0,\ A > 0,\ \varepsilon > 0,$$

where $j^\phi_{a,A,\alpha}(x) = (e^{r_\phi |x|} - 1)\, \mathbf{1}(r_\phi |x| > a)\, \mathbf{1}(|x| \le A)$.

Remark 5.2.11. *Theorems 5.2.9 and 2.8.5 imply Theorem 4.2.1.*

Next comes a locally bounded version. Let us define $\tau_N(\mathbf{x})$ by (5.1.3) and introduce the conditions

$$(A)_{loc} \quad \lim_{A\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\bigl(\nu^\phi([0, t \wedge \tau_N(X^\phi)], |x| > A)^{1/r_\phi} > \varepsilon\bigr) = 0,$$
$$t > 0,\ N \in \mathbb{N}, \varepsilon > 0,$$

$$(a)_{loc} \quad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Bigl(\frac{e^{\alpha r_\phi |x|}}{r_\phi}\, \mathbf{1}(r_\phi |x| > a)\, \mathbf{1}(|x| \le A) * \nu^\phi_{t\wedge\tau_N(X^\phi)} > \varepsilon\Bigr) = 0,$$
$$t > 0,\ N \in \mathbb{N},\ \alpha > 0,\ A > 0,\ \varepsilon > 0,$$

$$(\sup B)_{loc} \quad \sup_{t\le T} |B^\phi_{t\wedge\tau_N(X^\phi)} - B_{t\wedge\tau_N(X^\phi)}(X^\phi)| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi,$$
$$T > 0,\ N \in \mathbb{N},$$

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$$(C)_{loc} \quad \lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\, \| r_\phi\, C^{\phi,\delta}_{t\wedge\tau_N(X^\phi)} - C_{t\wedge\tau_N(X^\phi)}(X^\phi)\| > \varepsilon\big) = 0,$$
$$t\in U,\ N\in\mathbb{N}, \varepsilon>0,$$

$$(\tilde C)_{loc} \qquad \| r_\phi \tilde C^\phi_{t\wedge\tau_N(X^\phi)} - \tilde C_{t\wedge\tau_N(X^\phi)}(X^\phi)\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi\in\Phi,$$
$$t\in U,\ N\in\mathbb{N},$$

$$(\nu)_{loc} \quad f^\phi(x) * \nu^\phi_{t\wedge\tau_N(X^\phi)} - f(x) * \nu_{t\wedge\tau_N(X^\phi)}(X^\phi) \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi\in\Phi,$$
$$t\in U,\ N\in\mathbb{N}, f\in\mathcal{C}_b,$$

$$(\hat\nu)_{loc} \quad \frac{1}{r_\phi}\sum_{s\le t\wedge\tau_N(X^\phi)} \big(f(r_\phi x)\bullet\nu^\phi_s\big)^k - \int_0^{t\wedge\tau_N(X^\phi)} \big(f(x)\bullet\hat\nu_s(X^\phi)\big)^k ds$$
$$\overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi\in\Phi,$$
$$t\in U,\ N\in\mathbb{N}, k=2,3,\ldots, f\in\mathcal{C}_b.$$

Theorem 5.2.12. *Let $h(x)$ be continuous. Let B, C (respectively, $\tilde C$), ν, and $\hat\nu$ satisfy the continuity conditions, and ν and $\hat\nu$ satisfy the $\mathbb{C}$–local boundedness conditions. Let the local majoration conditions on B, C (respectively, $\tilde C$) and ν hold. Let condition (NE) hold.*

If conditions $(0), (A)_{loc} + (a)_{loc}, (\sup B)_{loc}, (C)_{loc}$ (respectively, $(\tilde C)_{loc}$), $(\nu)_{loc}$, and $(\hat\nu)_{loc}$ hold, then the net $\{\mathcal{L}(X^\phi), \phi\in\Phi\}$ is $\mathbb{C}$–exponentially tight and its every LD accumulation point solves the maxingale problem (x_0, G).

Remark 5.2.13. *Condition $(a)_{loc}$ can be replaced with the condition*

$$(a')_{loc} \qquad \lim_{a\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi} j^\phi_{a,A,\alpha}(x) * \nu^{\phi,c}_{t\wedge\tau_N(X^\phi)}$$
$$+\frac{1}{r_\phi}\sum_{s\le t\wedge\tau_N(X^\phi)} \ln\big(1 + j^\phi_{a,A,\alpha}(x)\bullet\nu^\phi_s\big) > \varepsilon\Big) = 0,$$
$$t>0,\ N\in\mathbb{N},\ \alpha>0,\ A>0,\ \varepsilon>0.$$

Definition 5.2.14. *We say that $b_s(\mathbf{x})$, respectively, $c_s(\mathbf{x})$, meets the linear-growth condition if there exists an $\mathbb{R}_+$-valued Lebesgue measurable function l_s such that $\int_0^t l_s\, ds < \infty, t\in\mathbb{R}_+$, and*

$$|b_s(\mathbf{x})| \le (1+\mathbf{x}^*_s) l_s, \tag{5.2.10}$$

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respectively,

$$\|c_s(\mathbf{x})\| \leq (1+(\mathbf{x}_s^*)^2)l_s. \tag{5.2.11}$$

We say that ν meets the linear-growth condition if

$$\int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda \cdot x)\nu_s(dx;\mathbf{x}) \leq \int_{\mathbb{R}^d} (e^{(1+\mathbf{x}_s^*)\lambda\cdot x} - 1 - (1+\mathbf{x}_s^*)\lambda\cdot x)m_s(dx),\ \lambda \in \mathbb{R}^d, \tag{5.2.12}$$

where $m_s(dx)$ is a transition kernel from $(\mathbb{R}_+, \overline{\mathcal{B}}(\mathbb{R}_+))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\int_0^t \int_{\mathbb{R}^d} (\exp(\alpha|x|) - 1 - \alpha|x|)m_s(dx)ds < \infty$, $t > 0, \alpha > 0$.

Theorem 5.2.15. *Let $h(x)$ be continuous, B, C (respectively, $\tilde{C}$), ν, and $\hat{\nu}$ satisfy the continuity conditions. Let the linear-growth conditions on $b_s(\mathbf{x})$, $c_s(\mathbf{x})$, and $\nu_s(\Gamma;\mathbf{x})$ hold. Let $\hat{\nu}$ satisfy the $\mathbb{C}$–local boundedness condition.*

If conditions $(0), (A)_{loc} + (a)_{loc}, (\sup B)_{loc}, (C)_{loc}$ (respectively, $(\tilde{C})_{loc}$), $(\nu)_{loc}$, and $(\hat{\nu})_{loc}$ hold, then the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight and its every LD accumulation point solves the maxingale problem (x_0, G).

Remark 5.2.16. *Condition $(a)_{loc}$ can be replaced with condition $(a')_{loc}$.*

Remark 5.2.17. *In the above theorems we can equivalently describe the accumulation points by saying that if Π is an LD accumulation point of $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$, then the canonical process X on $(\mathbb{C}, \mathbf{C}, \Pi)$ is a Luzin-continuous semimaxingale with characteristics $(B, C, \nu, \hat{\nu})$ starting at x_0. If $(B, C, \nu, \hat{\nu})$ and x_0 uniquely specify Π, then $X^\phi \overset{ld}{\to} X$.*

The proofs use the ideas of the proofs of Theorems 4.2.1, 5.1.5, and 5.1.10. An outline is as follows: we introduce truncated processes $\hat{X}^{\phi,a}$ as in the proof of Theorem 4.2.1, establish that the pairs $(\hat{X}^{\phi,a}, X^\phi)$ as random elements of $\mathbb{D}' (= \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^d))$ satisfy the conditions of Theorem 5.1.16, then observe in view of Lemma 4.2.16

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that condition $(A) + (a)$ implies that the nets $\{(X^\phi, X^\phi), \phi \in \Phi\}$ and $\{(\hat{X}^{\phi,a}, X^\phi), \phi \in \Phi\}$ asymptotically (as $a \to \infty$) have the same LD limit, and derive the statements of Theorems 5.2.9, 5.2.12 and 5.2.15 by taking the limit as $a \to \infty$ in the maxingale problems associated with the nets $\{(\hat{X}^{\phi,a}, X^\phi), \phi \in \Phi\}$. This turns out to be quite a long way. The next subsection studies required exponential tightness properties. After, LD accumulation points are identified as solutions to certain maxingale problems, and finally the proofs of the above results are given. We assume in the proofs that $x_0 = 0$.

5.2.1 Exponential tightness results

We develop exponential tightness results for $(\hat{X}^{\phi,a}, X^\phi)$. The next lemma extends Lemma 4.2.6.

Lemma 5.2.18. *Let $Z^{\phi,\delta} = (Z_t^{\phi,\delta}, t \in \mathbb{R}_+), Z_0^{\phi,\delta} = 0, \delta > 0, \phi \in \Phi$, and $Z^\phi = (Z_t^\phi, t \in \mathbb{R}_+), Z_0^\phi = 0, \phi \in \Phi$, be $\mathbb{R}^d$–valued, componentwise increasing processes with paths in $\mathbb{D}$ defined on respective probability spaces $(\Omega_\phi, \mathcal{F}_\phi, P_\phi)$ such that for all $t \in U, \varepsilon > 0$,*

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(|Z_t^{\phi,\delta} - Z_t^\phi| > \varepsilon) = 0.$$

If the net $\{\mathcal{L}(Z^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight, the latter convergence is uniform on bounded intervals, i.e.,

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\sup_{t\le T} |Z_t^{\phi,\delta} - Z_t^\phi| > \varepsilon) = 0,\ T > 0, \varepsilon > 0.$$

Proof. Acting as in the proof of Lemma 4.2.6 for $N \in \mathbb{N}$ we choose $t_i^N \in U$, $i = 0, \ldots, k^N$, such that $0 = t_0^N < t_1^N < \ldots < t_{k^N-1}^N < T \le t_{k^N}^N < T + 1$ and $|t_i^N - t_{i-1}^N| \le 1/N$, $i = 1, \ldots, k^N$. Then by the same argument

$$\sup_{t\le T} |Z_t^{\phi,\delta} - Z_t^\phi| \le \max_{i=1,\ldots,k^N} |Z_{t_i^N}^{\phi,\delta} - Z_{t_i^N}^\phi| + \sup_{\substack{s,t\le T+1:\\ |s-t|\le 1/N}} |Z_t^\phi - Z_s^\phi|.$$

Therefore,

$$P_\phi^{1/r_\phi}\Big(\sup_{t\le T} |Z_t^{\phi,\delta} - Z_t| > \varepsilon\Big) \le \sum_{i=1}^{k^N} P_\phi^{1/r_\phi}\Big(|Z_{t_i^N}^{\phi,\delta} - Z_{t_i^N}| > \varepsilon/2\Big) + P_\phi^{1/r_\phi}\Big(\sup_{\substack{s,t\le T+1:\\ |s-t|\le 1/N}} |Z_t^\phi - Z_s^\phi| > \epsilon/2\Big)$$

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so that by hypotheses

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\sup_{t\le T}|Z_t^{\phi,\delta}-Z_t^\phi|>\varepsilon)$$
$$\le \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{\substack{s,t\le T+1:\\ |s-t|\le 1/N}}|Z_t^\phi - Z_s^\phi|>\epsilon/2\Big)$$

The right-hand side tends to 0 as $N\to\infty$ by Theorem 3.2.3. □

We now study, as in the proof of Theorem 4.2.1, when one can replace conditions (C), $(\tilde C)$ and (ν) with the associated uniform versions. Let us introduce the conditions:

(sup C) $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\sup_{t\le T}\|r_\phi C_t^{\phi,\delta}-C_t(X^\phi)\| >\varepsilon)=0,\quad \varepsilon>0, T>0,$$

(sup $\tilde C$) $$\sup_{t\le T}\|r_\phi \tilde C_t^\phi - \tilde C_t(X^\phi)\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ T>0,$$

(sup ν) $$\sup_{t\le T}|f^\phi(x)*\nu_t^\phi - f(x)*\nu_t\,(X^\phi)| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\quad f\in\mathcal{C}_b, T>0.$$

In some of the statements below we say, with a slight abuse of terminology, that nets of laws of $\mathbb{R}^m$–valued random processes are $\mathbb{C}$–exponentially tight if they are $\mathbb{C}(\mathbb{R}_+,\mathbb{R}^m)$–exponentially tight. The meaning should be clear from the context.

Lemma 5.2.19. *1. Let the net* $\{\mathcal{L}(C(X^\phi)),\phi\in\Phi\}$ *(respectively,* $\{\mathcal{L}(\tilde C(X^\phi)),\phi\in\Phi\}$*;* $\{\mathcal{L}(f*\nu\,(X^\phi)),\phi\in\Phi\}, f\in\mathcal{C}_b$*) be* $\mathbb{C}$*–exponentially tight. Then condition* (C) *(respectively,* $(\tilde C)$*;* (ν)*) is equivalent to condition* (sup C) *(respectively,* (sup $\tilde C$)*;* (sup ν)*).*

2. If the nets $\{\mathcal{L}(f(x)*\nu\,(X^\phi)),\phi\in\Phi\}, f\in\mathcal{C}_b$*, are* $\mathbb{C}$*–exponentially tight and condition* (ν) *holds, then condition* (sup B) *does not depend on the particular choice of a continuous limiter* h*.*

Proof. The first part follows by Lemma 5.2.18. The second part follows from the first and is proved in the same way as Lemma 4.2.8. □

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Now, we state and prove an exponential tightness result. We recall the definition of $\hat{X}^{\phi,a} = (\hat{X}^{\phi,a}_t, t \in \mathbb{R}_+)$, $a \in \mathbb{R}_+$, from Subsection 4.2.2:

$$\hat{X}^{\phi,a}_t = X^{\phi}_t - \check{X}^{\phi,a}_t, \tag{5.2.13}$$

where

$$\check{X}^{\phi,a}_t = \sum_{s \le t} (\Delta X^{\phi}_s - h^{\phi}_a(\Delta X^{\phi}_s)), \tag{5.2.14}$$

$$h_a(x) = \Bigl(\frac{a}{|x|} \wedge 1\Bigr) x, \; h^{\phi}_a(x) = \frac{1}{r_\phi} h_a(r_\phi x). \tag{5.2.15}$$

We also recall that $\mathbb{C}' = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}^d)$.

Theorem 5.2.20. *1. Let the nets* $\{\mathcal{L}(B(X^\phi)), \phi \in \Phi\}$, $\{\mathcal{L}(C(X^\phi)), \phi \in \Phi\}$ *(respectively* $\{\mathcal{L}(\tilde{C}(X^\phi)), \phi \in \Phi\}$ *), and* $\{\mathcal{L}(f(x) * \nu\,(X^\phi)), \phi \in \Phi\}, f \in \mathcal{C}_b$, *be* $\mathbb{C}$*–exponentially tight.*

If conditions (0), $(\sup B)$, (C) *(respectively,* $(\tilde{C})$*), and* (ν) *hold, then the net* $\{\mathcal{L}(\hat{X}^{\phi,a}), \phi \in \Phi\}$ *is* $\mathbb{C}$*–exponentially tight. If, in addition, conditions* (A) + (a) *hold, then the net* $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ *is* $\mathbb{C}$*–exponentially tight, so, the net* $\{\mathcal{L}((\hat{X}^{\phi,a}, X^\phi)), \phi \in \Phi\}$ *is* $\mathbb{C}'$*–exponentially tight.*

2. Let the net $\{\mathcal{L}(X^\phi),\ \phi \in \Phi\}$ *be* $\mathbb{C}$*–exponentially tight. If the function* B *(respectively,* C*;* $\tilde{C}$*;* ν*) satisfies the uniform continuity condition, then the net* $\{\mathcal{L}(B(X^\phi)), \phi \in \Phi\}$ *(respectively,* $\{\mathcal{L}(C(X^\phi)), \phi \in \Phi\}$*;* $\{\mathcal{L}(\tilde{C}(X^\phi)), \phi \in \Phi\}$*;* $\{\mathcal{L}(f(x) * \nu\,(X^\phi)), \phi \in \Phi\}, f \in \mathcal{C}_b$*) is* $\mathbb{C}$*–exponentially tight.*

Proof. Part 2 follows from Theorem 3.2.3 via a diagonal argument. Let, given $T > 0$ and $\eta > 0$, $\{\mathcal{L}(X^{\phi'}), r_{\phi'}, p_{\phi'},\ \phi' \in \Phi'\}$ be a subnet of $\{\mathcal{L}(X^\phi), r_\phi, P_\phi^{1/r_\phi}\bigl(\sup_{\substack{s,t \in [0,T]: \\ |s-t| \le \delta}} |B_t(X^\phi) - B_s(X^\phi)| > \eta\bigr), (\phi, \delta) \in \Phi \times (0,\infty)\}$ such that $\mathcal{L}(X^{\phi'}) \overset{ld}{\to} \Pi'$ at rate $r_{\phi'}$, where Π' is supported by $\mathbb{C}$, and

$$\lim_{\phi' \in \Phi'} p_{\phi'} = \lim_{\delta \to 0} \limsup_{\phi \in \Phi} P_\phi^{1/r_\phi}\Bigl(\sup_{\substack{s,t \in [0,T]: \\ |s-t| \le \delta}} |B_t(X^\phi) - B_s(X^\phi)| > \eta\Bigr).$$

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Then for arbitrary $\hat{\delta} > 0$ by the contraction principle (Corollary 3.1.15) and uniform continuity condition for B

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\delta}} |B_t(X^\phi)-B_s(X^\phi)|>\eta\big) = \lim_{\phi'\in\Phi'} p_{\phi'}$$
$$\le \limsup_{\phi'\in\Phi'} P_{\phi'}^{1/r_{\phi'}}\big(\sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\hat{\delta}}} |B_t(X^{\phi'})-B_s(X^{\phi'})|>\eta\big)$$
$$\le \Pi'\big(\mathbf{x}\in\mathbb{C}: \sup_{\substack{s,t\in[0,T]:\\ |s-t|\le\hat{\delta}}} |B_t(\mathbf{x})-B_s(\mathbf{x})|\ge\eta\big).$$

By τ-smoothness of Π' the latter deviability tends to 0 as $\hat{\delta}\to 0$. This checks condition I(ii) of Theorem 3.2.3. Condition I(i) holds since $B_0(\mathbf{x})=0$ completing the proof of $\mathbb{C}$-exponential tightness of $\{\mathcal{L}(B(X^\phi)),\phi\in\Phi\}$. Proofs for the other processes are similar.

We prove part 1. Let us assume, first, that the nets $\{\mathcal{L}(B(X^\phi)),\phi\in\Phi\}$, $\{\mathcal{L}(C(X^\phi)),\phi\in\Phi\}$ and $\{\mathcal{L}(f(x)*\nu\,(X^\phi)),\phi\in\Phi\}, f\in\mathcal{C}_b$, are $\mathbb{C}$–exponentially tight. We begin with a proof of $\mathbb{C}$–exponential tightness for $\{\mathcal{L}(\hat{X}^{\phi,a}),\phi\in\Phi\}$. Let

$$X_t^\phi = X_0^\phi + B_t^{\phi,\delta} + M_t^{\phi,\delta} + x\,\mathbf{1}(r_\phi|x|>\delta)*\mu_t^\phi$$

be the canonical representation of X^ϕ associated with the truncation function $x\,\mathbf{1}(r_\phi|x|\le\delta)$, where $\delta<a$, so that
$B^{\phi,\delta}=(B_t^{\phi,\delta},t\in\mathbb{R}_+)$, $B_0^{\phi,\delta}=0$, is an $\mathbf{F}_\phi$–predictable process with bounded variation over bounded intervals;
$M^{\phi,\delta}=(M_t^{\phi,\delta},t\in\mathbb{R}_+)$, $M_0^{\phi,\delta}=0$, is the $\mathbf{F}_\phi$–locally square-integrable martingale defined by

$$M_t^{\phi,\delta} = X_t^{\phi,c} + x\,\mathbf{1}(r_\phi|x|\le\delta)*(\mu^\phi-\nu^\phi)_t. \tag{5.2.16}$$

Since by (5.2.14) and (5.2.13) $\hat{X}_t^{\phi,a} = X_t^\phi - (x-h_a^\phi(x))*\mu_t^\phi$ and $\delta<a$, we have

$$\hat{X}_t^{\phi,a} = X_0^\phi + B_t^{\phi,\delta} + M_t^{\phi,\delta} + h_a^\phi(x)\,\mathbf{1}(r_\phi|x|>\delta)*\mu_t^\phi,$$

so, by Theorem 3.2.3 and condition (0) in order to prove $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(\hat{X}^{\phi,a}),\phi\in\Phi\}$ it suffices to prove that,

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for all $T > 0, \eta > 0$,

$$\lim_{A\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\sup_{t\le T} |B_t^{\phi,\delta}| > A) = 0, \tag{5.2.17a}$$

$$\lim_{\sigma\to 0} \limsup_{\phi\in\Phi} \sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}(\sup_{t\le\sigma} |B_{t+\tau}^{\phi,\delta} - B_t^{\phi,\delta}| > \eta) = 0, \tag{5.2.17b}$$

$$\lim_{\delta\to 0} \lim_{A\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\sup_{t\le T} \|M_t^{\phi,\delta}\| > A) = 0, \tag{5.2.17c}$$

$$\lim_{\delta\to 0} \limsup_{\sigma\to 0} \limsup_{\phi\in\Phi} \sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}(\sup_{t\le\sigma} \|M_{t+\tau}^{\phi,\delta} - M_t^{\phi,\delta}\| > \eta) = 0, \tag{5.2.17d}$$

$$\lim_{A\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(|h_a^\phi(x)|\,\mathbf{1}(r_\phi|x| > \delta) * \mu_T^\phi > A) = 0, \tag{5.2.17e}$$

$$\lim_{\sigma\to 0} \limsup_{\phi\in\Phi} \sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}\Big(\int_\tau^{\tau+\sigma}\int_{\mathbb{R}^d} |h_a^\phi(x)|\,\mathbf{1}(r_\phi|x| > \delta)\, \mu^\phi(ds,dx) > \eta\Big) = 0. \tag{5.2.17f}$$

We begin with $B^{\phi,\delta}$. Let $\tilde{B}^{\phi,\delta}$ be the first characteristic of X^ϕ associated with h_δ^ϕ (h_δ^ϕ is defined as h_a^ϕ with $\delta = a$). Then by (4.1.4)

$$B_t^{\phi,\delta} = \tilde{B}_t^{\phi,\delta} + (x\,\mathbf{1}(r_\phi|x| \le \delta) - h_\delta^\phi(x)) * \nu_t^\phi,$$

so, for $0 \le s < t$, using (5.2.15),

$$|B_t^{\phi,\delta} - B_s^{\phi,\delta}| \le |\tilde{B}_t^{\phi,\delta} - \tilde{B}_s^{\phi,\delta}| + \Big|\int_s^t\int_{\mathbb{R}^d} \frac{\delta}{r_\phi}\frac{x}{|x|}\,\mathbf{1}(r_\phi|x| > \delta)\nu^\phi(ds,dx)\Big|$$

$$\le |\tilde{B}_t^{\phi,\delta} - \tilde{B}_s^{\phi,\delta}| + \int_s^t\int_{\mathbb{R}^d} \tilde{f}_\delta^\phi(x)\nu^\phi(ds,dx), \tag{5.2.18}$$

where $\tilde{f}_\delta(x) = (2|x|/\delta - 1)^+ \wedge 1$ (Recall that, by the notation introduced in Section 4.2, $\tilde{f}_\delta^\phi(x) = \tilde{f}_\delta(r_\phi x)/r_\phi$.)

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Let $\tilde{B}_t^\delta(\mathbf{x})$ be the first characteristic of X associated with the limiter $h_\delta(x)$ so that it is defined by (5.2.4) with $h_\delta(x)$ as $h(x)$. Then in view of (5.2.6) $\{\mathcal{L}(\tilde{B}^\delta(X^\phi)), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight since $\{\mathcal{L}(B(X^\phi)), \phi \in \Phi\}$ and $\{\mathcal{L}((h_\delta(x) - h(x)) * \nu\ (X^\phi))), \phi \in \Phi\}$ are both $\mathbb{C}$–exponentially tight, hence, by (sup B) with h_δ in place of h and Theorem 3.2.3 the net $\{\mathcal{L}(\tilde{B}^{\phi,\delta}), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight.

Since $\tilde{f}_\delta \in \mathcal{C}_b$, the net $\{\mathcal{L}(\tilde{f}_\delta^\phi * \nu\,(X^\phi)), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight by hypotheses so that by Lemma 5.2.18 condition (sup ν) holds with $f = \tilde{f}_\delta$. Theorem 3.2.3 implies that the net $\{\mathcal{L}(\tilde{f}_\delta^\phi * \nu^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight.

Inequality (5.2.18), $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(\tilde{B}^{\phi,\delta}), \phi \in \Phi\}$ and $\{\mathcal{L}(\tilde{f}_\delta^\phi * \nu^\phi), \phi \in \Phi\}$ imply by Theorem 3.2.3 that the $B^{\phi,\delta}$ satisfy (5.2.17a) and (5.2.17b).

Now we prove (5.2.17e) and (5.2.17f). The process $(|h_a^\phi(x)|\,\mathbf{1}(r_\phi|x| > \delta) * \mu_t^\phi, t \in \mathbb{R}_+)$ has as its stochastic cumulant the process $\big((\exp(\alpha|h_a^\phi(x)|) - 1)\,\mathbf{1}(r_\phi|x| > \delta) * \nu_t^\phi, t \in \mathbb{R}_+\big), \alpha \in \mathbb{R}$. Then for $\tau \in \mathbf{S}_T(\mathbf{F}_\phi)$ and $c > 0$ by the second inequality in Lemma 5.1.15 with $d = 1$

$$P_\phi\Big(\int_\tau^{\tau+\sigma}\int_{\mathbb{R}^d} |h_a^\phi(x)|\,\mathbf{1}(r_\phi|x| > \delta)\mu^\phi(ds,dx) > \eta\Big)$$
$$\le 2\exp\Big(-r_\phi\frac{c\eta}{2}\Big)$$
$$+ 2\max_{i=1,2} P_\phi\Big(\frac{1}{r_\phi}\int_\tau^{\tau+\sigma}\int_{\mathbb{R}^d} \big(\exp((-1)^i|h_a(r_\phi x)|c) - 1\big)\,\mathbf{1}(r_\phi|x| > \delta)$$
$$\nu^\phi(ds,dx) > \frac{c\eta}{2}\Big),$$

hence,

$$P_\phi^{1/r_\phi}\Big(\int_\tau^{\tau+\sigma}\int_{\mathbb{R}^d} |h_a^\phi(x)|\,\mathbf{1}(r_\phi|x| > \delta)\mu^\phi(ds,dx) > \eta\Big)$$
$$\le 2^{1/r_\phi}\exp\Big(-\frac{c\eta}{2}\Big)$$

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$$+\, 2^{1/r_\phi} P_\phi^{1/r_\phi}\Big(\int_\tau^{\tau+\sigma} \int_{\mathbb{R}^d} \overline{f}^\phi(x)\nu^\phi(ds,dx) > \frac{c\eta}{2}\Big), \quad (5.2.19)$$

where $\overline{f}(x) = (\exp(c|h_a(x)|) - 1)\tilde{f}_\delta(x)$. Since $\overline{f}(x)$ belongs to $\mathcal{C}_b$, the net $\{\mathcal{L}(\overline{f}(x) * \nu\ (X^\phi)), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight by hypotheses. From (sup ν) and Theorem 3.2.3 we derive that the net $\{\mathcal{L}(\overline{f}^\phi(x) * \nu^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight. Then (5.2.19) and Theorem 3.2.3 imply (5.2.17f). Limit (5.2.17e) is proved similarly.

Now we prove (5.2.17c) and (5.2.17d). Denoting the stochastic cumulant associated with $M^{\phi,\delta}$ by $\tilde{G}^{\phi,\delta}(\lambda) = (\tilde{G}^{\phi,\delta}_t(\lambda), t \in \mathbb{R}_+)$ and applying Lemma 5.1.15, we reduce the proof of (5.2.17c) and (5.2.17d) to the proof of the respective limits ($i = 1,\ldots,2d$, $T > 0$, $\gamma > 0$)

$$\lim_{\delta\to 0}\lim_{A\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T}\frac{1}{r_\phi}\,\tilde{G}^{\phi,\delta}_t(r_\phi e_i) > A\Big) = 0, \quad (5.2.20a)$$

$$\lim_{\delta\to 0}\limsup_{\sigma\to 0}\limsup_{\phi\in\Phi}\sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}\Big(\sup_{t\le\sigma}\frac{1}{r_\phi}(\tilde{G}^{\phi,\delta}_{t+\tau}(r_\phi e_i) - \tilde{G}^{\phi,\delta}_\tau(r_\phi e_i)) > \gamma\Big) = 0. \quad (5.2.20b)$$

Now, by (5.2.16) the measure of jumps of $M^{\phi,\delta}$ is

$$\tilde{\mu}^{\phi,\delta}([0,t],\Gamma) = \sum_{0<s\le t}\mathbf{1}\Big(\int_{\mathbb{R}^d} x\,\mathbf{1}(r_\phi|x|\le\delta)(\mu^\phi(\{s\},dx) - \nu^\phi(\{s\},dx)) \in \Gamma\backslash\{0\}\Big),\ \Gamma\in\mathcal{B}(\mathbb{R}^d),$$

and then by (4.1.14) and the fact that the $\mathbf{F}_\phi$–predictable quadratic-variation process of $X^{\phi,c}$ is C^ϕ

$$\tilde{G}^{\phi,\delta}_t(\lambda) = \frac{1}{2}\lambda\cdot C^\phi_t\lambda + \big(\exp\big(\lambda\cdot(x\,\mathbf{1}(r_\phi|x|\le\delta) - x^{\phi,\delta}_s)\big) - 1 - \lambda\cdot(x\,\mathbf{1}(r_\phi|x|\le\delta) - x^{\phi,\delta}_s)\big) * \nu^\phi_t, \quad (5.2.21)$$

where

$$x^{\phi,\delta}_s = \int_{\mathbb{R}^d} x\,\mathbf{1}(r_\phi|x|\le\delta)\nu^\phi(\{s\},dx). \quad (5.2.22)$$

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We note that, since by (4.1.3a) $\nu^\phi(\{s\},\mathbb{R}^d) \le 1$, we have

$$|x_s^{\phi,\delta}| \le \frac{\delta}{r_\phi}. \tag{5.2.23}$$

Applying Taylor's formula to the integrand in (5.2.21), we obtain

$$\frac{1}{r_\phi}\tilde{G}_t^{\phi,\delta}(r_\phi\lambda) = T_t^{\phi,\delta}(\lambda) + r_t^{\phi,\delta}(\lambda), \tag{5.2.24}$$

where

$$T_t^{\phi,\delta}(\lambda) = \frac{1}{2}\,\lambda\cdot(r_\phi C_t^\phi)\lambda + \frac{1}{2}\,r_\phi\,(\lambda\cdot(x\,\mathbf{1}(r_\phi|x| \le \delta) - x_s^{\phi,\delta}))^2 * \nu_t^\phi \tag{5.2.25}$$

and

$$\sup_{s\le t}|r_s^{\phi,\delta}(\lambda)| \le \delta\frac{|\lambda|e^{2|\lambda|\delta}}{3} r_\phi(\lambda\cdot(x\,\mathbf{1}(r_\phi|x| \le \delta) - x_s^{\phi,\delta}))^2 * \nu_t^\phi \tag{5.2.26}$$

(for the last inequality we also used (5.2.23)).

Since by (5.2.22)

$$(\lambda\cdot(x\,\mathbf{1}(r_\phi|x| \le \delta) - x_s^{\phi,\delta}))^2 * \nu_t^\phi = (\lambda\cdot x)^2\,\mathbf{1}(r_\phi|x|\le\delta) * \nu_t^\phi \\ - \sum_{0<s\le t}(\lambda\cdot x_s^{\phi,\delta})^2\,(2 - \nu^\phi(\{s\},\mathbb{R}^d)),$$

where the sum is over s such that $\nu^\phi(\{s\},\mathbb{R}^d) > 0$, and $\nu^\phi(\{s\},\mathbb{R}^d) \le 1$, we have by (4.1.7), (5.2.22) and (5.2.25) that

$$0 \le T_t^{\phi,\delta}(\lambda) - T_s^{\phi,\delta}(\lambda) \le \lambda\cdot r_\phi(C_t^{\phi,\delta} - C_s^{\phi,\delta})\lambda,\ s < t, \tag{5.2.27}$$

and by (4.1.7) and (5.2.26) that

$$\sup_{s\le t}|r_s^{\phi,\delta}(\lambda)| \le \delta\frac{|\lambda|e^{2|\lambda|\delta}}{3}\lambda\cdot r_\phi C_t^{\phi,\delta}\lambda. \tag{5.2.28}$$

Since $\{\mathcal{L}(C(X^\phi)), \phi\in\Phi\}$ is $\mathbb{C}$–exponentially tight, an application of (sup C) yields in view of Theorem 3.2.3

$$\lim_{\substack{\delta\to 0\\ A\to\infty}}\ \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\,\|r_\phi C_{T+1}^{\phi,\delta}\| > A\big) = 0,$$

$$\lim_{\substack{\delta\to 0\\ \sigma\to 0}}\ \limsup_{\phi\in\Phi}\ \sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}\big(\sup_{t\le\sigma} r_\phi\,\|C_{t+\tau}^{\phi,\delta} - C_\tau^{\phi,\delta}\| > \gamma\big) = 0,$$

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for every $\gamma > 0$. The first of these relations together with (5.2.28) implies that

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\sup_{t\le T+1}|r_t^{\phi,\delta}(\lambda)| > \varepsilon) = 0,\ \varepsilon > 0, \tag{5.2.29}$$

and together with (5.2.27) that

$$\lim_{\delta\to 0}\lim_{A\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\sup_{t\le T+1} T_t^{\phi,\delta}(\lambda) > A) = 0, \tag{5.2.30}$$

while the second one and (5.2.27) yield

$$\lim_{\delta\to 0}\limsup_{\sigma\to 0}\limsup_{\phi\in\Phi}\sup_{\tau\in\mathbf{S}_T(\mathbf{F}_\phi)} P_\phi^{1/r_\phi}(\sup_{t\le\sigma}|T_{t+\tau}^{\phi,\delta}(\lambda) - T_\tau^{\phi,\delta}(\lambda)| > \varepsilon) = 0,\ \varepsilon > 0. \tag{5.2.31}$$

In view of (5.2.24), limits (5.2.29) and (5.2.30) prove (5.2.20a), while limits (5.2.29) and (5.2.31) prove (5.2.20b). $\mathbb{C}$–exponential tightness of the net $\{\mathcal{L}(\hat{X}^{\phi,a}),\phi\in\Phi\}$ has been proved. $\mathbb{C}$–exponential tightness of the net $\{\mathcal{L}(X^\phi),\phi\in\Phi\}$ under $(A)+(a)$ follows by Theorem 3.2.3 and Lemma 4.2.16. Finally, by Corollary 3.2.7 $\mathbb{C}'$–exponential tightness of the net $\{\mathcal{L}((\hat{X}^{\phi,a},X^\phi)),\phi\in\Phi\}$ is implied by $\mathbb{C}$–exponential tightness of both $\{\mathcal{L}(\hat{X}^{\phi,a}),\phi\in\Phi\}$ and $\{\mathcal{L}(X^\phi),\phi\in\Phi\}$.

We now assume that instead of condition (C) and $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(C(X^\phi)),\phi\in\Phi\}$ we have condition $(\tilde{C})$ and $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(\tilde{C}(X^\phi)),\phi\in\Phi\}$. As above, it is sufficient to prove $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(\hat{X}^{\phi,a}),\phi\in\Phi\}, a>0$. We again consider a canonical representation of X^ϕ, but this time with respect to $h_\delta^\phi(x)$ from (5.2.15) with $\delta < a$ so we replace a by δ in (5.2.15) and substitute throughout in the preceding argument $h_\delta^\phi(x)$ for $x\,\mathbf{1}(r_\phi|x|\le\delta)$. Retaining the above notation for the components of the representation, we again reduce the task to proving (5.2.17a)–(5.2.17f).

Limits (5.2.17a) and (5.2.17b) follow by $(\sup B)$ and $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(B(X^\phi)),\phi\in\Phi\}$ and $\{\mathcal{L}(f*\nu(X^\phi)),\phi\in\Phi\}, f\in\mathcal{C}_b$. The proofs of (5.2.17e) and (5.2.17f) do not change. The proofs of (5.2.17c) and (5.2.17d) also proceed along the same lines, (5.2.27) and (5.2.28) being replaced by (with $h_\delta^\phi(x)$ in place of

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$x\,\mathbf{1}(r_\phi|x| \le \delta)$ in (5.2.22), (5.2.25) and (5.2.26))

$$\begin{aligned} 0 \le T_t^{\phi,\delta}(\lambda) - T_s^{\phi,\delta}(\lambda) &\le \lambda \cdot r_\phi(\tilde{C}_t^{\phi,\delta} - \tilde{C}_s^{\phi,\delta})\lambda, & (5.2.32)\\ \sup_{s\le t} |r_s^{\phi,\delta}(\lambda)| &\le \delta\,\frac{|\lambda|}{3}\,e^{2|\lambda|\delta}\lambda \cdot r_\phi \tilde{C}_t^{\phi,\delta}\lambda, & (5.2.33)\end{aligned}$$

where $\tilde{C}_t^{\phi,\delta}$ is defined by (4.1.8) with $h_\delta(x)$ as $h(x)$.

By (4.1.8) for $0 \le s \le t$

$$\begin{aligned} &\lambda \cdot r_\phi(\tilde{C}_t^{\phi,\delta} - \tilde{C}_s^{\phi,\delta})\lambda \le \lambda \cdot r_\phi(\tilde{C}_t^{\phi} - \tilde{C}_s^{\phi})\lambda \\ &+ r_\phi \int_{(s,t]}\int_{\mathbb{R}^d} \left|(\lambda \cdot h^\phi(x))^2 - (\lambda \cdot h_\delta^\phi(x))^2\right| \nu^\phi(du,dx) \\ &\quad + r_\phi \sum_{s<u\le t} \left|\left(\lambda \cdot h^\phi(x) \bullet \nu_u^\phi\right)^2 - \left(\lambda \cdot h_\delta^\phi(x) \bullet \nu_u^\phi\right)^2\right|. \qquad (5.2.34)\end{aligned}$$

Let $c > 0$ be such that $h(x) = x, |x| \le c$. Then we obtain for $\delta < c$, recalling the notation $||h|| = \sup_{x\in\mathbb{R}^d} |h(x)|$,

$$\begin{aligned} &\int_{(s,t]}\int_{\mathbb{R}^d} |(\lambda \cdot h^\phi(x))^2 - (\lambda \cdot h_\delta^\phi(x))^2|\,\nu^\phi(du,dx) \\ &\le \frac{|\lambda|^2}{r_\phi^2}\,(||h||^2 + \delta^2) \int_{(s,t]}\int_{\mathbb{R}^d} \mathbf{1}(r_\phi|x| > c)\,\nu^\phi(du,dx) \\ &\qquad + |\lambda|^2 \int_{(s,t]}\int_{\mathbb{R}^d} |x|^2\,\mathbf{1}(\delta < r_\phi|x| \le c)\,\nu^\phi(du,dx)\end{aligned}$$

and

$$\begin{aligned} &\sum_{s<u\le t} \left|\left(\lambda \cdot h^\phi(x) \bullet \nu_u^\phi\right)^2 - \left(\lambda \cdot h_\delta^\phi(x) \bullet \nu_u^\phi\right)^2\right| \\ &\qquad \le \sum_{s<u\le t} \left(\lambda \cdot (h^\phi(x) - h_\delta^\phi(x)) \bullet \nu_u^\phi\right)^2\end{aligned}$$

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$$+ 2 \sum_{s<u\le t} \left|\lambda \cdot (h^\phi(x) - h^\phi_\delta(x)) \bullet \nu^\phi_u\right| \left|\lambda \cdot h^\phi_\delta(x) \bullet \nu^\phi_u\right|$$

$$\le 2\frac{|\lambda|^2}{r_\phi^2} (||h||^2 + \delta^2) \int_{(s,t]}\int_{\mathbb{R}^d} \mathbf{1}(r_\phi|x| > c)\,\nu^\phi(du, dx)$$

$$+ |\lambda|^2 \int_{(s,t]}\int_{\mathbb{R}^d} |x|^2\,\mathbf{1}(\delta < r_\phi|x| \le c)\,\nu^\phi(du, dx)$$

$$+ 2|\lambda|\,\frac{\delta}{r_\phi}\Big[\frac{|\lambda|}{r_\phi}(||h|| + \delta)\int_{(s,t]}\int_{\mathbb{R}^d} \mathbf{1}(r_\phi|x| > c)\,\nu^\phi(du, dx)$$

$$+ |\lambda| \int_{(s,t]}\int_{\mathbb{R}^d} |x|\,\mathbf{1}(\delta < r_\phi|x| \le c)\,\nu^\phi(du, dx)\Big].$$

Therefore, using (5.2.34), we have for $T > 0, \eta > 0, \sigma > 0$, and $\delta < 1 \wedge c \wedge ||h||$, by (sup $\tilde{C}$) and (sup ν)

$$\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\sup_{\substack{|s-t|\le\sigma\\ s,t\le T}} |\lambda \cdot r_\phi(\tilde{C}^{\phi,\delta}_t - \tilde{C}^{\phi,\delta}_s)\lambda| > \eta\big)$$

$$\le \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{\substack{|s-t|\le\sigma\\ s,t\le T}} |\lambda \cdot r_\phi(\tilde{C}_t(X^\phi) - \tilde{C}_s(X^\phi))\lambda| > \frac{\eta}{6}\Big)$$

$$+ \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(10|\lambda|^2\,||h||^2 \sup_{\substack{|s-t|\le\sigma\\ s,t\le T}} \int_{(s,t]}\int_{\mathbb{R}^d} \mathbf{1}(|x| > c/2)$$

$$\nu(du, dx)(X^\phi) > \frac{\eta}{6}\Big)$$

$$+ \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(4|\lambda|^2 \sup_{\substack{|s-t|\le\sigma\\ s,t\le T}} \int_{(s,t]}\int_{\mathbb{R}^d} |x|^2\,\mathbf{1}(\delta/2 < |x| \le 2c)$$

$$\nu(du, dx)(X^\phi) > \frac{\eta}{6}\Big),$$

where the right-hand side goes to 0 as $\sigma \to 0$ by $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(\tilde{C}(X^\phi)), \phi \in \Phi\}$ and $\{\mathcal{L}(f * \nu(X^\phi)), \phi \in \Phi\}, f \in \mathcal{C}_b$, and Theorem 3.2.3. By (5.2.32) this proves (5.2.31). Limits (5.2.29) and (5.2.30) are proved similarly. Thus, $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(\hat{X}^{\phi,a}), \phi \in \Phi\}$ under the new set of assumptions has been proved. The theorem has been proved. □

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Remark 5.2.21. *According to the theorem, under* (0)*,* $(a) + (A)$*,* $(\sup B)$*,* (C) *(or* $(\tilde{C})$*), and* (ν)*, and the uniform continuity conditions on* B*,* C *(or* $\tilde{C}$*), and* ν*,* $\mathbb{C}$*–exponential tightness of the net* $\{\mathcal{L}(X^\phi),\ \phi \in \Phi\}$ *is equivalent to* $\mathbb{C}$*–exponential tightness of the nets* $\{\mathcal{L}(B(X^\phi)), \phi \in \Phi\}$*,* $\{\mathcal{L}(C(X^\phi)), \phi \in \Phi\}$ *(or* $\{\mathcal{L}(\tilde{C}(X^\phi)), \phi \in \Phi\}$ *), and* $\{\mathcal{L}(f(x) * \nu\ (X^\phi)), \phi \in \Phi\}, f \in \mathcal{C}_b$*.*

5.2.2 LD accumulation points as solutions to maxingale problems

In this subsection, assuming that either the net $\{\mathcal{L}((\hat{X}^{\phi,a}, X^\phi)), \phi \in \Phi\}$ is $\mathbb{C}'$–exponentially tight or the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight, we characterise their LD accumulation points as solutions to maxingale problems.

The first step is to consider small jumps as in Theorem 4.2.11. As in Subsection 4.2.2 the triplet of $\hat{X}^{\phi,a}$ without truncation is $(B^{\phi,a}, C^\phi, \nu^{\phi,a})$, where $B^{\phi,a}$ is the first characteristic of B^ϕ corresponding to h_a and

$$\nu^{\phi,a}([0,t],\Gamma) = \nu^\phi([0,t],(h_a^\phi)^{-1}(\Gamma)),\ \Gamma \in \mathcal{B}(\mathbb{R}^d), t \in \mathbb{R}_+ . \tag{5.2.35}$$

We also note that by (5.2.15) and (5.2.35)

$$\nu^{\phi,a}([0,t],\{r_\phi|x| > a\}) = 0,\ t \in \mathbb{R}_+, \phi \in \Phi. \tag{5.2.36}$$

Since the jumps of $\hat{X}^{\phi,a}$ are bounded in modulus by a/r_ϕ, its stochastic exponential is well defined. We denote it by $\hat{\mathcal{E}}^{\phi,a}(\lambda) = (\hat{\mathcal{E}}_t^{\phi,a}(\lambda), t \in \mathbb{R}_+), \lambda \in \mathbb{R}^d$.

The “limiting” semimaxingale X^a is defined in analogy with Subsection 4.2.2 as having characteristics $(B^a, C, \nu^a, \hat{\nu}^a)$ relative to h_a, which are defined for $\mathbf{x} \in \mathbb{D}$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ by

$$B_t^a(\mathbf{x}) = B_t'(\mathbf{x}) + (h_a(x) - x) * \nu_t(\mathbf{x}), \tag{5.2.37}$$

$$\nu_t^a(\Gamma;\mathbf{x}) = \nu_t(h_a^{-1}(\Gamma);\mathbf{x}),\ \ \hat{\nu}_t^a(\mathbf{x}) = \hat{\nu}_t(h_a^{-1}(\Gamma);\mathbf{x}). \tag{5.2.38}$$

We also note that

$$\nu_t^a(\{|x| > a\};\mathbf{x}) = 0. \tag{5.2.39}$$

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The associated cumulant is given by

$$\hat{G}_t^a(\lambda;\mathbf{x}) = \lambda\cdot B_t^a(\mathbf{x}) + \frac{1}{2}\,\lambda\cdot C_t(\mathbf{x})\lambda + (e^{\lambda\cdot x} - 1 - \lambda\cdot h_a(x)) * \nu_t^a(\mathbf{x})$$
$$+ \int_0^t \Big(\ln\big(1 + (e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s^a(\mathbf{x})\big) - \big(e^{\lambda\cdot x} - 1\big) \bullet \hat{\nu}_s^a(\mathbf{x})\Big)\,ds. \tag{5.2.40}$$

Obviously $\hat{G}_t^a(\lambda;\mathbf{x})$ is continuous in t and $\mathcal{D}_t$–measurable in $\mathbf{x}$.

Let $(\hat{M}^a)$ denote the maxingale problem (M') introduced in Subsection 5.1.1 with $G(\lambda)$ replaced with $\hat{G}^a(\lambda) = (\hat{G}_t^a(\lambda;\mathbf{x}),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D}), \lambda \in \mathbb{R}^d$, i.e., deviability $\hat{\Pi}^a$ on $\mathbb{C}'$ solves $(\hat{M}^a)$ if

	$\mathbf{x}_0 = 0$	$\hat{\Pi}^a$–a.e.
$(\hat{M}^a)$	$\hat{Y}^a(\lambda), \lambda \in \mathbb{R}^d,$	is a $\mathbb{C}'$–local exponential maxingale on $(\mathbb{C}', \hat{\Pi}^a)$,

where $\hat{Y}^a(\lambda) = (\hat{Y}_t^a(\lambda;(\mathbf{x},\mathbf{x}')), t \in \mathbb{R}_+, (\mathbf{x},\mathbf{x}') \in \mathbb{C}')$ is defined by

$$\hat{Y}_t^a(\lambda;(\mathbf{x},\mathbf{x}')) = \exp(\lambda\cdot\mathbf{x}_t - \hat{G}_t^a(\lambda;\mathbf{x}')). \tag{5.2.41}$$

Theorem 5.2.22. *Let the net* $\{\mathcal{L}((\hat{X}^{\phi,a}, X^\phi)), \phi \in \Phi\}$ *be* $\mathbb{C}'$*–exponentially tight, let B satisfy the uniform continuity condition, and let C (respectively $\tilde{C}$), ν and $\hat{\nu}$ satisfy the continuity conditions. If conditions* (0), (sup B), (C) *(respectively $(\tilde{C})$), (ν), and $(\hat{\nu})$ hold, then every LD accumulation point of* $\{\mathcal{L}((\hat{X}^{\phi,a}, X^\phi)), \phi \in \Phi\}$ *solves* $(\hat{M}^a)$.

The idea of the proof is to apply Theorem 5.1.16 to the pair $(\hat{X}^{\phi,a}, X^\phi)$ (note that $\hat{X}^{\phi,a}$ plays the role of X^ϕ in Theorem 5.1.16, and X^ϕ plays the role of X'^ϕ). We again need auxiliary results. The second part of the next lemma extends Lemma 4.2.9.

Lemma 5.2.23. I. *Let the net* $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ *be* $\mathbb{C}$*–exponentially tight and ν satisfy the continuity condition. Then, for $c > 0, \varepsilon > 0$ and $t > 0$*

1. $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(|x|^2\,\mathbf{1}(|x| \le \delta) * \nu_t\,(X^\phi) \ge \varepsilon\big) = 0,$$

2. $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\delta(|x| \wedge c)\,\mathbf{1}(|x| \ge \delta) * \nu_t\,(X^\phi) \ge \varepsilon\big) = 0.$$

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II. *If, in addition, conditions* (ν) *and* $(\hat{\nu})$ *hold, then for* $\varepsilon > 0$ *and* $t \in U$

1. $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\Big|\frac{1}{r_\phi} f(r_\phi x)\,\mathbf{1}(r_\phi|x|>\delta)*\nu_t^\phi - f(x)*\nu_t(X^\phi)\Big| > \varepsilon\Big) = 0$$

for all $\mathbb{R}_+$*-valued bounded continuous functions* $f(x), x \in \mathbb{R}^d$*, such that* $f(x) \le c|x|^2$ *in a neighbourhood of* 0 *for some* $c > 0$*;*

2. $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{\delta}{r_\phi}|g(r_\phi x)|\,\mathbf{1}(r_\phi|x|>\delta)*\nu_s^\phi > \varepsilon\Big) = 0,$$

and

3. $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\Big|\frac{1}{r_\phi}\sum_{0<s\le t}\big(g(r_\phi x)\,\mathbf{1}(r_\phi|x|>\delta)\bullet\nu_s^\phi\big)^k - \int_0^t \big(g(x)\bullet\hat{\nu}_s(X^\phi)\big)^k ds\Big| > \varepsilon\Big) = 0,\ k = 2,3,\ldots$$

for all $\mathbb{R}$*-valued, bounded and continuous functions* $g(x), x \in \mathbb{R}^d$*, such that* $|g(x)| \le c|x|$ *in a neighbourhood of* 0 *for some* $c > 0$*.*

Proof. We begin with part I. We denote

$$H_\delta(\mathbf{x}) = |x|^2\,\mathbf{1}(|x|\le\delta)*\nu_t(\mathbf{x}),\ \mathbf{x}\in\mathbb{D}. \tag{5.2.42}$$

Let a subnet $\{(\mathcal{L}(X^{\phi'}), r_{\phi'}, p_{\phi'}),\ \phi' \in \Phi'\}$ of $\{(\mathcal{L}(X^\phi), r_\phi, P_\phi^{1/r_\phi}(H_\delta(X^\phi)\ge\varepsilon)),\ (\phi,\delta)\in\Phi\times(0,\infty)\}$ be such that

$$\lim_{\phi'\in\Phi'} p_{\phi'} = \limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(H_\delta(X^\phi)\ge\varepsilon\big)$$

and $\mathcal{L}(X^{\phi'}) \overset{ld}{\to} \Pi'$ at rate $r_{\phi'}$, where deviability Π' is supported by $\mathbb{C}$. Then for arbitrary $\eta > 0$ and $t \in U$

$$\limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(H_\delta(X^\phi)\ge\varepsilon\big) = \lim_{\phi'\in\Phi'} p_{\phi'}$$
$$\le \limsup_{\phi'\in\Phi'} P_{\phi'}^{1/r_{\phi'}}\big(H_\eta(X^{\phi'})\ge\varepsilon\big) \le \Pi'\big(\mathbf{x}\in\mathbb{C} : H_\eta(\mathbf{x})\ge\varepsilon\big),$$

where the latter inequality follows by Corollary 3.1.9 since the continuity condition on ν implies that $H_\eta(\mathbf{x})$ is $\mathbb{C}$–upper-semi-continuous. Upper semi-continuity of $H_\eta(\mathbf{x})$ on $\mathbb{C}$ implies that the sets $\{\mathbf{x}\in\mathbb{C} :$

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$H_\eta(\mathbf{x}) \geq \varepsilon\}$ are closed. They converge to $\emptyset$ as $\eta \to 0$ by Lebesgue's bounded convergence theorem. The τ-smoothness property of deviability then implies that $\lim_{\eta\to 0} \Pi'\big(\mathbf{x} \in \mathbb{C} : H_\eta(\mathbf{x}) \geq \varepsilon\big) = 0$. The first assertion of part I is proved.

For the second, picking $\sigma > 0$, we have for δ small enough and $\mathbf{x} \in \mathbb{D}$,

$$\begin{aligned} &\delta(|x| \wedge c)\,\mathbf{1}(|x| \geq \delta) * \nu_t\,(\mathbf{x}) \\ &\quad \leq \delta(|x| \wedge c)\,\mathbf{1}(|x| \geq \sigma) * \nu_t\,(\mathbf{x}) \\ &\qquad\qquad + |x|^2\,\mathbf{1}(|x| \leq \sigma) * \nu_t\,(\mathbf{x}), \end{aligned} \tag{5.2.43}$$

where for the second summand, which is well defined by (2.7.53a) and (2.7.54), we used Chebyshev's inequality. Let $k(x) = (2|x|/\sigma - 1)^+ \wedge 1$, $x \in \mathbb{R}^d$. Since the function $(|x|\wedge c)k(x)$ belongs to $\mathcal{C}_b$ and we can assume that $t \in U$, $\mathbb{C}$–exponential tightness of $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ and the continuity condition on ν imply by Corollary 3.1.22 that the net $\{\mathcal{L}((|x| \wedge c)k(x) * \nu_t\,(X^\phi)), \phi \in \Phi\}$ is exponentially tight in $\mathbb{R}$; hence, by Theorem 3.2.3, since $\mathbf{1}(|x| \geq \sigma) \leq k(x)$,

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\delta(|x|\wedge c)\,\mathbf{1}(|x| \geq \sigma) * \nu_t(X^\phi) \geq \frac{\varepsilon}{2}\Big) = 0.$$

Inequality (5.2.43) and the first assertion of part I yield the required. Part I is proved.

Part II is proved in analogy with Lemma 4.2.9, necessary modifications make use of part I and are obvious. □

The next lemma follows by Lemma 5.2.23 in the same way as Lemma 4.2.10 follows by Lemma 4.2.9.

Lemma 5.2.24. *Let the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ be $\mathbb{C}$-exponentially tight and ν satisfy the continuity condition. Then, under conditions (ν) and ($\hat{\nu}$), condition (C) is equivalent to condition ($\tilde{C}$), which, hence, does not depend on the choice of h.*

Lemma 5.2.25. *Under the uniform continuity condition on B and the continuity conditions on C (or $\tilde{C}$), ν and $\hat{\nu}$, the function $\hat{G}^a(\lambda)$ satisfies the uniform continuity condition.*

Proof. Since under the continuity conditions on ν and $\hat{\nu}$ the continuity conditions for C and $\tilde{C}$ are equivalent, we can assume that the

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continuity condition on C holds. We prove that each of the functions on the right of (5.2.40) is $\mathbb{C}$–continuous as a map from $\mathbb{D}$ into $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$. The function B^a has the required property by hypotheses and the fact that, in view of the continuity condition on ν, the uniform continuity condition for B does not depend on a limiter. The same fact is clearly also true for the next two terms on the right-hand side of (5.2.40) (use (5.2.38) for the third term).

Let us consider the last term. Recalling the notation $\psi(x) = x - \ln(1+x)$, $x > -1$ (see (4.2.26)), we have, since $\psi(x) \geq 0$, that it is sufficient to show that $\int_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s^a(\mathbf{x})\big)ds$ is $\mathbb{C}$–continuous for each $t \in U$. Let $\mathbf{x}_n \to \hat{\mathbf{x}} \in \mathbb{C}$ and $\varepsilon > 0$. By (5.2.39)

$$e^{-|\lambda|a} - 1 \leq \int\limits_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1)\hat{\nu}_s^a(dx;\mathbf{x}) \leq e^{|\lambda|a} - 1, \ \mathbf{x} \in \mathbb{D}.$$

Also by Weierstrass' theorem there exists a polynomial $q(u) = \sum_{k=2}^{l} d_k \mathbf{x}^k$, $l \geq 2, u \in \mathbb{R}_+$, such that $|\psi(u) - q(u)| \leq \varepsilon$ for $x \in [\exp(-|\lambda|a) - 1, \exp(|\lambda|a) - 1]$. Since by the continuity condition on $\hat{\nu}$ and (5.2.38)

$$\lim_{n\to\infty} \int\limits_0^t q\big((e^{\lambda\cdot x} - 1)\bullet\hat{\nu}_s^a(\mathbf{x}_n)\big)ds = \int\limits_0^t q\big((e^{\lambda\cdot x} - 1)\bullet\hat{\nu}_s^a(\hat{\mathbf{x}})\big)ds,$$

we obtain that

$$\limsup_{n\to\infty}\Big|\int\limits_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s^a(\mathbf{x}_n)\big)ds - \int\limits_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s^a(\hat{\mathbf{x}})\big)ds\Big| \leq 2\varepsilon.$$

Since ε is arbitrary, the lemma is proved. □

Proof of Theorem 5.2.22. Since by Lemma 5.2.25 the function $\hat{G}^a(\lambda)$ satisfies the uniform continuity condition, it is sufficient to prove in view of Theorem 5.1.16 that

$$\sup_{t\leq T} \Big|\frac{1}{r_\phi} \ln \hat{\mathcal{E}}_t^{\phi,a}(r_\phi\lambda) - \hat{G}_t^a(\lambda; X^\phi)\Big| \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi \in \Phi, \ T > 0. \tag{5.2.44}$$

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The idea is to follow the proof of Theorem 4.2.1 in order to derive (5.2.44) from convergence of the characteristics of the $\hat{X}^{\phi,a}, \phi \in \Phi$. Let $\hat{C}_t^{\phi,a,\delta}$ denote the modified second characteristic of $\hat{X}^{\phi,a}$ corresponding to the truncation function $x\,\mathbf{1}(|x| \le \delta)$ so that

$$\lambda \cdot \hat{C}_t^{\phi,a,\delta}\lambda = \lambda \cdot C_t^{\phi}\lambda + (\lambda \cdot x)^2\,\mathbf{1}(r_\phi|x|\le\delta) * \nu_t^{\phi,a} - \sum_{0<s\le t}\bigl(\lambda\cdot x\;\mathbf{1}(r_\phi|x|\le\delta)\bullet\nu_s^{\phi,a}\bigr)^2,\ \lambda \in \mathbb{R}^d. \quad (5.2.45)$$

We note that the following conditions hold

(sup B^a) $$\sup_{t\le T}|B_t^{\phi,a} - B_t^a(X^\phi)| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ T>0,$$

(C^a) $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\,\|r_\phi\,\hat{C}_t^{\phi,a,\delta} - C_t(X^\phi)\| > \varepsilon) = 0,\ t\in U, \varepsilon>0,$$

(ν^a) $$f^\phi(x) * \nu_t^{\phi,a} - f(x) * \nu_t^a(X^\phi) \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\ \ t\in U, f\in\mathcal{C}_b,$$

($\hat{\nu}^a$) $$\frac{1}{r_\phi}\sum_{0<s\le t}\bigl(f(r_\phi x)\bullet\nu_s^{\phi,a}\bigr)^k - \int_0^t\bigl(f(x)\bullet\hat{\nu}_s^a(X^\phi)\bigr)^k ds \overset{P_\phi^{1/r_\phi}}{\to} 0$$
$$\text{as } \phi\in\Phi,\ \ t\in U, k=2,3,\ldots, f\in\mathcal{C}_b.$$

Proof is by the argument of the proof of Lemma 4.2.13. Condition (sup B^a) is actually condition (sup B) with $h_a(x)$ as $h(x)$ and holds since (sup B) does not depend on the choice of a limiter by Lemma 5.2.19 and Theorem 5.2.20. Since by (5.2.35) for $\delta < a$

$$\nu^{\phi,a}([0,t],\Gamma\cap\{r_\phi|x|\le\delta\}) = \nu^\phi([0,t],\Gamma\cap\{r_\phi|x|\le\delta\}),$$

(5.2.45) and (4.1.7) imply that $\hat{C}_t^{\phi,a,\delta} = C_t^{\phi,\delta}$ when $\delta < a$, so that conditions (C^a) and (C) coincide. Similarly, the argument of the proof of Lemma 4.2.13 shows in view of (5.2.38) and (5.2.35) that (ν^a) and ($\hat{\nu}^a$) are implied by (ν) and ($\hat{\nu}$).

We now prove that conditions (sup B^a), (C^a), (ν^a), and ($\hat{\nu}^a$) imply (5.2.44). This is carried out analogously to the proof of Theorem 4.2.11. Therefore, we do not give all the details but only indicate modifications that have to be made in that proof. For this reason, we extensively use the notation of the proof of Theorem 4.2.11.

Let us first note that by (5.2.37) and (5.2.38) the functions $B_t^a(\mathbf{x})$ and $\nu_t^a(\Gamma;\mathbf{x})$ satisfy the same continuity conditions as imposed on

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$B_t(\mathbf{x})$ and $\nu_t(\Gamma;\mathbf{x})$ in the statement of Theorem 5.2.22. Also by Lemma 5.2.19 conditions (sup C^a) and (sup ν^a) hold (with obvious notation).

We now define in analogy with (4.2.14), (4.2.15a)– (4.2.15h), substituting $\nu^{\phi,a}$ for ν^ϕ,

$$a_s^\phi = \nu^{\phi,a}(\{s\},\mathbb{R}^d),$$

and, for $\delta > 0$ and $\lambda \in \mathbb{R}^d$,

$$\begin{aligned}
x_s^{\phi,\delta} &= x\,\mathbf{1}(r_\phi|x| \le \delta) \bullet \nu_s^{\phi,a},\\
D_s^{\phi,\delta}(\lambda) &= (e^{\lambda\cdot x} - 1)\ \mathbf{1}(r_\phi|x| > \delta) \bullet \nu_s^{\phi,a},\\
R_s^{\phi,\delta}(\lambda) &= \big(\exp(\lambda\cdot(x\,\mathbf{1}(r_\phi|x| \le \delta) - x_s^{\phi,\delta})) - 1\\
&\quad -\lambda\cdot(x\,\mathbf{1}(r_\phi|x| \le \delta) - x_s^{\phi,\delta})\big) \bullet \nu_s^{\phi,a},\\
Q_s^{\phi,\delta}(\lambda) &= (\exp(-\lambda\cdot x_s^{\phi,\delta}) - 1 + \lambda\cdot x_s^{\phi,\delta})(1 - a_s^\phi),\\
G_s^{\phi,\delta}(\lambda) &= \exp(-\lambda\cdot x_s^{\phi,\delta}) D_s^{\phi,\delta}(\lambda) + R_s^{\phi,\delta}(\lambda) + Q_s^{\phi,\delta}(\lambda),\\
U_t^{\phi,\delta}(\lambda) &= (e^{\lambda\cdot x} - 1 - \lambda\cdot x)\ \mathbf{1}(r_\phi|x|\le\delta) * \nu_s^{\phi,a,c},\\
V_t^{\phi,\delta}(\lambda) &= (e^{\lambda\cdot x} - 1 - \lambda\cdot x)\ \mathbf{1}(r_\phi|x|>\delta) * \nu_s^{\phi,a},
\end{aligned}$$

where $t \in \mathbb{R}_+, s \in \mathbb{R}_+$, and $\nu^{\phi,a,c}(ds,dx)$ is the continuous part of $\nu^{\phi,a}(ds,dx)$.

Let, as in Lemma 4.2.12,

$$\begin{aligned}
Y_t^{\phi,\delta}(\lambda) &= \sum_{0<s\le t} \psi(D_s^{\phi,\delta}(\lambda)), \qquad (5.2.46)\\
Z_t^{\phi,\delta}(\lambda) &= \sum_{0<s\le t} \ln(1 + G_s^{\phi,\delta}(\lambda)) + U_t^{\phi,\delta}(\lambda) + \frac{1}{2}\,\lambda\cdot C_t^\phi \lambda\\
&\quad - \sum_{0<s\le t} \ln(1 + D_s^{\phi,\delta}(\lambda)),
\end{aligned}$$

and, as in (4.2.23)–(4.2.25),

$$\begin{aligned}
V_t(\lambda;\mathbf{x}) &= (e^{\lambda\cdot x} - 1 - \lambda\cdot x) * \nu_t^a(\mathbf{x}),\\
\bar{Y}_t(\lambda;\mathbf{x}) &= \int_0^t \psi\big((e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s^a(\mathbf{x})\big)ds,\\
Z_t(\lambda;\mathbf{x}) &= \frac{1}{2}\,\lambda\cdot C_t(\mathbf{x})\lambda
\end{aligned}$$

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(we "bar" here $\bar{Y}_t(\lambda;\mathbf{x})$ not to confuse it with earlier notation). All the quantities above are well defined by the same argument as in Subsection 4.2.1.

Then exactly as in the proof of Theorem 4.2.11 convergence (5.2.44) would hold provided for every $T>0$ and $\varepsilon>0$

$$\alpha)\qquad \sup_{t\le T}|B_t^{\phi,a}-B_t^a(X^\phi)|\overset{P_\phi^{1/r_\phi}}{\to}0\ \text{ as }\phi\in\Phi,$$

$$\beta)\qquad \lim_{\delta\to0}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(\sup_{t\le T}|\frac{1}{r_\phi}V_t^{\phi,\delta}(r_\phi\lambda)-V_t(\lambda;X^\phi)|>\varepsilon\Big)=0,$$

$$\gamma)\qquad \lim_{\delta\to0}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(\sup_{t\le T}|\frac{1}{r_\phi}Y_t^{\phi,\delta}(r_\phi\lambda)-\bar{Y}_t(\lambda;X^\phi)|>\varepsilon\Big)=0,$$

$$\delta)\qquad \lim_{\delta\to0}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(\sup_{t\le T}|\frac{1}{r_\phi}Z_t^{\phi,\delta}(r_\phi\lambda)-Z_t(\lambda;X^\phi)|>\varepsilon\Big)=0.$$

Limit α) is just ($\sup B^a$) which we have already proved. For part β), we first note that by part II.1 of Lemma 5.2.23 applied to $\{\mathcal{L}(\hat{X}^{\phi,a}),\phi\in\Phi\}$ and ν^a, in which hypotheses boundedness of the associated function f follows by (5.2.36) and (5.2.39), we have that

$$\lim_{\delta\to0}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(|\frac{1}{r_\phi}V_t^{\phi,\delta}(r_\phi\lambda)-V_t(\lambda;X^\phi)|>\varepsilon\Big)=0.$$

Since by Theorem 3.2.3 and (5.2.39) the net $\{\mathcal{L}((V_t(\lambda;X^\phi),\,t\in\mathbb{R}_+)),\phi\in\Phi\}$ is $\mathbb{C}$–exponentially tight, Lemma 5.2.18 implies β).

We prove γ) by the argument of the proof of part γ) in Theorem 4.2.11 (a similar argument we have already used in the proof of Lemma 5.2.25). Theorem 3.2.3 implies in view of (5.2.2) and (5.2.39) that the net $\{\mathcal{L}(\bar{Y}_t(\lambda;X^\phi),t\in\mathbb{R}_+),\phi\in\Phi\}$ is $\mathbb{C}$–exponentially tight. Therefore, by (5.2.46) we have, in view of Lemma 5.2.18, that γ) would follow from

$$\lim_{\delta\to0}\limsup_{\phi\in\Phi}P_\phi^{1/r_\phi}\Big(|\frac{1}{r_\phi}Y_t^{\phi,\delta}(r_\phi\lambda)-\bar{Y}_t(\lambda;X^\phi)|>\varepsilon\Big)=0,$$
$$\varepsilon>0,t\in U.\quad (5.2.47)$$

Next, noting that by (5.2.36) $e^{-|\lambda|a}-1\le(e^{r_\phi\lambda\cdot x}-1)\bullet\nu_s^{\phi,a}\le e^{|\lambda|a}-1$ and in view of (5.2.39), we have as in the proof of γ) while proving

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Theorem 4.2.11 that (5.2.47) is implied by

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\Big|\frac{1}{r_\phi}\sum_{0<s\le t} D_s^{\phi,\delta}(r_\phi\lambda)^k - \int_0^t \big((e^{\lambda\cdot x}-1)\bullet\hat\nu_s^a(X^\phi)\big)^k ds\Big| > \eta\Big) = 0,$$

$$\eta > 0, k = 2,3,\ldots, \quad (5.2.48)$$

$$\lim_{A\to\infty}\limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}\sum_{0<s\le t} D_s^{\phi,\delta}(r_\phi\lambda)^2 > A\Big) = 0, \quad (5.2.49)$$

and

$$\lim_{A\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\int_0^t \big(|e^{\lambda\cdot x}-1|\bullet\hat\nu_s^a(X^\phi)\big)^2 ds > A\Big) = 0, \quad (5.2.50)$$

where $t \in U$. Limit (5.2.50) follows by (5.2.39). Limit (5.2.49) is easily deduced from ($\hat\nu^a$) and (5.2.50). Limit (5.2.48) follows by part II.2 of Lemma 5.2.23, (5.2.36) and (5.2.39). Part γ) is proved.

To prove δ), we introduce as in the proof of Theorem 4.2.11

$$\begin{aligned} H_s^{\phi,\delta}(\lambda) &= \frac{1}{2}|\lambda\cdot(x\,\mathbf{1}(r_\phi|x|\le\delta) - x_s^{\phi,\delta})|^2\bullet\nu_s^{\phi,a} \\ &\quad + \frac{1}{2}|\lambda\cdot x_s^{\phi,\delta}|^2(1-a_s^\phi), \\ W_t^{\phi,\delta}(\lambda) &= \frac{1}{2}|\lambda\cdot x|^2\ \mathbf{1}(r_\phi|x|\le\delta) * \nu_t^{\phi,a,c}. \end{aligned}$$

Then by (5.2.45) and the definitions of $x_s^{\phi,\delta}$ and a_s^ϕ

$$\frac{1}{2}\lambda\cdot\hat C_t^{\phi,a,\delta}\lambda = \frac{1}{2}\lambda\cdot C_t^\phi\lambda + W_t^{\phi,\delta}(\lambda) + \sum_{0<s\le t} H_s^{\phi,\delta}(\lambda)$$

so that by the definitions of $Z_t^{\phi,\delta}(\lambda)$ and $Z_t(\lambda;\mathbf{x})$ convergence (sup C^a) implies that δ) would follow by

δ') $$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\sup_{t\le T}\Big|\frac{1}{r_\phi}(U_t^{\phi,\delta}(r_\phi\lambda) - W_t^{\phi,\delta}(r_\phi\lambda))\Big| > \varepsilon\Big) = 0,$$

$$\varepsilon > 0, T > 0,$$

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$$\delta'')\qquad \lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{1}{r_\phi}\sum_{0<s\le t} |\ln(1+G_s^{\phi,\delta}(r_\phi\lambda)) - (H_s^{\phi,\delta}(r_\phi\lambda) + \ln(1+D_s^{\phi,\delta}(r_\phi\lambda)))| > \varepsilon\Big) = 0,\quad \varepsilon>0, t>0.$$

Limit $\delta')$ is proved as $\delta')$ in the proof of Theorem 4.2.11 if we note that by Theorem 5.2.20 the net $\{\mathcal{L}(C(X^\phi)),\phi\in\Phi\}$ is $\mathbb{C}$–exponentially tight so that by (C^a) and Theorem 3.2.3

$$\lim_{A\to\infty}\limsup_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(|r_\phi \hat{C}_t^{\phi,a,\delta}| > A) = 0. \tag{5.2.51}$$

As for $\delta'')$, the argument is again as in the proof of Theorem 4.2.11. We first note that by part II.3 of Lemma 5.2.23 and (5.2.36)

$$\lim_{\delta\to 0}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\Big(\frac{\delta}{r_\phi}\sum_{0<s\le t}|D_s^{\phi,\delta}(r_\phi\lambda)| > \varepsilon\Big) = 0,\ \varepsilon>0.$$

Now the rest of the proof is the same as the proof of $\delta'')$ in the proof of Theorem 4.2.11 (with the use of (5.2.51) in due place).

Thus, $\alpha)$, $\beta)$, $\gamma)$, and $\delta)$ have been proved, and (5.2.44) has been proved. By Theorem 5.1.16, we have thus proved Theorem 5.2.22 under conditions (sup B), (C), (ν), and $(\hat{\nu})$. The fact that (C) can be replaced by $(\tilde{C})$ follows by Lemma 5.2.24. Theorem 5.2.22 has been proved. □

Theorem 5.2.26. *Let the net $\{\mathcal{L}(X^\phi),\ \phi\in\Phi\}$ be $\mathbb{C}$–exponentially tight, let B satisfy the uniform continuity condition, and let C (respectively, $\tilde{C}$), ν, and $\hat{\nu}$ satisfy the continuity conditions. Let also ν and $\hat{\nu}$ satisfy the $\mathbb{C}$–local boundedness conditions* (5.2.8) *and* (5.2.9). *If conditions* (0), $(A)+(a)$, (sup B), (C) *(respectively,* $(\tilde{C})$*),* (ν)*, and* $(\hat{\nu})$ *hold, then every LD accumulation point of* $\{\mathcal{L}(X^\phi),\ \phi\in\Phi\}$ *is a solution of* $(0,G)$.

The idea of the proof is to "take the limit as $a\to\infty$" in Theorem 5.2.22. The next lemma proves that limits in the weak topology of solutions to $(\hat{M}^a)$ solve (x_0,G).

Lemma 5.2.27. *Let ν and $\hat{\nu}$ satisfy the $\mathbb{C}$–local boundedness conditions* (5.2.8) *and* (5.2.9)*, B satisfy the uniform continuity condition, and C, ν and $\hat{\nu}$ satisfy the continuity conditions. Let deviabilities $\hat{\Pi}^a$, $a>0$, on $\mathbb{C}'$ solve $(\hat{M}^a)$. Let Π be a deviability on $\mathbb{C}$ and deviability $\hat{\Pi}$ on $\mathbb{C}'$ be defined as $\hat{\Pi}(\mathbf{x},\mathbf{x}') = \Pi(\mathbf{x})\,\mathbf{1}(\mathbf{x}=\mathbf{x}')$, $(\mathbf{x},\mathbf{x}')\in\mathbb{C}'$. If $\hat{\Pi}^a \overset{iw}{\to} \hat{\Pi}$ as $a\to\infty$, then Π solves $(0,G)$.*

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Proof. We begin by proving that for every compact $K \subset \mathbb{C}$

$$\lim_{a\to\infty} \sup_{\mathbf{x}\in K} \sup_{s\le t} |\hat{G}^a_s(\lambda;\mathbf{x}) - G_s(\lambda;\mathbf{x})| = 0,\ t > 0. \tag{5.2.52}$$

By the definitions of $\hat{G}^a(\lambda)$ and $G(\lambda)$

$$|\hat{G}^a_s(\lambda;\mathbf{x}) - G_s(\lambda;\mathbf{x})| \le e^{|\lambda\cdot x|}\,\mathbf{1}(|x|>a) * \nu_s(\mathbf{x}) + \Big|\int_0^t \psi\big(1 + (e^{\lambda\cdot h_a(x)} - 1) \bullet \hat{\nu}_s(\mathbf{x})\big)\,ds - \int_0^t \psi\big(1 + (e^{\lambda\cdot x} - 1) \bullet \hat{\nu}_s(\mathbf{x})\big)\,ds\Big| \tag{5.2.53}$$

The first term on the right tends to 0 as $a \to \infty$ uniformly over $\mathbf{x} \in K$ by (5.2.8). Since by (5.2.9) as in the proof of Lemma 4.2.17

$$\lim_{a\to\infty} \sup_{\mathbf{x}\in K} \sup_{s\le t} e^{\lambda\cdot x}\,\mathbf{1}(|x|>a) \bullet \hat{\nu}_s(\mathbf{x}) = 0,$$

$$\liminf_{a\to\infty} \inf_{\mathbf{x}\in K} \inf_{s\le t} \Big(1 + \int_{\mathbb{R}^d} (e^{\lambda\cdot h_a(x)} - 1) \bullet \hat{\nu}_s(\mathbf{x})\Big) > 0,$$

the second term on the right of (5.2.53) also tends to 0. Limit (5.2.52) has been proved.

Now, for $r \in \mathbb{R}_+$ and $(\mathbf{x},\mathbf{x}') \in \mathbb{C}'$ we introduce the $\mathbf{C}'$–stopping times

$$\gamma^r(\mathbf{x},\mathbf{x}') = \inf\{t \in \mathbb{R}_+ : G^*_t(\lambda;\mathbf{x}') \vee \mathbf{x}^*_t + t \ge r\} \tag{5.2.54}$$

and

$$\gamma^{r,a}(\mathbf{x},\mathbf{x}') = \inf\{t \in \mathbb{R}_+ : \hat{G}^{a\,*}_t(\lambda;\mathbf{x}') \vee \mathbf{x}^*_t + t \ge r\},\ a > 0. \tag{5.2.55}$$

By Lemma 5.2.25 $\hat{G}^a(\lambda)$ satisfies the uniform continuity condition, and (5.2.52) then implies that the map $\mathbf{x} \to (G_t(\lambda;\mathbf{x}), t \in \mathbb{R}_+)$ is a continuous map from $\mathbb{C}$ into $\mathbb{C}(\mathbb{R}_+,\mathbb{R})$. Therefore, by Lemma 5.1.9 γ^r and $\gamma^{r,a}$ are continuous in $(\mathbf{x},\mathbf{x}') \in \mathbb{C}'$, also by (5.2.52) for every compact $K' \subset \mathbb{C}'$

$$\lim_{a\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in K'} |\gamma^{r,a}(\mathbf{x},\mathbf{x}') - \gamma^r(\mathbf{x},\mathbf{x}')| = 0,\ r \in \mathbb{R}_+. \tag{5.2.56}$$

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Since $\hat{\Pi}^a$ is a solution to $(\hat{M}^a)$, $\hat{Y}^a(\lambda)$ defined by (5.2.41) is a $\mathbf{C}'$–local exponential maxingale on $(\mathbb{C}', \hat{\Pi}^a)$. It is also continuous in the time variable, hence, by part 2 of Lemma 2.3.13 and continuity of $\gamma^{r,a}(\mathbf{x}, \mathbf{x}')$ the function $\hat{Y}^{a,N}(\lambda) = \big(\hat{Y}^a_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}')), t \in \mathbb{R}_+\big)$ is a $\mathbf{C}'$–local exponential maxingale on $(\mathbb{C}', \hat{\Pi}^a)$. Since by (5.2.41) and (5.2.55) $\hat{Y}^{a,N}(\lambda)$ is bounded, we conclude that $\hat{Y}^{a,N}(\lambda)$ is a $\mathbf{C}'$–uniformly maximable exponential maxingale on $(\mathbb{C}', \hat{\Pi}^a)$. Hence, for all $0 \le s < t$ and every $\mathbb{R}_+$-valued continuous and bounded $\mathcal{C}'_s$–measurable function $f(\mathbf{x}, \mathbf{x}')$

$$\sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} \hat{Y}^a_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))f(\mathbf{x}, \mathbf{x}')\hat{\Pi}^a(\mathbf{x}, \mathbf{x}') \\ = \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} \hat{Y}^a_{s\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))f(\mathbf{x}, \mathbf{x}')\hat{\Pi}^a(\mathbf{x}, \mathbf{x}'). \quad (5.2.57)$$

We now prove that the equality is preserved on taking in both sides the limits as $a \to \infty$. More precisely, we prove that for every $\mathbb{R}_+$-valued bounded continuous function $f(\mathbf{x}, \mathbf{x}')$ on $\mathbb{C}'$ and $t > 0$

$$\lim_{a\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} \hat{Y}^a_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))f(\mathbf{x}, \mathbf{x}')\hat{\Pi}^a(\mathbf{x}, \mathbf{x}') \\ = \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} Y'_{t\wedge\gamma^{r}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))f(\mathbf{x}, \mathbf{x}')\hat{\Pi}(\mathbf{x}, \mathbf{x}'), \quad (5.2.58)$$

where $Y'(\lambda)$ is defined by (5.1.4), i.e., $Y'_t(\lambda; (\mathbf{x}, \mathbf{x}')) = \exp(\lambda \cdot \mathbf{x}_t - G_t(\lambda; \mathbf{x}'))$. As a first step, we prove that

$$\lim_{a\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} |\hat{Y}^a_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}')) - Y'_{t\wedge\gamma^{r}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))| \\ \hat{\Pi}^a(\mathbf{x}, \mathbf{x}') = 0. \quad (5.2.59)$$

Let K' be a compact in $\mathbb{C}'$. By (5.2.56) and Arzelà–Ascoli's theorem

$$\lim_{a\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in K'} |\mathbf{x}_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')} - \mathbf{x}_{t\wedge\gamma^{r}(\mathbf{x},\mathbf{x}')}| = 0,$$

and by (5.2.52), (5.2.56) and continuity of $G_t(\lambda; \mathbf{x})$ in $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{C}$

$$\lim_{a\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in K'} |\hat{G}^a_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda; \mathbf{x}') - G_{t\wedge\gamma^{r}(\mathbf{x},\mathbf{x}')}(\lambda; \mathbf{x}')| = 0,$$

which imply that

$$\lim_{a\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in K'} |\hat{Y}^a_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}')) - Y'_{t\wedge\gamma^{r}(\mathbf{x},\mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}'))| = 0. \\ (5.2.60)$$

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Let $\{a_k,\ k \in \mathbb{N}\}$ be a subsequence, along which $\limsup_{a\to\infty}$ of the supremums in (5.2.59) is attained. Since $\hat{\Pi}^{a_k} \overset{iw}{\to} \hat{\Pi}$ as $k \to \infty$, by Theorem 1.9.27 the sequence $\{\hat{\Pi}^{a_k},\ k \in \mathbb{N}\}$ is tight. Given $\varepsilon > 0$, we choose K' such that $\limsup_{k\to\infty} \hat{\Pi}^{a_k}(\mathbb{C}' \setminus K') < \varepsilon$. Then, since $\hat{Y}^a_{t\wedge\gamma^{r,a}(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))$ and $Y'_{t\wedge\gamma^r(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))$ are bounded above by $e^{(1+|\lambda|)r}$,

$$\limsup_{k\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'\setminus K'} \hat{Y}^{a_k}_{t\wedge\gamma^{r,a_k}(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))\hat{\Pi}^{a_k}(\mathbf{x},\mathbf{x}') < \varepsilon e^{(1+|\lambda|)r},$$

$$\limsup_{k\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'\setminus K'} Y'_{t\wedge\gamma^{r,a_k}(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))\hat{\Pi}^{a_k}(\mathbf{x},\mathbf{x}') < \varepsilon e^{(1+|\lambda|)r}.$$

These inequalities and (5.2.60) imply (5.2.59) (recall that $\hat{\Pi}^a(\mathbf{x},\mathbf{x}') \le 1$).

Next, using the convergence $\hat{\Pi}^a \overset{iw}{\to} \hat{\Pi}$ and the fact that $Y'_{t\wedge\gamma^r(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))$ is bounded and continuous in $(\mathbf{x},\mathbf{x}') \in \mathbb{C}'$, we have by the definition of idempotent weak convergence

$$\lim_{a\to\infty} \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} Y'_{t\wedge\gamma^r(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))f(\mathbf{x},\mathbf{x}')\hat{\Pi}^a(\mathbf{x},\mathbf{x}')$$
$$= \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} Y'_{t\wedge\gamma^r(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))f(\mathbf{x},\mathbf{x}')\hat{\Pi}(\mathbf{x},\mathbf{x}'),$$

which by (5.2.59) concludes the proof of (5.2.58).

Equalities (5.2.57) and (5.2.58) imply that

$$\sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} Y'_{t\wedge\gamma^r(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))f(\mathbf{x},\mathbf{x}')\hat{\Pi}(\mathbf{x},\mathbf{x}')$$
$$= \sup_{(\mathbf{x},\mathbf{x}')\in\mathbb{C}'} Y'_{s\wedge\gamma^r(\mathbf{x},\mathbf{x}')}(\lambda;(\mathbf{x},\mathbf{x}'))f(\mathbf{x},\mathbf{x}')\hat{\Pi}(\mathbf{x},\mathbf{x}'),$$

and, since $\hat{\Pi}(\mathbf{x},\mathbf{x}') = \Pi(\mathbf{x})\,\mathbf{1}(\mathbf{x} = \mathbf{x}')$, we conclude that $(Y_{t\wedge\gamma^r(\mathbf{x},\mathbf{x})}(\lambda;\mathbf{x}), \mathbf{x} \in \mathbb{C}, t \in \mathbb{R}_+)$ satisfies the maxingale property with respect to Π. Being bounded, it is a $\mathbf{C}$–uniformly maximable exponential maxingale on $(\mathbb{C},\Pi)$. Since also $(Y_t(\lambda;\mathbf{x}), \mathbf{x} \in \mathbb{C}, t \in \mathbb{R}_+)$ is $\mathbf{C}$–adapted and $\gamma^r(\mathbf{x},\mathbf{x}), \mathbf{x} \in \mathbb{C}$, is a continuous $\mathbf{C}$–stopping time, it follows that $Y(\lambda)$ is a $\mathbf{C}$–local exponential maxingale on $(\mathbb{C},\Pi)$. The equality $\Pi(\mathbf{x}_0 \ne 0) = 0$ holds since

$$0 = \liminf_{a\to\infty} \hat{\Pi}^a((\mathbf{x},\mathbf{x}') : \mathbf{x}_0 \ne 0) \ge \hat{\Pi}((\mathbf{x},\mathbf{x}') : \mathbf{x}_0 \ne 0)$$
$$= \Pi(\mathbf{x} :\ \mathbf{x}_0 \ne 0).$$

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□

Proof of Theorem 5.2.26. By (5.2.7) and Lemma 5.2.24 it is sufficient to prove the part of the statement that concerns C. Let Π be a deviability on $\mathbb{D}$ supported by $\mathbb{C}$ that is an LD accumulation point of $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ so that along a subnet

$$\mathcal{L}(X^{\phi'}) \stackrel{ld}{\to} \Pi. \tag{5.2.61}$$

By Theorem 5.2.20 the nets $\{\mathcal{L}(\hat{X}^{\phi',a}, X^{\phi'}), \phi \in \Phi\}$ are $\mathbb{C}'$–exponentially tight for all $a > 0$. Therefore, by Corollary 3.1.20 there exists a subnet $\{(\mathcal{L}(\hat{X}^{\phi'',a}, X^{\phi''}), a > 0), \phi'' \in \Phi''\}$ of $\{(\mathcal{L}(\hat{X}^{\phi',a}, X^{\phi'}), a > 0), \phi' \in \Phi'\}$ such that for every $a > 0$

$$\mathcal{L}(\hat{X}^{\phi'',a}, X^{\phi''}) \stackrel{ld}{\to} \hat{\Pi}^a, \tag{5.2.62}$$

where $\hat{\Pi}^a$ are deviabilities on $\mathbb{D}'$ supported by $\mathbb{C}'$. Convergence (5.2.61) implies that in $\mathbb{D}'$

$$\mathcal{L}(X^{\phi''}, X^{\phi''}) \stackrel{ld}{\to} \hat{\Pi}, \tag{5.2.63}$$

where $\hat{\Pi}$ is the deviability on $\mathbb{D}'$ defined by

$$\hat{\Pi}(\mathbf{x}, \mathbf{x}') = \Pi(\mathbf{x})\, \mathbf{1}(\mathbf{x} = \mathbf{x}'),\ (\mathbf{x}, \mathbf{x}') \in \mathbb{D}'.$$

Also by Lemma 4.2.16 we have that for every $\eta > 0$

$$\lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\rho'_S((\hat{X}^{\phi,a}, X^\phi), (X^\phi, X^\phi)) > \eta\big) = 0, \tag{5.2.64}$$

where ρ'_S is the Skorohod–Prohorov–Lindvall metric on $\mathbb{D}'$. Limits (5.2.62), (5.2.63) and (5.2.64) imply by Lemma 3.1.37 that $\hat{\Pi}^a \stackrel{iw}{\to} \hat{\Pi}$ as $a \to \infty$. By Theorem 5.2.22 $\hat{\Pi}^a$ solves problem $(\hat{M}^a)$. An application of Lemma 5.2.27 ends the proof. □

5.2.3 Proofs of the main results

Proof of Theorem 5.2.9. By the majoration conditions in the theorem and Theorem 3.2.3 the nets $\{\mathcal{L}(B(X^\phi)), \phi \in \Phi\}$, $\{\mathcal{L}(C(X^\phi)), \phi \in \Phi\}$ (respectively, $\{\mathcal{L}(\tilde{C}(X^\phi)), \phi \in \Phi\}$), and $\{\mathcal{L}(f * \nu\ (X^\phi)), \phi \in \Phi\}, f \in \mathcal{C}_b$, are $\mathbb{C}$–exponentially tight. Then by Theorem 5.2.20 the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight. Also the continuity and majoration conditions on B imply the uniform continuity condition. An application of Theorem 5.2.26 concludes the proof. □

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Proof of Theorem 5.2.12. The argument is similar to the one in the proof of Theorem 5.1.10. We give only the main points. Let $X^{\phi,N}, \phi \in \Phi, N \in \mathbb{N}$, be defined by (5.1.21). Then it is verified analogously to the proof of Theorem 5.1.10 that the net $\{X^{\phi,N}, \phi \in \Phi\}$ satisfies the conditions of Theorem 5.2.9 for every $N \in \mathbb{N}$ with $B_t(\mathbf{x}), C_t(\mathbf{x}), \tilde{C}_t(\mathbf{x}), \nu_t(dx;\mathbf{x})$, and $\hat{\nu}_t(dx;\mathbf{x})$ replaced, respectively, by $B_t^N(\mathbf{x}), C_t^N(\mathbf{x}), \tilde{C}_t^N(\mathbf{x}), \nu_t^N(dx;\mathbf{x})$, and $\hat{\nu}_t^N(dx;\mathbf{x})$ defined as

$$\begin{aligned} B_t^N(\mathbf{x}) &= B_{t\wedge\tau_N(\mathbf{x})}(\mathbf{x}), \\ C_t^N(\mathbf{x}) &= C_{t\wedge\tau_N(\mathbf{x})}(\mathbf{x}), \\ \tilde{C}_t^N(\mathbf{x}) &= \tilde{C}_{t\wedge\tau_N(\mathbf{x})}(\mathbf{x}), \\ \nu_t^N(dx;\mathbf{x}) &= \nu_t(dx;\mathbf{x})\,\mathbf{1}(t \le \tau_N(\mathbf{x})), \\ \hat{\nu}_t^N(dx;\mathbf{x}) &= \hat{\nu}_t(dx;\mathbf{x})\,\mathbf{1}(t \le \tau_N(\mathbf{x})). \end{aligned}$$

In particular, the majoration conditions are checked in a manner similar to the proof of Theorem 5.1.10; checking the continuity conditions is simple (note, however, that the proof of the continuity condition for $\hat{\nu}^N$ uses the inequality $\hat{\nu}_s(\mathbb{R}^d;\mathbf{x}) \le 1$).

By Theorem 5.2.9 the nets $\{\mathcal{L}(X^{\phi,N}), \phi \in \Phi\}$, $N \in \mathbb{N}$, are $\mathbb{C}$–exponentially tight. Let Π^N, $N \in \mathbb{N}$, be their respective LD accumulation points. It follows as in the proof of Theorem 5.1.10 that Π^N solves (M^N). Condition (NE) yields by Lemma 5.1.17

$$\lim_{N\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(\tau_N(X^\phi) \le t) = 0, \ t \in \mathbb{R}_+,$$

which implies, again as in the proof of Theorem 5.1.10, that $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight. The uniform continuity condition for B follows by the continuity and local majoration conditions. An application of Theorem 5.2.26 ends the proof. □

Proof of Theorem 5.2.15. By the definition of the cumulant for $0 < s < t$

$$\begin{aligned} G_t(\lambda;\mathbf{x}) - G_s(\lambda;\mathbf{x}) \le \int_s^t \lambda \cdot b_u(\mathbf{x})\,du + \frac{1}{2}\int_s^t \lambda \cdot c_u(\mathbf{x})\lambda\,du \\ + \int_s^t \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda \cdot x)\nu_u(dx;\mathbf{x})\,du, \end{aligned}$$

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so the linear-growth conditions on $b_s(\mathbf{x})$, $c_s(\mathbf{x})$, and $\nu_s(dx;\mathbf{x})$ imply that $G(\lambda)$ satisfies the linear-growth condition with

$$F_t^l(\lambda) = \lambda \int_0^t l_s\,ds + \frac{1}{2}|\lambda|^2 \int_0^t l_s\,ds + \int_0^t \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda\cdot x) m_s(dx)\,ds$$

Thus, by Lemma 2.8.12 and Remark 2.8.13 condition (NE) holds.

Also equalities (5.2.3), (5.2.5), (5.2.4), and (5.2.7) show that the linear-growth conditions on $b_s(\mathbf{x})$, $c_s(\mathbf{x})$, and $\nu_s(\Gamma;\mathbf{x})$ imply the local majoration conditions for B, C, $\tilde{C}$, and ν. Finally, the linear-growth condition on $\nu_s(dx;\mathbf{x})$ implies the $\mathbb{C}$–local boundedness condition for ν. Thus, all the hypotheses of Theorem 5.2.12 are satisfied and an application of that theorem completes the proof of Theorem 5.2.15.

□

5.3 Large deviation convergence results

This section contains corollaries and versions of the results of the preceding section. In addition, we use results on uniqueness for maxingale problems from Section 2.8 in order to state the results in the form of LD convergence.

We first show how one can adapt the method of stochastic exponentials of Section 5.1 to the case when the Cramér condition is not met. For $A > 0$, we introduce

$$X_t^{\phi,A} = X_t^\phi - \sum_{0<s\le t} \Delta X_s^\phi\,\mathbf{1}(|\Delta X_s^\phi| > A).$$

Since $X^{\phi,A} = (X_t^{\phi,A}, t \in \mathbb{R}_+)$ has bounded jumps, it satisfies the Cramér condition. Let $G^{\phi,A}(\lambda) = (G_t^{\phi,A}(\lambda), t \in \mathbb{R}_+)$ be the associated stochastic cumulant and $\mathcal{E}^{\phi,A}(\lambda) = (\mathcal{E}_t^{\phi,A}(\lambda), t \in \mathbb{R}_+)$ be the stochastic exponential of $G^{\phi,A}(\lambda)$. Then the following holds.

Theorem 5.3.1. *Let the cumulant $G(\lambda)$ satisfy the uniform continuity condition and the linear-growth condition. Let condition* (0) *hold and, for some $A > 0$,*

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$$(A_0) \qquad \nu^\phi([0,t],|x|>A)^{1/r_\phi} \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \textit{as}\ \phi\in\Phi,\ t>0,$$

$$(\sup\ \mathcal{E}^A) \qquad \sup_{t\le T}\Big|\frac{1}{r_\phi}\ln\mathcal{E}^{\phi,A}_{t\wedge\tau_N(X^\phi)}(r_\phi\lambda) - G_{t\wedge\tau_N(X^\phi)}(\lambda;X^\phi)\Big| \overset{P_\phi^{1/r_\phi}}{\to} 0 \quad \textit{as}\ \phi\in\Phi,\ T>0,\ N\in\mathbb{N},\ \lambda\in\mathbb{R}^d.$$

Then the net $\{\mathcal{L}(X^\phi),\phi\in\Phi\}$ *is* $\mathbb{C}$*–exponentially tight, and its every LD accumulation point solves the maxingale problem* (x_0,G)*. If the latter problem has the unique solution* $\mathbf{\Pi}_{x_0}$*, then* $\mathcal{L}(X^\phi) \overset{ld}{\to} \mathbf{\Pi}_{x_0}$.

Proof. According to the proof of Lemma 4.2.16 condition (A_0) implies that

$$P_\phi\big(\sup_{t\le T}|X^\phi_t - X^{\phi,A}_t|>0\big)\to 0 \ \text{as}\ \phi\in\Phi,\ T>0. \tag{5.3.1}$$

Therefore, condition $(\sup\ \mathcal{E}^A)$ implies that

$$\sup_{t\le T}\Big|\frac{1}{r_\phi}\ln\mathcal{E}^{\phi,A}_{t\wedge\tau_N(X^{\phi,A})}(r_\phi\lambda) - G_{t\wedge\tau_N(X^{\phi,A})}(\lambda;X^{\phi,A})\Big| \overset{P_\phi^{1/r_\phi}}{\to} 0 \quad \text{as}\ \phi\in\Phi,\ T>0,\ N\in\mathbb{N},\ \lambda\in\mathbb{R}^d.$$

By Theorem 5.1.12 the net $\{\mathcal{L}(X^{\phi,A}),\phi\in\Phi\}$ is $\mathbb{C}$–exponentially tight, and its every LD accumulation point solves the maxingale problem (x_0,G). By Theorem 3.2.3 and (5.3.1) the net $\{\mathcal{L}(X^\phi),\phi\in\Phi\}$ is $\mathbb{C}$–exponentially tight. By Lemma 3.1.38 and (5.3.1) every LD accumulation point of $\{\mathcal{L}(X^\phi),\phi\in\Phi\}$ is an LD accumulation point of $\{\mathcal{L}(X^{\phi,A}),\phi\in\Phi\}$. □

Theorems 5.1.5 and 5.1.10 admit similar versions. We next concentrate on consequences of Theorem 5.2.15 as the most useful one for applications. Since considerations below are along the lines of the content of Section 4.3 and the proofs use similar ideas, we omit details. As in Section 4.3, we begin with integrable versions when one can consider nontruncated characteristics. We introduce the following localised versions of conditions (I_1) and (I_2).

$$(I_1)_{loc} \qquad \lim_{a\to\infty}\limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(|x|\,\mathbf{1}(r_\phi|x|>a)*\nu^\phi_{t\wedge\tau_N(X^\phi)}>\varepsilon\big)=0, \quad \varepsilon>0, t>0, N\in\mathbb{N},$$

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$$(I_2)_{loc} \qquad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(r_\phi|x|^2\,\mathbf{1}(r_\phi|x|>a)*\nu^\phi_{t\wedge\tau_N(X^\phi)}>\varepsilon\big)=0,$$
$$\varepsilon>0, t>0, N\in\mathbb{N}.$$

Clearly, $(I_2)_{loc}$ implies $(I_1)_{loc}$. We recall that the modified second characteristic without truncation $\tilde{C}' = (\tilde{C}'_t(\mathbf{x}),\, t\in\mathbb{R}_+, \mathbf{x}\in\mathbb{C})$ of X is specified by the equalities

$$\lambda\cdot\tilde{C}'_t(\mathbf{x})\lambda = \lambda\cdot C_t(\mathbf{x})\lambda + (\lambda\cdot x)^2 * \nu_t\,(\mathbf{x}) - \int_0^t (\lambda\cdot x \bullet \hat{\nu}_s(\mathbf{x}))^2 ds,\ \lambda\in\mathbb{R}^d.$$

Lemma 5.3.2. *1. Let the X^ϕ be special semimartingales. If, in addition, condition $(I_1)_{loc}$ holds, then condition* (sup $B)_{loc}$ *is equivalent to the condition*

$$(\sup\ B')_{loc} \qquad \sup_{t\le T}|B'^\phi_{t\wedge\tau_N(X^\phi)} - B'_{t\wedge\tau_N(X^\phi)}(X^\phi)| \overset{P_\phi^{1/r_\phi}}{\to} 0$$
as $\phi\in\Phi,\ \ T>0, N\in\mathbb{N}.$

2. Let the X^ϕ be locally square-integrable semimartingales. If, in addition, condition $(I_2)_{loc}$ holds, then condition $(\tilde{C})_{loc}$ is equivalent to the condition

$$(\tilde{C}')_{loc} \qquad \|r_\phi\tilde{C}'^\phi_{t\wedge\tau_N(X^\phi)} - \tilde{C}'_{t\wedge\tau_N(X^\phi)}(X^\phi)\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \ \textit{as}\ \phi\in\Phi,$$
$$t\in U, N\in\mathbb{N}.$$

The proof is similar to the proof of Theorem 4.3.2. As in Section 4.3, in view of the lemma Remark 4.3.3 applies to the setting of this section as well.

We now introduce simplified versions of the other conditions. For $(\hat{\nu})_{loc}$ and $(\nu)_{loc}$, we consider the conditions

$$(QC)_{loc} \qquad \frac{1}{r_\phi}\sum_{0<s\le t\wedge\tau_N(X^\phi)} \nu^\phi(\{s\},\{r_\phi|x|>\epsilon\})^2 \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi\in\Phi,$$
$$t>0, \epsilon>0, N\in\mathbb{N},$$

$$(MD)_{loc} \qquad \frac{1}{r_\phi}\nu^\phi([0,t\wedge\tau_N(X^\phi)],\{r_\phi|x|>\epsilon\}) \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi\in\Phi,$$
$$t>0, \epsilon>0, N\in\mathbb{N}.$$

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Obviously, condition $(QC)_{loc}$ implies condition $(\hat{\nu})_{loc}$ with $\hat{\nu}(\Gamma;\mathbf{x}) = 0$. Condition $(MD)_{loc}$, which is stronger than $(QC)_{loc}$, implies both $(\hat{\nu})_{loc}$ and $(\nu)_{loc}$ with $\nu(\Gamma;\mathbf{x}) = \hat{\nu}(\Gamma;\mathbf{x}) = 0$. It thus defines the case of moderate deviations considered in more detail below for the Markov setting. Then by Theorem 5.2.15 we have the following generalisation of Corollary 4.3.4.

Theorem 5.3.3. *Let the limiter $h(x)$ be continuous, and B, C (respectively, $\tilde{C}$), ν, and $\hat{\nu}$ satisfy the continuity conditions. Let the linear-growth conditions (5.2.10), (5.2.11) and (5.2.12) hold. If conditions* $(0), (A)_{loc} + (a)_{loc}, (\sup B)_{loc}, (C)_{loc}$ *(respectively, $(\tilde{C})_{loc}$), $(\nu)_{loc}$, and $(QC)_{loc}$ hold, then the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight, and its every LD accumulation point solves problem (x_0, G) with cumulant*

$$G_t(\lambda;\mathbf{x}) = \lambda\cdot B'_t(\mathbf{x}) + \frac{1}{2}\lambda\cdot C_t(\mathbf{x})\lambda + (e^{\lambda\cdot x} - 1 - \lambda\cdot x) * \nu_t(\mathbf{x}).$$

If the latter problem has the unique solution $\mathbf{\Pi}_{x_0}$ (e.g., either Theorem 2.8.33 or Theorem 2.8.34 applies), then $\mathcal{L}(X^\phi) \overset{ld}{\to} \mathbf{\Pi}_{x_0}$ as $\phi \in \Phi$.

By Remark 4.3.3 the theorem also has locally integrable and locally square-integrable versions.

The next result follows by Theorem 5.2.15 with $B_t(\mathbf{x}) = \int_0^t u_s(\mathbf{x})\, ds$, $C_t(\mathbf{x}) = 0$, and $\nu_s(\Gamma;\mathbf{x}) = u_s(\mathbf{x})\mathbf{1}(1 \in \Gamma), \Gamma \in \mathcal{B}(\mathbb{R}^d)$, and Theorem 2.8.10. It extends Corollary 4.3.5.

Theorem 5.3.4. *Let $d = 1$ and conditions* (0), $(A)_{loc} + (a)_{loc}$ *and* $(QC)_{loc}$ *hold. Let the limiter $h(x)$ be continuous at $x = 1$. Let $(u_s(\mathbf{x}), \mathbf{x} \in \mathbb{D}, s \in \mathbb{R}_+)$ be a $\mathbf{D}$-progressively measurable $\mathbb{R}_+$-valued function, which is $\mathbb{C}$–continuous in $\mathbf{x}$ and satisfies the linear-growth condition $u_s(\mathbf{x}) \le (1 + \mathbf{x}_s^*)l_s$, where $\int_0^t l_s\, ds < \infty$, $t \in \mathbb{R}_+$.*

If

$$\sup_{t\le T} |B^\phi_{t\wedge\tau_N(X^\phi)} - h(1) \int_0^{t\wedge\tau_N(X^\phi)} u_s(X^\phi)ds| \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{as } \phi \in \Phi,$$

$$T > 0, N \in \mathbb{N},$$

$$\lim_{\delta\to 0} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}(r_\phi \, \|C^{\phi,\delta}_{t\wedge\tau_N(X^\phi)}\| > \varepsilon) = 0,$$

$$t \in U, \varepsilon > 0, N \in \mathbb{N},$$

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and, for all $\varepsilon \in (0,1/2), t \in \mathbb{R}_+$ *and* $N \in \mathbb{N}$, *as* $\phi \in \Phi$,

$$\frac{1}{r_\phi}\,\nu^\phi([0,t\wedge\tau_N(X^\phi)],\{|r_\phi x - 1| < \varepsilon\}) - \int\limits_0^{t\wedge\tau_N(X^\phi)} u_s(X^\phi)ds \overset{P_\phi^{1/r_\phi}}{\to} 0,$$

$$\frac{1}{r_\phi}\,\nu^\phi([0,t\wedge\tau_N(X^\phi)],\{|r_\phi x| > \varepsilon\}\bigcap\{|r_\phi x - 1| > \varepsilon\}) \overset{P_\phi^{1/r_\phi}}{\to} 0,$$

then the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ *is* $\mathbb{C}$*–exponentially tight, and its every LD accumulation point solves problem* (x_0, G) *with cumulant* $G_t(\lambda;\mathbf{x}) = (e^\lambda - 1)\int_0^t u_s(\mathbf{x})\,ds$. *If the latter problem has a unique solution (e.g., by Theorem 2.8.28* $\inf\limits_{s\le t}\inf\limits_{\mathbf{x}\in K} u_s(\mathbf{x}) > 0$ *and* $\sup\limits_{s\le t}\sup\limits_{\mathbf{x}\in K} u_s(\mathbf{x}) < \infty$ *for every compact* $K \subset \mathbb{C}$ *and* $t \in \mathbb{R}_+$*), then* $X^\phi \overset{ld}{\to} X$ *as* $\phi \in \Phi$, *where* X *is the Luzin-continuous idempotent Poisson process of rate* $u_s(X)$ *starting at* x_0 *with idempotent distribution* $\mathbf{\Pi}_{x_0}$, *whose density is given by*

$$\mathbf{\Pi}_{x_0}(\mathbf{x}) = \exp\Big(-\int\limits_0^\infty \sup_{\lambda\in\mathbb{R}}\big(\lambda\dot{\mathbf{x}}_t - (e^\lambda - 1)u_t(\mathbf{x})\big)\,dt\Big)$$

if $\mathbf{x}$ *is absolutely continuous and* $\mathbf{x}_0 = x_0$, *and* $\mathbf{\Pi}_{x_0}(\mathbf{x}) = 0$ *otherwise.*

Let us introduce the following conditions.

$(\sup\ B_0')_{loc}$ $$\sup_{t\le T}|B^\phi_{t\wedge\tau_N(X^\phi)} - B'_{t\wedge\tau_N(X^\phi)}(X^\phi)| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\quad T>0,\ N\in\mathbb{N},$$

$(C_0)_{loc}$ $$\|r_\phi \tilde{C}^\phi_{t\wedge\tau_N(X^\phi)} - C_{t\wedge\tau_N(X^\phi)}(X^\phi)\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\quad t\in U,\ N\in\mathbb{N},$$

$(C_0')_{loc}$ $$\|r_\phi \tilde{C}'^\phi_{t\wedge\tau_N(X^\phi)} - C_{t\wedge\tau_N(X^\phi)}(X^\phi)\| \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\quad t\in U, N\in\mathbb{N},$$

$(L_2)_{loc}$ $$r_\phi|x|^2\,\mathbf{1}(r_\phi|x|>\epsilon) * \nu^\phi_{t\wedge\tau_N(X^\phi)} \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi,\quad t>0, N\in\mathbb{N}, \epsilon>0.$$

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Note that the latter condition is a localised version of the Lindeberg condition. As above, it implies both $(I_2)_{loc}$ and $(MD)_{loc}$ and allows us to do without truncation. The following result extends Corollaries 4.3.7 and 4.3.8. The proof is similar and also uses Theorem 2.8.9.

Theorem 5.3.5. *Let the functions* $(b_s(\mathbf{x}), s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ *and* $(c_s(\mathbf{x}), s \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D})$ *be* $\mathbb{C}$*–continuous and satisfy the linear-growth conditions*

$$|b_t(\mathbf{x})| \le l_t(1+\mathbf{x}_t^*), \quad \|c_t(\mathbf{x})\| \le l_t(1+\mathbf{x}_t^{*2}),$$

where l_t *is Lebesgue measurable and* $\int_0^t l_s\,ds < \infty, \quad t \in \mathbb{R}_+$*. Let conditions* (0) *and* $(A)_{loc} + (a)_{loc}$ *hold. If, in addition, either conditions* $(MD)_{loc}$*,* $(\sup B_0')_{loc}$ *and* $(C_0)_{loc}$ *hold for some limiter* $h(x)$ *or conditions* $(L_2)_{loc}$*,* $(\sup B')_{loc}$*, and* $(C_0')_{loc}$ *hold, then the net* $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ *is* $\mathbb{C}$*–exponentially tight, and its every LD accumulation point solves problem* (x_0, G) *with cumulant*

$$G_t(\lambda;\mathbf{x}) = \lambda \cdot B_t'(\mathbf{x}) + \frac{1}{2}\lambda \cdot C_t(\mathbf{x})\lambda.$$

If, in addition, uniqueness holds for problem (x_0, G) *(e.g., according to Theorem 2.8.21,* $\inf_{|\lambda|=1} \inf_{s\le t} \inf_{\mathbf{x}\in K} \lambda \cdot c_s(\mathbf{x})\lambda > 0$ *and* $\sup_{s\le t} \sup_{\mathbf{x}\in K} \|c_s(\mathbf{x})\| < \infty$ *for every* $t \in \mathbb{R}_+$ *and compact* $K \subset \mathbb{C}$*), then* $X^\phi \overset{ld}{\to} X$ *as* $\phi \in \Phi$*, where* $X = (X_t, t \in \mathbb{R}_+)$ *is the Luzin-continuous idempotent diffusion*

$$\dot{X}_t = b_t(X) + c_t^{1/2}(X)\dot{W}_t, \; X_0 = x_0,$$

whose deviability distribution is given by

$$\mathbf{\Pi}_{x_0}(\mathbf{x}) = \exp\Bigl(-\frac{1}{2}\int_0^\infty (\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\cdot c_t(\mathbf{x})^\oplus(\dot{\mathbf{x}}_t - b_t(\mathbf{x}))\,dt\Bigr)$$

if $\mathbf{x}_0 = x_0$*,* $\mathbf{x}$ *is absolutely continuous and* $\dot{\mathbf{x}}_t - b_t(\mathbf{x})$ *is in the range of* $c_t(\mathbf{x})$ *a.e., and* $\mathbf{\Pi}_{x_0}(\mathbf{x}) = 0$ *otherwise.*

Now, we turn our attention to conditions $(A)_{loc} + (a)_{loc}$. Let us introduce the condition

$$(VS_0)_{loc} \qquad \nu^\phi([0, t \wedge \tau_N(X^\phi)], \{r_\phi|x| > \epsilon\})^{1/r_\phi} \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi \in \Phi, \quad t > 0, \epsilon > 0, N \in \mathbb{N}.$$

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Condition $(VS_0)_{loc}$ implies both $(A)_{loc} + (a)_{loc}$ and $(MD)_{loc}$ so that by Lemma 5.3.2 and Theorem 5.3.5 we have the following result.

Theorem 5.3.6. *Let the X^ϕ be locally square-integrable. Then the assertion of Theorem 5.3.5 holds if instead of conditions* (0), $(A)_{loc}+(a)_{loc}$, $(MD)_{loc}$, $(\sup B'_0)_{loc}$, *and* $(C_0)_{loc}$ *one requires conditions* (0), $(I_2)_{loc}$, $(VS_0)_{loc}$, $(\sup\ B')_{loc}$, *and* $(C'_0)_{loc}$.

Conditions $(A)_{loc} + (a)_{loc}$ are also implied by the following condition $(VS)_{loc}$, which is weaker than $(VS_0)_{loc}$:

$$(VS)_{loc} \quad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\nu^\phi([0, t\wedge\tau_N(X^\phi)], \{r_\phi|x| > a\})^{1/r_\phi} > \varepsilon\big) = 0, \quad t > 0,\, \varepsilon > 0,\, N \in \mathbb{N}.$$

For the sequel, we note that conditions $(A)_{loc} + (a)_{loc}$ are implied by the conditions

$$(A_0)_{loc} \qquad \nu^\phi([0, t\wedge\tau_N(X^\phi)], |x| > A)^{1/r_\phi} \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi\in\Phi, \quad t > 0, N\in\mathbb{N}, \exists A > 0,$$

$$(a_0)_{loc} \qquad \frac{1}{r_\phi}\, e^{\alpha r_\phi |x|}\,\mathbf{1}(r_\phi|x| > a)\,\mathbf{1}(|x|\le A) * \nu^\phi_{t\wedge\tau_N(X^\phi)} \overset{P_\phi^{1/r_\phi}}{\to} 0 \ \text{ as } \phi\in\Phi, \quad t > 0, \alpha > 0, A > 0, N\in\mathbb{N}, \exists a > 0.$$

If, in addition, the convergence in $(a_0)_{loc}$ holds for every $a > 0$, then $(MD)_{loc}$ holds.

Let us assume that the Cramér condition (Cr) holds, i.e., $e^{\alpha|x|}\,\mathbf{1}(|x| > 1) * \nu_t^\phi < \infty,\ t > 0, \alpha > 0$. Then moment conditions can be used to check $(A)_{loc} + (a)_{loc}$. More specifically, let us introduce the conditions

$$(I_e)_{loc} \quad \lim_{a\to\infty} \limsup_{\phi\in\Phi} P_\phi^{1/r_\phi}\big(\frac{1}{r_\phi}\, e^{\alpha r_\phi|x|}\,\mathbf{1}(r_\phi|x| > a) * \nu^\phi_{t\wedge\tau_N(X^\phi)} > \varepsilon\big) = 0, \quad t > 0, \varepsilon > 0, \alpha > 0, N\in\mathbb{N},$$

$$(L_e)_{loc} \qquad \frac{1}{r_\phi}\, e^{\alpha r_\phi|x|}\,\mathbf{1}(r_\phi|x| > \epsilon) * \nu^\phi_{t\wedge\tau_N(X^\phi)} \overset{P_\phi^{1/r_\phi}}{\to} 0 \text{ as } \phi\in\Phi, \quad t > 0,\, \varepsilon > 0,\, N\in\mathbb{N}.$$

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Note that $(L_e)_{loc}$ is an exponential analogue of the Lindeberg condition. Then $(L_e)_{loc} \Rightarrow (I_e)_{loc} \Rightarrow (A)_{loc} + (a)_{loc}, (I_e)_{loc} \Rightarrow (I_2)_{loc}$ and $(L_e)_{loc} \Rightarrow (L_2)_{loc} \Rightarrow (MD)_{loc}$. In particular, we can check $(A)_{loc}+(a)_{loc}$ by checking $(I_e)_{loc}$ (e.g., in Theorem 5.3.3). As another illustration, Theorem 5.3.4 allows us to state the following extension of part b) of Corollary 4.3.12.

Theorem 5.3.7. *Let $X_t^\phi = N_t^\phi / r_\phi$, where $N^\phi = (N_t^\phi, t \in \mathbb{R}_+)$ are one-dimensional point processes with respective compensators $A^\phi = (A_t^\phi, t \in \mathbb{R}_+)$. Let $(u_s(\mathbf{x}), \mathbf{x} \in \mathbb{D}, s \in \mathbb{R}_+)$ be a $\mathbf{D}$-progressively measurable $\mathbb{R}_+$-valued function, which is $\mathbb{C}$–continuous in $\mathbf{x}$ and satisfies the linear-growth condition $u_s(\mathbf{x}) \le (1 + \mathbf{x}_s^*) l_s$, where $\int_0^t l_s\, ds < \infty$, $t \in \mathbb{R}_+$. Let the maxingale problem $(0, G)$ with cumulant $G_t(\lambda; \mathbf{x}) = (e^\lambda - 1) \int_0^t u_s(\mathbf{x})\, ds$ has the unique solution $\mathbf{\Pi}_0$ (e.g., by Theorem 2.8.28 $\inf\limits_{s \le t} \inf\limits_{\mathbf{x} \in K} u_s(\mathbf{x}) > 0$ and $\sup\limits_{s \le t} \sup\limits_{\mathbf{x} \in K} u_s(\mathbf{x}) < \infty$ for every compact $K \subset \mathbb{C}$ and $t \in \mathbb{R}_+$). Let X be the Poisson idempotent process of rate $u_s(X)$ with idempotent distribution $\mathbf{\Pi}_0$.*

If, as $\phi \in \Phi$,

$$\frac{1}{r_\phi} A^\phi_{t \wedge \tau_N(X^\phi)} - \int\limits_0^{t \wedge \tau_N(X^\phi)} u_s(X^\phi) ds \overset{P_\phi^{1/r_\phi}}{\to} 0$$

and

$$\frac{1}{r_\phi} \sum_{0 < s \le t \wedge \tau_N(X^\phi)} (\Delta A_s^\phi)^2 \overset{P_\phi^{1/r_\phi}}{\to} 0,$$

then $X^\phi \overset{ld}{\to} X \quad$ as $\phi \in \Phi$.

Since $(L_e)_{loc}$ implies $(A)_{loc} + (a)_{loc}$, $(I_2)_{loc}$ and $(MD)_{loc}$, Lemma 5.3.2 and Theorem 5.3.5 result in the following version of Theorem 5.3.6.

Theorem 5.3.8. *Let the Cramér condition hold. Then in Theorem 5.3.5 one can replace conditions $(A)_{loc} + (a)_{loc}$, $(MD)_{loc}$, $(\sup B'_0)_{loc}$, and $(C_0)_{loc}$ with conditions $(L_e)_{loc}$, $(\sup\ B')_{loc}$, and $(C'_0)_{loc}$.*

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5.4 Large deviation convergence of Markov processes

We now consider implications of the above results for the Markov setting. In the next theorem we assume that X^ϕ are generally speaking non-time homogeneous "continuous-time Markov processes with generators A_t^ϕ" in the sense that the A_t^ϕ map the functions $(e^{\lambda\cdot x}, x \in \mathbb{R}^d), \lambda \in \mathbb{R}^d$, into $\overline{\mathcal{B}}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable functions of (t, x) and the processes $(\exp(\lambda\cdot X_t^\phi) - \exp(\lambda\cdot X_0^\phi) - \int_0^t A_s^\phi \exp(\lambda\cdot X_s^\phi)\, ds,\ t \in \mathbb{R}_+)$ are well-defined local martingales on $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$. Let S^ϕ denote the state space of X^ϕ.

Theorem 5.4.1. *Let $g_t(\lambda; x), t \in \mathbb{R}_+, \lambda \in \mathbb{R}^d, x \in \mathbb{R}^d$, be a $\overline{\mathcal{B}}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable $\mathbb{R}$-valued function such that $g_t(0; x) = 0$. Let us assume that $g_t(\lambda; x)$ is continuous in x and meets the linear-growth condition $g_t(\lambda; x) \le k_t(|\lambda|(1 + |x|))$, where $t \in \mathbb{R}_+$, $\lambda \in \mathbb{R}^d$, and the function $k_t(\alpha)$ is $\mathbb{R}_+$-valued, Lebesgue measurable in t, increasing in α, and $\int_0^t k_s(\alpha)\, ds < \infty$, $t \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+$.*

If, as $\phi \in \Phi$, $X_0^\phi \overset{P_\phi^{1/r_\phi}}{\to} x_0$ and, for $T > 0, N \in \mathbb{N}$,

$$\sup_{t\le T}\ \sup_{x\in S^\phi:|x|\le N} \left|\frac{1}{r_\phi} \exp(-r_\phi\lambda\cdot x) A_t^\phi \exp(r_\phi\lambda\cdot x) - g_t(\lambda; x)\right| \to 0,$$

then the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ is $\mathbb{C}$–exponentially tight, and its every LD accumulation point solves maxingale problem (x_0, G) associated with the cumulant

$$G_t(\lambda; \mathbf{x}) = \int_0^t g_s(\lambda; \mathbf{x}_s)\, ds.$$

Proof. Since the processes $(\exp(r_\phi\lambda\cdot(X_t^\phi - X_0^\phi))\exp(-\int_0^t \exp(-r_\phi\lambda\cdot X_s^\phi) A_s^\phi \exp(r_\phi\lambda \cdot X_s^\phi) ds), t \in \mathbb{R}_+)$ are local martingales on $(\Omega_\phi, \mathcal{F}_\phi, \mathbf{F}_\phi, P_\phi)$, Liptser and Shiryaev [79, Theorem 2.5.1] (see also Ethier and Kurtz [48, Lemma 3.2]), the claim follows by Theorem 5.1.13. □

We next consider applications of Theorem 5.2.15. The semimartingales X^ϕ are assumed to be Markov processes. To simplify

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the conditions, we distinguish between the continuous- and discrete-time cases. Let $\{\alpha_\phi, \phi \in \Phi\}$ and $\{\beta_\phi,\ \phi \in \Phi\}$ be nets of real numbers tending to ∞ as $\phi \in \Phi$. In the continuous-time case, we assume that the predictable triplets of X^ϕ corresponding to a limiter $h(x)$ are given by:

$$B_t^\phi = \int_0^t b_s^\phi(X_s^\phi)ds, \tag{5.4.1}$$

$$C_t^\phi = \frac{1}{r_\phi}\int_0^t c_s^\phi(X_s^\phi)ds, \tag{5.4.2}$$

$$\nu^\phi([0,t],dx) = \alpha_\phi \int_0^t \nu_s^\phi(\beta_\phi\, dx; X_s^\phi)ds, \tag{5.4.3}$$

where $b_s^\phi(u)$ is an $\mathbb{R}^d$–valued $\overline{\mathcal{B}}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$–measurable function such that $\int_0^t |b_s^\phi(\mathbf{x}_s)|\, ds < \infty$ for $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{D}$, $c_s^\phi(u)$ is a function with values in the space of symmetric positive semi-definite $d \times d$–matrices, which is $\overline{\mathcal{B}}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^{d\times d})$–measurable and such that $\int_0^t \|c_s^\phi(\mathbf{x}_s)\|\, ds < \infty$ for $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{D}$, $\nu_s^\phi(dx;u)$ is a transition kernel from $(\mathbb{R}_+ \times \mathbb{R}^d, \overline{\mathcal{B}}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\nu_t^\phi(\{0\};u) = 0, \quad \int_{\mathbb{R}^d} 1 \wedge |x|^2\, \nu_t^\phi(dx;u) < \infty,$$

$$\int_0^t \int_{\mathbb{R}^d} 1 \wedge |x|^2\, \nu_s^\phi(dx;\mathbf{x}_s)\, ds < \infty,\ t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{D}, u \in \mathbb{R}^d.$$

In the discrete-time case, we assume that X^ϕ is a pure jump process with predictable measure of jumps

$$\nu^\phi([0,t],dx) = \sum_{i=1}^{\lfloor \alpha_\phi t \rfloor} \hat{\nu}_{i/\alpha_\phi}^\phi(\beta_\phi\, dx; X_{(i-1)/\alpha_\phi}^\phi), \tag{5.4.4}$$

where $\hat{\nu}_{i/\alpha_\phi}^\phi(dx;u)$, for every $i \in \mathbb{N}$, is a transition kernel from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\hat{\nu}_{i/\alpha_\phi}^\phi(\{0\};u) = 0, \quad \hat{\nu}_{i/\alpha_\phi}^\phi(\mathbb{R}^d;u) \le 1, \quad i \in \mathbb{N}, u \in \mathbb{R}^d.$$

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The parameter α_ϕ can be interpreted as the frequency of jumps of X^ϕ and the parameter $1/\beta^\phi$ as the size of the jumps. Depending on the relative speed at which α_ϕ and β_ϕ go to ∞, there are two different asymptotics, which are referred to as "very large deviations", when α_ϕ and β_ϕ are of the same order and one takes $r_\phi = \alpha_\phi = \beta_\phi$, and "moderate deviations", when $\alpha_\phi/\beta_\phi \to \infty$ but $\alpha_\phi/\beta_\phi^2 \to 0$, and one takes $r_\phi = \beta_\phi^2/\alpha_\phi$.

Let us recall the definition of the essential supremum of a collection of measurable functions, see, e.g., Neveu [94, II.4]. Let $f_j = (f_j(x), x \in \mathbb{R}_+), j \in J$, be a collection of $\overline{\mathcal{B}}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R})$–measurable $\mathbb{R}$-valued functions on $\mathbb{R}_+$; for a $\overline{\mathcal{B}}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R})$–measurable $\mathbb{R}$-valued function $f = (f(x), x \in \mathbb{R}_+)$, we say that $f = \operatorname{ess\,sup}_{j\in J} f_j$ if, for every $j \in J$, $f(x) \geq f_j(x)$ for almost all $x \in \mathbb{R}_+$ and $f(x) \leq g(x)$ for almost all $x \in \mathbb{R}_+$ for every $\overline{\mathcal{B}}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R})$–measurable $\mathbb{R}$-valued function $g = (g(x), x \in \mathbb{R}_+)$ such that, for every $j \in J$, $g(x) \geq f_j(x)$ for almost all $x \in \mathbb{R}_+$. Note that this usage is different from the interpretation above.

We consider, first, the case of very large deviations: $\alpha_\phi = \beta_\phi = r_\phi$. Let us introduce the following versions of the Cramér condition:

$$\operatorname*{ess\,sup}_{|u|\leq v} \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda \cdot x)\, \nu_t^\phi(dx;u) < \infty,$$

$$\int_0^t \Big(\operatorname*{ess\,sup}_{|u|\leq v} \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda \cdot x)\, \nu_s^\phi(dx;u)\Big)\, ds < \infty,$$

$$\lambda \in \mathbb{R}^d, t \in \mathbb{R}_+, v \in \mathbb{R}_+, \quad (5.4.5)$$

in the continuous-time case, and

$$\int_{\mathbb{R}^d} e^{\lambda\cdot x}\, \hat{\nu}^\phi_{i/r_\phi}(dx;u) < \infty, \lambda \in \mathbb{R}^d, i \in \mathbb{N}, u \in \mathbb{R}^d, \quad (5.4.6)$$

in the discrete-time case.

Under these conditions, the X^ϕ are locally square integrable semimartingales, so, we may and will take $h(x) = x$ so that B^ϕ is the first characteristic "without truncation" ($B^\phi = B'^\phi$), and define

$$\overline{g}_s^\phi(\lambda;u) = \lambda\cdot b_s^\phi(u) + \frac{1}{2}\lambda\cdot c_s^\phi(u)\lambda + \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda\cdot x)\, \nu_s^\phi(dx;u)$$

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in the continuous-time case and

$$\overline{g}_s^{\phi}(\lambda;u)=\ln\Big(1+\int\limits_{\mathbb{R}^d}(e^{\lambda\cdot x}-1)\,\hat{\nu}^{\phi}_{(\lfloor r_\phi s\rfloor+1)/r_\phi}(dx;u)\Big)$$

in the discrete-time case.

Let $\overline{g}_s(\lambda;u)$ be a $\overline{\mathcal{B}}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable function, which is continuous in u and satisfies the following linear-growth condition

$$|\overline{g}_s(\lambda;u)|\le\tilde{g}_s(|\lambda|(1+|u|)),$$

where $\tilde{g}_s(y)$ is $\mathbb{R}_+$-valued, $\overline{\mathcal{B}}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R})$-measurable in s, increasing in y, $\int_0^t\tilde{g}_s(y)\,ds<\infty$, $t\in\mathbb{R}_+, y\in\mathbb{R}_+$, and $\overline{g}_s(0;u)=0$.

We then have the following version of Theorem 5.4.1.

Theorem 5.4.2. *Let* $\alpha_\phi=\beta_\phi=r_\phi$ *and the Cramér condition* (5.4.5) *in the continuous-time case, respectively, the Cramér condition* (5.4.6) *in the discrete-time case, hold. Let* $X_0^\phi\overset{P_\phi^{1/r_\phi}}{\to}x_0$ *as* $\phi\in\Phi$. *If, for all* $\lambda\in\mathbb{R}^d$, $t\in\mathbb{R}_+$ *and* $v\in\mathbb{R}_+$, *as* $\phi\in\Phi$,

$$\int\limits_0^t\operatorname*{ess\,sup}_{|u|\le v}\;|\overline{g}_s^{\phi}(\lambda;u)-\overline{g}_s(\lambda;u)|\,ds\to0,$$

then the net $\{\mathcal{L}(X^\phi),\phi\in\Phi\}$ *is* $\mathbb{C}$*–exponentially tight, and its every LD accumulation point solves problem* (x_0,G) *with cumulant*

$$G_t(\lambda;\mathbf{x})=\int\limits_0^t\overline{g}_s(\lambda;\mathbf{x}_s)\,ds.$$

If the latter problem has the unique solution $\mathbf{\Pi}_{x_0}$ *(e.g., the conditions of Theorem 2.8.32 hold for* $g_s(\lambda;\mathbf{x})=\overline{g}_s(\lambda;\mathbf{x}_s)$*), then* $\mathcal{L}(X^\phi)\overset{ld}{\to}\mathbf{\Pi}_{x_0}$ *as* $\phi\in\Phi$.

Proof. We have the following representations for the stochastic exponential $\mathcal{E}^\phi(\lambda)=(\mathcal{E}_t^\phi(\lambda),t\in\mathbb{R}_+),\lambda\in\mathbb{R}^d$, associated with X^ϕ. In the continuous-time case,

$$\frac{1}{r_\phi}\ln\mathcal{E}_t^\phi(r_\phi\lambda)=\int\limits_0^t\overline{g}_s^{\phi}(\lambda;X_s^\phi)\,ds.$$

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In the discrete-time case, by the equalities $\overline{g}_s^\phi = \overline{g}^\phi_{\lfloor r_\phi s\rfloor/r_\phi}$ and $X_s^\phi = X^\phi_{\lfloor r_\phi s\rfloor/r_\phi}$,

$$\frac{1}{r_\phi}\ln \mathcal{E}_t^\phi(r_\phi\lambda) = \frac{1}{r_\phi}\sum_{i=0}^{\lfloor r_\phi t\rfloor -1} \overline{g}^\phi_{i/r_\phi}(\lambda; X^\phi_{i/r_\phi}) = \int\limits_0^{\lfloor r_\phi t\rfloor/r_\phi} \overline{g}^\phi_s(\lambda; X_s^\phi)\, ds.$$

Therefore, in both cases for $N \in \mathbb{N}$

$$\begin{aligned}&\left|\frac{1}{r_\phi}\ln \mathcal{E}^\phi_{t\wedge\tau_N(X^\phi)}(r_\phi\lambda) - G_{t\wedge\tau_N(X^\phi)}(\lambda; X^\phi)\right| \\ &\le \int\limits_0^t \operatorname*{ess\,sup}_{|u|\le N}\, |\overline{g}_s^\phi(\lambda;u) - \overline{g}_s(\lambda;u)|\, ds \\ &\qquad\qquad + \int\limits_{\lfloor r_\phi t\rfloor/r_\phi}^t \sup_{|u|\le N} |\overline{g}_s(\lambda;u)|\, ds,\end{aligned}$$

and the claim follows by Theorem 5.1.12. □

We next state a "very large deviation" result in terms of characteristics, which does not require the Cramér condition. We confine ourselves to the continuous-time case. We assume all the above conditions on $b_s^\phi(u)$, $c_s^\phi(u)$ and $\nu_s^\phi(dx;u)$ to hold except the Cramér condition (5.4.5). Instead, we assume that the first characteristic B^ϕ corresponds to a continuous limiter $h(x)$. We define positive semi-definite symmetric matrices $\tilde{c}_s^\phi(u)$ by

$$\lambda\cdot\tilde{c}_s^\phi(u)\lambda \;=\; \lambda\cdot c_s^\phi(u)\lambda + \int\limits_{\mathbb{R}^d} (\lambda\cdot h^\phi(x))^2\nu_s^\phi(dx;u),\ \lambda\in\mathbb{R}^d.$$

We next introduce the limit idempotent process. Let $b_s(u)$ be an $\mathbb{R}^d$–valued $\overline{\mathcal{B}}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$–measurable function, $c_s(u)$ be a $\overline{\mathcal{B}}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^{d\times d})$–measurable function with values in the space of positive semi-definite symmetric $d\times d$–matrices and $\nu_s(dx;u)$ be a transition kernel from $(\mathbb{R}_+\times\mathbb{R}^d, \overline{\mathcal{B}}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d))$ into

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$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\nu_s(\{0\};u) = 0, \int_{\mathbb{R}^d} (e^{\lambda\cdot x} - 1 - \lambda\cdot x)\, \nu_s(dx;u) < \infty, \qquad \lambda \in \mathbb{R}^d,\ s \in \mathbb{R}_+,\ u \in \mathbb{R}^d.$$

Let the following linear-growth conditions be satisfied:

$$|b_s(u)| \le l_s(1+|u|), \ \ \|c_s(u)\| \ \le l_s(1+|u|^2), \tag{5.4.7}$$

$$\int_{\mathbb{R}^d} (e^{\alpha|x|} - 1 - \alpha|x|)\nu_s(dx;u) \le \int_{\mathbb{R}^d} (e^{\alpha|x|(1+|u|)} - 1 - \alpha|x|(1+|u|))m_s(dx), \ \alpha \in \mathbb{R}_+,$$

where l_s is an $\mathbb{R}_+$-valued $\overline{\mathcal{B}}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R}_+)$–measurable function such that $\int_0^t l_s\, ds < \infty$ and $m_s(dx)$ is a transition kernel from $(\mathbb{R}_+, \overline{\mathcal{B}}(\mathbb{R}_+))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\int_0^t \int_{\mathbb{R}^d} (\exp(\alpha|x|) - 1 - \alpha|x|)m_s(dx)ds < \infty,\ t \in \mathbb{R}_+, \alpha \in \mathbb{R}_+$.

In addition, we assume that the functions

$$u \to b_s(u), \quad u \to c_s(u), \quad u \to \int_{\mathbb{R}^d} f(x)\nu_s(dx;u),$$

for f continuous and such that $|f(x)| \le 1 \wedge |x|^2$,

are continuous in $u \in \mathbb{R}^d$.

We also define positive semi-definite symmetric matrices $\tilde{c}_s(u)$ by

$$\lambda \cdot \tilde{c}_s(u)\lambda \ = \ \lambda \cdot c_s(u)\lambda + \int_{\mathbb{R}^d} (\lambda \cdot h(x))^2 \nu_s(dx;u), \ \lambda \in \mathbb{R}^d.$$

Theorem 5.4.3. *Let $\alpha_\phi = \beta_\phi = r_\phi$ and the above conditions hold.*

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Let $X_0^\phi \overset{P_\phi^{1/r_\phi}}{\to} x_0$ *and, for all* $t \in \mathbb{R}_+$ *and* $v \in \mathbb{R}_+$,

$$\int_0^t \operatorname*{ess\,sup}_{|u|\le v} |b_s^\phi(u) - b_s(u)|\, ds \to 0,\ \int_0^t \operatorname*{ess\,sup}_{|u|\le v} \|\tilde c_s^\phi(u) - \tilde c_s(u)\|\ ds \to 0,$$

$$\int_0^t \operatorname*{ess\,sup}_{|u|\le v} \Big|\int_{\mathbb{R}^d} f(x)\nu_s^\phi(dx;u) - \int_{\mathbb{R}^d} f(x)\nu_s(dx;u)\Big|\, ds \to 0,\ f \in \mathcal{C}_b.$$

Let also

$$\lim_{a\to\infty} \limsup_{\phi\in\Phi}\Big(\int_0^t \operatorname*{ess\,sup}_{|u|\le v} \nu_s^\phi(\{|x|>a\};u)\, ds\Big)^{1/r_\phi} = 0,$$
$$v \in \mathbb{R}_+,\, t \in \mathbb{R}_+. \quad (5.4.8)$$

Then the net $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$ *is* $\mathbb{C}$*-exponentially tight. If* Π *is an LD accumulation point of* $\{\mathcal{L}(X^\phi), \phi \in \Phi\}$, *then the canonical idempotent process* X *is a Luzin-continuous semimaxingale with local characteristics* $(b, c, \nu, 0)$ *on* $(\mathbb{C}, \mathbf{C}, \Pi)$. *If the idempotent distribution* $\mathcal{L}_i(X)$ *of* X *is specified uniquely (e.g., Theorem 2.8.34 applies), then* $\mathcal{L}_i(X) = \mathbf{\Pi}_{x_0}$ *and* $X^\phi \overset{ld}{\to} X$ *as* $\phi \in \Phi$.

If, in addition, the Cramér condition (5.4.5) *holds, then condition* (5.4.8) *can be replaced with the condition*

$$\lim_{a\to\infty} \limsup_{\phi\in\Phi} \int_0^t \Big(\operatorname*{ess\,sup}_{|u|\le v} \int_{\mathbb{R}^d} e^{\alpha|x|}\,\mathbf{1}(|x|>a)\nu_s^\phi(dx;u)\Big) ds = 0,$$
$$t \in \mathbb{R}_+, \alpha \in \mathbb{R}_+, v \in \mathbb{R}_+. \quad (5.4.9)$$

In the latter case we can take $h(x) = x$, *i.e., consider nontruncated characteristics.*

Proof. Condition (5.4.8) implies condition $(VS)_{loc}$, and condition (5.4.9) implies condition $(I_e)_{loc}$. Either one of these conditions implies conditions $(a)_{loc} + (A)_{loc}$. Therefore, the claim follows by Theorem 5.2.15. We can consider nontruncated characteristics under (5.4.9) by Lemma 5.3.2 and the fact that $(I_e)_{loc}$ implies $(I_2)_{loc}$. ☐

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We consider now moderate deviations so $\alpha_\phi \to \infty$ and $\beta_\phi \to \infty$ in such a way that $\alpha_\phi/\beta_\phi \to \infty$ and $\alpha_\phi/\beta_\phi^2 \to 0$. Let $r_\phi = \beta_\phi^2/\alpha_\phi$.

We assume the locally square-integrable case, i.e.,

$$\operatorname*{ess\,sup}_{|u|\le v} \int_{\mathbb{R}^d} |x|^2 \nu_t^\phi(dx;u) < \infty,$$

$$\int_0^t \Big(\operatorname*{ess\,sup}_{|u|\le v} \int_{\mathbb{R}^d} |x|^2 \nu_s^\phi(dx;u)\Big) ds < \infty,\ t \in \mathbb{R}_+, v \in \mathbb{R}_+, \phi \in \Phi,$$

in the continuous-time case, and

$$\int_{\mathbb{R}^d} |x|^2 \hat{\nu}^\phi_{i/\alpha_\phi}(dx;u) < \infty,\ i \in \mathbb{N}, u \in \mathbb{R}^d, \phi \in \Phi,$$

in the discrete-time case.

In the discrete-time case we also assume that the X^ϕ are martingales, i.e.,

$$\int_{\mathbb{R}^d} x \hat{\nu}^\phi_{i/\alpha_\phi}(dx;u) = 0,\ i \in \mathbb{N},\ u \in \mathbb{R}^d,\ \phi \in \Phi.$$

Then the X^ϕ are locally square integrable semimartingales, so we choose nontruncated predictable characteristics. According to (5.4.2), (5.4.3), (5.4.4), and the equality $r_\phi \alpha_\phi = \beta_\phi^2$, the (nontruncated) modified predictable second characteristics of the X^ϕ are of the form:

in the continuous-time case

$$\tilde{C}_t'^\phi = \frac{\alpha_\phi}{\beta_\phi^2} \int_0^t \tilde{c}_s'^\phi(X_s^\phi)\, ds,$$

where $\tilde{c}_s'^\phi(u)$ are positive semi-definite symmetric matrices defined by

$$\lambda \cdot \tilde{c}_s'^\phi(u)\lambda = \lambda \cdot c_s^\phi(u)\lambda + \int_{\mathbb{R}^d} (\lambda \cdot x)^2 \nu_s^\phi(dx;u);$$

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in the discrete-time case

$$\tilde{C}'^{\phi}_t = \frac{\alpha_\phi}{\beta_\phi^2} \int_0^{\lfloor \alpha_\phi t \rfloor / \alpha_\phi} \tilde{c}'^{\phi}_s (X^\phi_s)\, ds,$$

where $\tilde{c}'^{\phi}_s(u)$ are positive semi-definite symmetric matrices defined by

$$\lambda \cdot \tilde{c}'^{\phi}_s(u) \lambda = \int_{\mathbb{R}^d} (\lambda \cdot x)^2 \hat{\nu}^{\phi}_{(\lfloor \alpha_\phi s \rfloor + 1)/\alpha_\phi}(dx; u).$$

Let us assume that $b_s(u)$ and $c_s(u)$ satisfy the conditions stated before Theorem 5.4.3 (i.e., measurability, linear growth and continuity in u).

We introduce the following conditions on the predictable measures of jumps and rates of convergence.

(P) For some $\delta > 0$

$$\limsup_{\phi} \int_0^t \Big(\operatorname*{ess\,sup}_{|u| \le v} \int_{\mathbb{R}^d} |x|^{2+\delta} \nu^\phi_s(dx; u) \Big) ds < \infty, \ t \in \mathbb{R}_+, v \in \mathbb{R}_+,$$

in the continuous-time case, respectively,

$$\limsup_{\phi} \frac{1}{\alpha_\phi} \sum_{i=1}^{\lfloor \alpha_\phi t \rfloor} \sup_{|u| \le v} \int_{\mathbb{R}^d} |x|^{2+\delta} \hat{\nu}^\phi_{i/\alpha_\phi}(dx; u) < \infty, \ t \in \mathbb{R}_+, v \in \mathbb{R}_+,$$

in the discrete-time case, and $\beta_\phi^2 / (\alpha_\phi \ln \alpha_\phi) \to 0$ as $\phi \in \Phi$.

(SE) For some $\beta \in (0, 1]$ and $\gamma > 0$

$$\limsup_{\phi} \int_0^t \Big(\operatorname*{ess\,sup}_{|u| \le v} \int_{\mathbb{R}^d} \exp(\gamma |x|^\beta) \nu^\phi_s(dx; u) \Big) ds < \infty,$$

$$t \in \mathbb{R}_+, v \in \mathbb{R}_+,$$

in the continuous-time case, respectively,

$$\limsup_{\phi} \frac{1}{\alpha_\phi} \sum_{i=1}^{\lfloor \alpha_\phi t \rfloor} \sup_{|u| \le v} \int_{\mathbb{R}^d} \exp(\gamma |x|^\beta) \hat{\nu}^\phi_{i/\alpha_\phi}(dx; u) < \infty,$$

$$t \in \mathbb{R}_+, v \in \mathbb{R}_+,$$

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in the discrete-time case, and $\beta_\phi^{2-\beta}/\alpha_\phi \to 0$ as $\phi \in \Phi$.

The next theorem extends Theorem 4.4.8.

Theorem 5.4.4. *Let $\alpha_\phi/\beta_\phi \to \infty$ and $\alpha_\phi/\beta_\phi^2 \to 0$ as $\phi \in \Phi$. Let either condition (P) or condition (SE) hold. Let the law of a Luzin-continuous semimaxingale X with local characteristics $(b, c, 0, 0)$ starting at x_0 be specified uniquely (e.g., according to Theorem 2.8.21, $\inf_{|\lambda|=1} \inf_{s \le t} \inf_{|u| \le v} \lambda \cdot c_s(u)\lambda > 0$ and $\sup_{s \le t} \inf_{|u| \le v} \|c_s(u)\| < \infty$, $t \in \mathbb{R}_+, v \in \mathbb{R}_+$).*

If, as $\phi \in \Phi$, $X_0^\phi \overset{P_\phi^{1/r_\phi}}{\to} x_0$, where $r_\phi = \beta_\phi^2/\alpha_\phi$, and for all $t \in \mathbb{R}_+$ and $v \in \mathbb{R}_+$

$$\int_0^t \operatorname*{ess\,sup}_{|u| \le v} |b_s^\phi(u) - b_s(u)|\, ds \to 0, \quad \int_0^t \operatorname*{ess\,sup}_{|u| \le v} \|\tilde{c}_s'^\phi(u) - c_s(u)\|\, ds \to 0,$$

then $X^\phi \overset{ld}{\to} X$ as $\phi \in \Phi$ at rate r_ϕ.

Proof. The proof is almost the same as for Theorem 4.4.8. In some more detail, either one of conditions (P) or (SE) implies $(L_2)_{loc}$. Since by hypotheses conditions $(\sup B')_{loc}$ and $(C_0')_{loc}$ hold, according to Theorem 5.3.5 one needs to check conditions $(A)_{loc} + (a)_{loc}$. If condition (P) is satisfied, then condition $(VS_0)_{loc}$ holds, which implies $(A)_{loc} + (a)_{loc}$. If condition (SE) is satisfied, then conditions $(A_0)_{loc} + (a_0)_{loc}$ can be verified as in the proof of Theorem 4.4.8. □

Remark 5.4.5. *We recall that by Theorem 2.8.9 under the hypotheses X is a Luzin-continuous idempotent process satisfying the equation*

$$\dot{X}_t = b_t(X_t) + \sigma_t(X_t)\dot{W}_t, \; X_0 = x_0,$$

and the deviability distribution of X has density given by

$$\Pi^X(\mathbf{x}) = \exp\Big(-\frac{1}{2}\int_0^\infty (\dot{\mathbf{x}}_s - b_s(\mathbf{x}_s)) \cdot c_s(\mathbf{x}_s)^\oplus (\dot{\mathbf{x}}_s - b_s(\mathbf{x}_s))\, ds\Big)$$

if $\mathbf{x}$ is absolutely continuous, $\mathbf{x}_0 = x_0$ and $\dot{\mathbf{x}}_s - b_s(\mathbf{x}_s)$ is in the range of $c_s(\mathbf{x}_s)$ a.e., and $\Pi^X(\mathbf{x}) = 0$ otherwise.

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We conclude the section with some illustrative examples. To simplify notation, we consider one-dimensional settings.

Example 5.4.6.

Let $\mathbb{R}$-valued processes $X^{\varepsilon,\delta,\gamma} = (X_t^{\varepsilon,\delta,\gamma}, t \in \mathbb{R}_+)$, indexed by $\varepsilon > 0, \delta > 0$ and $\gamma > 0$, be defined on respective stochastic bases $(\Omega_{\varepsilon,\delta,\gamma}, \mathcal{F}_{\varepsilon,\delta,\gamma}, \mathbf{F}_{\varepsilon,\delta,\gamma}, P_{\varepsilon,\delta,\gamma})$ and satisfy the equations

$$X_t^{\varepsilon,\delta,\gamma} = x_0 + \int_0^t b_s(X_s^{\varepsilon,\delta,\gamma})\, ds + \sqrt{\varepsilon} \int_0^t \sigma_s(X_s^{\varepsilon,\delta,\gamma})\, dW_s^\varepsilon + \delta \int_0^t \int_G f_s(X_{s-}^{\varepsilon,\delta,\gamma}, y) \left[\mathcal{N}^\gamma(ds, dy) - \gamma^{-1} ds\, m(dy)\right],$$

where $(G, \mathcal{G})$ is a measurable space, $b_s(u), \sigma_s(u)$ and $f_s(u, y)$ are respective $\overline{\mathcal{B}}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$, $\overline{\mathcal{B}}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ and $\overline{\mathcal{B}}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{G}/\mathcal{B}(\mathbb{R})$-measurable functions, $W^\varepsilon = (W_s^\varepsilon, s \in \mathbb{R}_+)$ are $\mathbb{R}$-valued Wiener processes, $m(dy)$ is a non-negative σ-finite measure on $(G, \mathcal{G})$, and $\mathcal{N}^\gamma = (\mathcal{N}^\gamma(ds, dy))$ are Poisson random measures on $\mathbb{R}_+ \times G$ with intensity measures $\gamma^{-1}\, ds\, m(dy)$. We also assume that $b_s(u)$ and $\sigma_s(u)$ are continuous in u,

$$\lim_{u \to v} \int_G |f_s(u, y) - f_s(v, y)|^2 m(dy) = 0,$$

the following linear-growth conditions are met:

$$b_s(u)^2 + \sigma_s(u)^2 \le l_s(1 + u^2), \quad |f_s(u, y)| \le h_s(y)(1 + |u|),$$

where l_s and $h_s(y)$ are $\mathbb{R}_+$-valued and increasing in s, $h_s(y)$ is $\mathcal{G}/\mathcal{B}(\mathbb{R}_+)$–measurable for every $s \in \mathbb{R}_+$, $\int_G h_s(y)^2\, m(dy) < \infty$, and the non-degeneracy condition holds:

$$\inf_{s \le t} \inf_{|u| \le v} \left(\sigma_s(u)^2 + \int_G f_s(u, y)^2\, m(dy)\right) > 0, \ t \in \mathbb{R}_+, \, v \in \mathbb{R}_+.$$

For existence of $X^{\varepsilon,\delta,\gamma}$ see, e.g., Gihman and Skorohod [54, Chapter 5].

Let us consider the following moment conditions on the jumps of the $X^{\varepsilon,\delta,\gamma}$:

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($\tilde{\mathrm{P}}$) for some $\delta > 0$

$$\int_0^t \bigl(\operatorname*{ess\,sup}_{|u|\le v} \int_G |f_s(u,y)|^{2+\delta}\, m(dy)\bigr)ds < \infty,\ t \in \mathbb{R}_+,\ v \in \mathbb{R}_+,$$

($\widetilde{\mathrm{SE}}$) for some $\beta \in (0,1]$ and $\alpha > 0$

$$\int_0^t \bigl(\operatorname*{ess\,sup}_{|u|\le v} \int_G \exp\bigl(\alpha |f_s(u,y)|^{\beta}\bigr)\, m(dy)\bigr)ds < \infty,\ t \in \mathbb{R}_+,\ v \in \mathbb{R}_+.$$

Let an idempotent Luzin-continuous process X satisfy the equation

$$\dot{X}_t = b_t(X_t) + \bigl(\sigma_t(X_t)^2 + \int_G f_t(X_t,y)^2\, m(dy)\bigr)^{1/2} \dot{W}_t,\ X_0 = x_0,$$

where W is an $\mathbb{R}$-valued idempotent Wiener process. The process X is well defined by Theorems 2.6.24 and 2.8.21.

Theorem 5.4.7. *Let $\gamma \to 0$, $\epsilon \to 0$, and $\delta \to 0$ in such a way that $\epsilon = \delta^2/\gamma$. If, in addition, either $\delta^2\gamma^{-1}\ln(\gamma^{-1}) \to \infty$ and condition ($\tilde{P}$) holds, or $\delta^{2-\beta}\gamma^{-1} \to \infty$ and condition ($\widetilde{SE}$) holds, then $X^{\varepsilon,\delta,\gamma} \overset{ld}{\to} X$ at rate $1/\epsilon$.*

Proof. The predictable characteristics without truncation of $X^{\varepsilon,\delta,\gamma}$ are of the form

$$B'^{\varepsilon,\delta,\gamma}_t = \int_0^t b_s(X^{\varepsilon,\delta,\gamma}_s)\, ds,\ C^{\varepsilon,\delta,\gamma}_t = \varepsilon \int_0^t \sigma_s(X^{\varepsilon,\delta,\gamma}_s)^2\, ds,$$

$$g(x) * \nu^{\varepsilon,\delta,\gamma}_t = \gamma^{-1} \int_0^t \int_G g\bigl(\delta f_s(X^{\varepsilon,\delta,\gamma}_s, y)\bigr) m(dy)\, ds$$

for g Borel and bounded.

It is straightforward to see that the convergence hypotheses of Theorem 5.4.4 hold for $\phi = (\epsilon,\delta,\gamma)$, $\alpha_\phi = \gamma^{-1}$, $\beta_\phi = \delta^{-1}$, $b^{\epsilon,\delta,\gamma}_s(u) = b_s(u)$, and $\tilde{c}^{\epsilon,\delta,\gamma}_s(u) = c_s(u) = \sigma_s(u)^2 + \int_G f_s(u,y)^2 m(dy)$. The moment conditions on the jumps are the same as in Theorem 5.4.4. ☐

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Let us assume, in addition, that the function $b_s(u)$ is differentiable in u (for almost all s) and the derivative $b_s'(u)$ is bounded on bounded domains. We denote as $(x_t, t\in\mathbb{R}_+)$ the solution of the equation

$$x_t = x_0 + \int_0^t b_s(x_s)\,ds$$

(for existence and uniqueness of (x_t) see, e.g., Coddington and Levinson [26, Chapter II]).

We introduce the processes $\tilde{X}^{\epsilon,\eta} = (\tilde{X}^{\epsilon,\eta}_t, t\in\mathbb{R}_+)$ by

$$\tilde{X}^{\epsilon,\eta}_t = \sqrt{\frac{\eta}{\epsilon}}\big(X^{\epsilon,\epsilon,\epsilon}_t - x_t\big),$$

where $\eta > 0$ and define "a non-time-homogeneous idempotent Ornstein-Uhlenbeck process" $\tilde{X} = (\tilde{X}_t,\, t\in\mathbb{R}_+)$ by

$$\dot{\tilde{X}}_t = b_t'(x_t)\tilde{X}_t + \Big(\sigma_t(x_t)^2 + \int_G f_t(x_t,y)^2\,m(dy)\Big)^{1/2}\dot{\tilde{W}}_t,\ \tilde{X}_0 = 0,$$

where $\tilde{W}$ is an $\mathbb{R}$-valued idempotent Wiener process. By Theorem 2.6.26 the latter equation has a unique Luzin strong solution with idempotent distribution specified by the density

$$\Pi^{\tilde{X}}(\mathbf{x}) = \exp\Big(-\frac{1}{2}\int_0^\infty \frac{\big(\dot{\mathbf{x}}_t - b_t'(x_t)\mathbf{x}_t\big)^2}{\sigma_t(x_t)^2 + \int_G f_t(x_t,y)^2\,m(dy)}\,dt\Big)$$

if $\mathbf{x}_0 = 0$ and $\mathbf{x}$ is absolutely continuous, and $\Pi^{\tilde{X}}(\mathbf{x}) = 0$ otherwise.

Theorem 5.4.8. *Let $\epsilon\to 0$ and $\eta\to 0$ in such a way that $\eta/\epsilon\to\infty$. If, in addition, either condition $(\tilde{P})$ holds and $\eta\ln(\epsilon^{-1})\to\infty$, or condition $(\widetilde{SE})$ holds and $\eta^{2-\beta}\epsilon^{-\beta}\to\infty$, then $\tilde{X}^{\epsilon,\eta}\overset{ld}{\to}\tilde{X}$ at rate $1/\eta$.*

Proof. We again invoke Theorem 5.4.4. Since $\tilde{X}^{\epsilon,\eta}$ satisfies the equa-

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tion

$$\tilde{X}_t^{\epsilon,\eta} = \int_0^t \sqrt{\frac{\eta}{\epsilon}}\Big(b_s\Big(\sqrt{\frac{\epsilon}{\eta}}\tilde{X}_s^{\epsilon,\eta} + x_s\Big) - b_s(x_s)\Big)ds$$

$$+ \sqrt{\eta}\int_0^t \sigma_s\Big(\sqrt{\frac{\epsilon}{\eta}}\tilde{X}_s^{\epsilon,\eta} + x_s\Big)dW_s^\epsilon$$

$$+\sqrt{\epsilon\eta}\int_0^t\int_G f_s\Big(\sqrt{\frac{\epsilon}{\eta}}\tilde{X}_{s-}^{\epsilon,\eta}+x_s,y\Big)\left[\mathcal{N}^\epsilon(ds,dy) - \epsilon^{-1}ds\, m(dy)\right],$$

it follows that the predictable characteristics of $\tilde{X}^{\epsilon,\eta}$ without truncation are of the form

$$B'^{\epsilon,\eta}_t = \int_0^t \sqrt{\frac{\eta}{\epsilon}}\Big(b_s\Big(\sqrt{\frac{\epsilon}{\eta}}\tilde{X}_s^{\epsilon,\eta} + x_s\Big) - b_s(x_s)\Big)ds,$$

$$C_t^{\epsilon,\eta} = \eta\int_0^t \sigma_s\Big(\sqrt{\frac{\epsilon}{\eta}}\tilde{X}_s^{\epsilon,\eta} + x_s\Big)^2 ds,$$

$$g(x) * \nu_t^{\epsilon,\eta} = \epsilon^{-1}\int_0^t\int_G g\Big(\sqrt{\epsilon\eta} f_s\Big(\sqrt{\frac{\epsilon}{\eta}}\tilde{X}_s^{\epsilon,\eta} + x_s, y\Big)\Big)m(dy)\, ds,$$

for g Borel and bounded.

Therefore, letting $\phi = (\epsilon,\eta)$, $\alpha_\phi = \epsilon^{-1}$ and $\beta_\phi = (\epsilon\eta)^{-1/2}$, in the notation of Theorem 5.4.4

$$\begin{aligned} b_s^\phi(u) &= \sqrt{\frac{\eta}{\epsilon}}\Big(b_s\Big(\sqrt{\frac{\epsilon}{\eta}}u + x_s\Big) - b_s(x_s)\Big), \\ \tilde{c}'^{\phi}_s(u) &= \sigma_s\Big(\sqrt{\frac{\epsilon}{\eta}}u + x_s\Big)^2 \\ &\quad + \int_G f_s\Big(\sqrt{\frac{\epsilon}{\eta}}u + x_s, y\Big)^2 m(dy) \end{aligned}$$

so that we have the convergences

$$\int_0^t \operatorname*{ess\,sup}_{|u|\le v} |b_s^\phi(u) - b_s(u)|\, ds \to 0, \quad \int_0^t \operatorname*{ess\,sup}_{|u|\le v} |\tilde{c}'^{\phi}_s(u) - c_s|\, ds \to 0,$$

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where

$$b_s(u) = b_s'(x_s)u,\ c_s = \sigma_s(x_s)^2 + \int_G f_s(x_s, y)^2\, m(dy).$$

Now the claimed LD convergence follows by Theorem 5.4.4. □

Example 5.4.9.

Let $\mathbb{R}$-valued processes $X^n = (X_t^n, t \in \mathbb{R}_+)$, where $n \in \mathbb{N}$, be defined on respective stochastic bases $(\Omega_n, \mathcal{F}_n, \mathbf{F}_n = (\mathcal{F}_t^n, t \in \mathbb{R}_+), P_n)$ and have the form

$$X_t^n = \frac{1}{n}\mathcal{N}^n\Big(n\int_0^t f\big(X_s^n, Y_{ns}^n\big)\, ds\Big),$$

where $f(x,y)$ is an $\mathbb{R}_+$-valued Borel function, $\mathcal{N}^n = (\mathcal{N}_t^n,\, t \in \mathbb{R}_+)$ are Poisson processes on $(\Omega_n, \mathcal{F}_n, \mathbf{F}_n, P_n)$, and $Y^n = (Y_t^n, t \in \mathbb{R}_+)$ are Ornstein-Uhlenbeck processes on $(\Omega_n, \mathcal{F}_n, \mathbf{G}_n = (\mathcal{F}_{t/n}^n, t \in \mathbb{R}_+), P_n)$:

$$Y_t^n = -\int_0^t Y_s^n\, ds + \frac{1}{\sqrt{n}} W_t^n,$$

$W^n = (W_t^n,\, t \in \mathbb{R}_+)$ being Wiener processes on $(\Omega_n, \mathcal{F}_n, \mathbf{G}_n, P_n)$. The processes X^n are well defined since the $\mathcal{N}^n$ are piecewise constant.

We assume that $f(x,y)$ is continuous at points $(x,0)$ for $x \in \mathbb{R}_+$ and is such that $\sup_{0\le x\le a, y\in\mathbb{R}} f(x,y) < \infty$ for $a > 0$, $f(x,0) > 0$ for $x \in \mathbb{R}_+$, and the function $f(x,0)$ grows at most linearly as $x \to \infty$.

We prove that the X^n LD converge at rate n to the Luzin-continuous idempotent process X satisfying the equation

$$X_t = \mathcal{N}\Big(\int_0^t f(X_s, 0)\, ds\Big),$$

where $\mathcal{N}$ is a Poisson idempotent process, and having the idempotent distribution with density

$$\Pi^X(\mathbf{x}) = \exp\Big(-\int_0^\infty \sup_{\lambda\in\mathbb{R}}\big(\lambda\dot{\mathbf{x}}_t - (e^\lambda - 1) f(\mathbf{x}_t, 0)\big)\, dt\Big)$$

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if $\mathbf{x}$ is absolutely continuous, increasing and $\mathbf{x}_0 = 0$, and $\Pi^X(\mathbf{x}) = 0$ otherwise. The idempotent process X is well defined by Theorem 2.6.33.

Let us denote $\theta^n_t = Y^n_{nt}$. For θ^n_t we have the equation

$$\theta^n_t = -n \int_0^t \theta^n_s ds + \hat{W}^n_t, \tag{5.4.10}$$

where $\hat{W}^n = (\hat{W}^n_t, t \in \mathbb{R}_+)$ is a Wiener process on $(\Omega_n, \mathcal{F}_n, \mathbf{F}_n, P_n)$. One can show that the nX^n have $\mathbf{F}_n$-compensators $A^n_t = n \int_0^t f(X^n_s, \theta^n_s)\, ds$ so that by Theorems 2.8.10, 2.8.28, and 5.3.7 the claim would follow by

$$\sup_{t \le T} \left| \int_0^{t\wedge\tau_N(X^n)} f(X^n_s, \theta^n_s) ds - \int_0^{t\wedge\tau_N(X^n)} f(X^n_s, 0) ds \right| \overset{P_n^{1/n}}{\to} 0 \text{ as } n \to \infty, \quad T > 0, N \in \mathbb{N}.$$

By continuity of f at points $(x, 0)$, where $x \in \mathbb{R}_+$, and the boundedness condition $\sup_{x\in[0,a], y\in\mathbb{R}} f(x, y) < \infty$, for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\{ \sup_{t \le T} \left| \int_0^{t\wedge\tau_N(X^n)} f(X^n_s, \theta^n_s) ds - \int_0^{t\wedge\tau_N(X^n)} f(X^n_s, 0) ds \right| > \varepsilon \right\} \subset \left\{ \int_0^T \mathbf{1}(|\theta^n_s| > \delta) ds > \frac{\varepsilon}{2} \right\},$$

so by "the Chebyshev inequality" it is sufficient to show that

$$\lim_{n\to\infty} P_n^{1/n} \Big(\int_0^T |\theta^n_s|^2 \wedge 1 \; ds > \eta \Big) = 0, \; \eta > 0. \tag{5.4.11}$$

Let $g(x), x \in \mathbb{R}$, be a twice differentiable non-negative function with bounded first and second derivatives, and such that $g(x) = x^2/2, |x| \le 1$, and $xg'(x) \ge 1$ if $|x| \ge 1$ (e.g., $g(x) = 1/2 + \ln|x| +$

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$(\ln|x|)^2, |x| \geq 1)$. By Ito's formula and (5.4.10)

$$g(\theta_t^n) = g(0) - n \int_0^t g'(\theta_s^n)\theta_s^n \, ds + \int_0^t g'(\theta_s^n) \, d\hat{W}_s^n + \frac{1}{2} \int_0^t g''(\theta_s^n) ds.$$

Since g' is bounded, for all $\lambda \in \mathbb{R}$

$$E_n \exp\bigl(\lambda \int_0^T g'(\theta_s^n) d\hat{W}_s^n - \frac{\lambda^2}{2} \int_0^T g'(\theta_s^n)^2 ds\bigr) = 1$$

and hence

$$E_n \exp\bigl(\lambda g(\theta_T^n) - \lambda g(0) + \lambda n \int_0^T g'(\theta_s^n)\theta_s^n \, ds - \frac{\lambda}{2} \int_0^T g''(\theta_s^n) ds$$
$$- \frac{\lambda^2}{2} \int_0^T g'(\theta_s^n)^2 ds\bigr) = 1,$$

which implies, since g is non-negative, g' and g'' are bounded, $g'(x) = x, |x| \leq 1$, and $g'(x)x \geq 1$ for $|x| \geq 1$, that for some function $F(\lambda)$

$$E_n \exp\bigl(\lambda n \int_0^T |\theta_s^n|^2 \wedge 1 \, ds\bigr) \leq F(\lambda).$$

Hence,

$$\limsup_{n\to\infty} P_n^{1/n}\bigl(\int_0^T |\theta_s^n|^2 \wedge 1 \, ds > \eta\bigr)$$
$$\leq \exp(-\lambda\eta) \limsup_{n\to\infty} \bigl[E_n \exp(\lambda n \int_0^T |\theta_s^n|^2 \wedge 1 \, ds)\bigr]^{1/n}$$
$$\leq \exp(-\lambda\eta).$$

Since λ is arbitrary, (5.4.11) is proved.

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Example 5.4.10.

This example considers a discrete-time case and builds on Example 4.4.12. Let $n \to \infty$, $\alpha \to \infty$ and $\beta \to \infty$ in such a way that $n/\alpha \to \infty$, and $\xi_i^\phi, i = 1, 2, \ldots$, where $\phi = (n, \alpha, \beta)$, be i.i.d. indicator random variables, which equal 1 with probability $\alpha/(n\beta)$ and 0 with probability $1 - \alpha/(n\beta)$. We define random variables Y_k^ϕ by

$$Y_k^\phi = Y_{k-1}^\phi + \sum_{i=1}^{\lfloor \beta f(Y_{k-1}^\phi/\alpha) \rfloor} \xi_i^\phi, \quad Y_0^\phi = 0,$$

where $f(x)$ is a continuous positive function, growing no faster than linearly as $x \to \infty$. Let the process $X^\phi = (X_t^\phi, t \in \mathbb{R}_+)$ be defined by $X_t^\phi = Y_{\lfloor nt \rfloor}^\phi/\alpha$. Then αX^ϕ is a point process whose compensator $A^\phi = (A_t^\phi, t \in \mathbb{R}_+)$ relative to the filtration generated by X^ϕ is given by

$$A_t^\phi = E\xi_1^\phi \sum_{i=0}^{\lfloor nt \rfloor - 1} \lfloor \beta f(X_{i/n}^\phi) \rfloor.$$

We have

$$\left| \frac{A_t^\phi}{\alpha} - \int_0^t f(X_s^\phi)\, ds \right| \le \int_0^{\lfloor nt \rfloor/n} \left| \frac{\lfloor \beta f(X_s^\phi) \rfloor}{\beta} - f(X_s^\phi) \right| ds + \int_{\lfloor nt \rfloor/n}^t f(X_s^\phi)\, ds$$

and

$$\frac{1}{\alpha} \sum_{s \le t \wedge \tau_N(X^\phi)} \left(\Delta A_s^\phi \right)^2 \le \frac{\alpha}{n^2 \beta^2} \lfloor nt \rfloor \sup_{0 \le x \le N} (\beta f(x))^2.$$

Hence, by Theorems 5.3.7 and 2.8.28 the net $\{X^\phi,\, \phi \in \Phi\}$ LD converges at rate α to the semimaxingale X with local characteristics $(b, 0, \nu, 0)$, where $b_s(\mathbf{x}) = f(\mathbf{x}_s)$ and $\nu(\Gamma; \mathbf{x}) = \mathbf{1}(1 \in \Gamma) f(\mathbf{x}_s)$; equivalently X is the Luzin solution of the equation $X_t = \mathcal{N}\Big(\int_0^t f(X_s)\, ds\Big)$,

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where $\mathcal{N}$ is an idempotent Poisson process, whose idempotent distribution has density

$$\Pi^X(\mathbf{x}) = \exp\Big(-\int_0^\infty \sup_{\lambda\in\mathbb{R}}\big(\lambda\dot{\mathbf{x}}_t - (e^\lambda - 1)f(\mathbf{x}_t)\big)\,dt\Big)$$

if $\mathbf{x}$ is absolutely continuous, increasing and $\mathbf{x}_0 = 0$, and $\Pi^X(\mathbf{x}) = 0$ otherwise.

Remark 5.4.11. *It is straightforward to extend Examples 5.4.6, 5.4.9 and 5.4.10 to the case where the coefficients depend on the past.*

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Chapter 6

Large deviation convergence of queueing processes

In this chapter we apply the results on large deviation convergence of semimartingales for deriving large deviation asymptotics in queueing systems.

6.1 Moderate deviations in queueing networks

In this section we prove LD convergence of queueing processes in single server queues and networks of single server queues to idempotent diffusions.

6.1.1 Idempotent diffusion approximation for single server queues

We consider a sequence of FIFO single server queues indexed by n. For the nth system, we denote by A_t^n the number of arrivals by time t, by S_t^n the number of customers served for the first t units of the server's busy time, by D_t^n the number of departures by time t, by Q_t^n the queue length at time t, by W_t^n the unfinished work at time t, by C_t^n the completed work at time t, by H_k^n the waiting time of the kth customer, and by L_k^n the departure time of the kth

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customer. We also introduce

$$V_k^n = \min\{t \in \mathbb{R}_+ : S_t^n \geq k\},\ k \in \mathbb{Z}_+, \tag{6.1.1}$$

which, for $k \in \mathbb{N}$, is the cumulative service time of the first k customers. All the objects referring to the nth system are assumed to be defined on a complete probability space $(\Omega_n, \mathcal{F}_n, P_n)$. Also all the processes are assumed to have trajectories from the associated Skorohod space.

The above processes are connected by the following equalities

$$W_t^n = W_0^n + V^n \circ A_t^n - C_t^n,$$

$$C_t^n = \int_0^t \mathbf{1}(W_s^n > 0)\, ds = \int_0^t \mathbf{1}(Q_s^n > 0)\, ds, \tag{6.1.2}$$

$$Q_t^n = Q_0^n + A_t^n - D_t^n, \tag{6.1.3}$$

$$D_t^n = S^n \circ C_t^n, \tag{6.1.4}$$

Let $b_n \to \infty$ and $b_n/\sqrt{n} \to 0$ as $n \to \infty$, and λ_n and μ_n be positive numbers. We define the associated normalized and time-scaled processes by

$$\overline{A}^n = (\overline{A}_t^n,\, t \in \mathbb{R}_+),\ \overline{A}_t^n = \frac{1}{b_n\sqrt{n}}(A_{nt}^n - \lambda_n nt), \tag{6.1.5}$$

$$\overline{S}^n = (\overline{S}_t^n,\, t \in \mathbb{R}_+),\ \overline{S}_t^n = \frac{1}{b_n\sqrt{n}}(S_{nt}^n - \mu_n nt), \tag{6.1.6}$$

$$\overline{Q}^n = (\overline{Q}_t^n,\, t \in \mathbb{R}_+),\ \overline{Q}_t^n = \frac{1}{b_n\sqrt{n}}Q_{nt}^n, \tag{6.1.7}$$

$$\overline{C}^n = (\overline{C}_t^n,\, t \in \mathbb{R}_+),\ \overline{C}_t^n = \frac{1}{b_n\sqrt{n}}\left(C_{nt}^n - nt\right), \tag{6.1.8}$$

$$\overline{V}^n = (\overline{V}_t^n,\, t \in \mathbb{R}_+),\ \overline{V}_t^n = \frac{1}{b_n\sqrt{n}}(V_{\lfloor nt \rfloor}^n - \mu_n^{-1} nt),$$

$$\overline{D}^n = (\overline{D}_t^n,\, t \in \mathbb{R}_+),\ \overline{D}_t^n = \frac{1}{b_n\sqrt{n}}(D_{nt}^n - \mu_n nt),$$

$$\overline{W}^n = (\overline{W}_t^n,\, t \in \mathbb{R}_+),\ \overline{W}_t^n = \frac{1}{b_n\sqrt{n}}W_{nt}^n,$$

$$\overline{H}^n = (\overline{H}_t^n,\, t \in \mathbb{R}_+),\ \overline{H}_t^n = \frac{1}{b_n\sqrt{n}}H_{\lfloor nt \rfloor + 1}^n,$$

$$\overline{L}^n = (\overline{L}_t^n,\, t \in \mathbb{R}_+),\ \overline{L}_t^n = \frac{1}{b_n\sqrt{n}}(L_{\lfloor nt \rfloor + 1}^n - \mu_n^{-1} nt).$$

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We assume that $\lambda_n \to \lambda > 0$ and $\mu_n \to \mu > 0$ as $n \to \infty$, and "the near-heavy-traffic condition" holds:

$$\frac{1}{b_n}\sqrt{n}(\lambda_n - \mu_n) \to c,\ c \in \mathbb{R}. \tag{6.1.9}$$

Note that (6.1.9) implies that $\lambda = \mu$.

We recall that the one-dimensional Skorohod reflection map $\mathbf{x} \to \mathcal{R}(\mathbf{x})$ is characterised by the property that $\mathbf{z} = \mathcal{R}(\mathbf{x})$ is an only $\mathbb{R}_+$-valued function such that $\mathbf{z} = \mathbf{x} + \mathbf{y}$, where $\mathbf{y}$ is increasing, $\mathbf{y}_0 = 0$ and $\int_0^\infty \mathbf{1}(\mathbf{z}_t > 0)\, d\mathbf{y}_t = 0$. It is a continuous map from $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ to $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and can explicitly be written as

$$\mathcal{R}(\mathbf{x})_t = \mathbf{x}_t - \inf_{0 \le s \le t} \mathbf{x}_s \wedge 0,\ t \in \mathbb{R}_+\ , \tag{6.1.10}$$

where $\mathbf{x} \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and $\mathbf{x}_0 \in \mathbb{R}_+$, see, e.g., Ikeda and Watanabe [66].

Let $W_A = (W_{A,t},\, t \in \mathbb{R}_+)$ and $W_S = (W_{S,t},\, t \in \mathbb{R}_+)$ be independent idempotent Wiener processes on an idempotent probability space (Ω, Π). Let σ_A and σ_S be real numbers. As above we denote $\mathbf{e} = (t,\, t \in \mathbb{R}_+)$. In the theorems below LD convergence refers to the Skorohod topology and rate $r_n = b_n^2$.

Theorem 6.1.1. *Let* $(\overline{A}^n, \overline{S}^n) \stackrel{ld}{\to} (\sigma_A W_A, \sigma_S W_S)$ *and* $\overline{Q}_0^n \stackrel{P_n^{1/b_n^2}}{\to} q_0$. *Then* $\overline{Q}^n \stackrel{ld}{\to} Q$, *where* $Q = (Q_t,\, t \in \mathbb{R}_+)$ *is an* $\mathbb{R}_+$*-valued Luzin-continuous idempotent process defined by*

$$Q = \mathcal{R}\big(q_0 + \sigma_A W_A - \sigma_S W_S + c\mathbf{e}\big).$$

Proof. Let us denote $A = \sigma_A W_A$ and $S = \sigma_S W_S$. By (6.1.3), (6.1.4), (6.1.2), (6.1.7), (6.1.5), (6.1.6), and (6.1.8)

$$\overline{Q}_t^n = \overline{Q}_0^n + \overline{A}_t^n - \overline{S}^n \circ \overline{C}_t^{\prime n} + \frac{\sqrt{n}}{b_n}(\lambda_n - \mu_n)t + \frac{\sqrt{n}}{b_n}\mu_n \int_0^t \mathbf{1}(\overline{Q}_s^n = 0)\, ds, \tag{6.1.11}$$

$$\overline{C}_t^n = -\frac{\sqrt{n}}{b_n} \int_0^t \mathbf{1}(\overline{Q}_s^n = 0)\, ds, \tag{6.1.12}$$

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where

$$\overline{C}'^n_t = \frac{1}{n} C^n_{nt} = \int_0^t \mathbf{1}(\overline{Q}^n_s > 0)\, ds. \qquad (6.1.13)$$

Since $\overline{Q}^n_t$ is non-negative and $\int_0^t \mathbf{1}(\overline{Q}^n_s = 0)\, ds$ increases only when $\overline{Q}^n_t = 0$, (6.1.11) allows us to conclude that

$$\overline{Q}^n = \mathcal{R}\left(\overline{Q}^n_0 + \overline{A}^n - \overline{S}^n \circ \overline{C}'^n + \frac{\sqrt{n}}{b_n}(\lambda_n - \mu_n)\mathbf{e}\right). \qquad (6.1.14)$$

Since by (6.1.12) and (6.1.11)

$$\mu_n \overline{C}^n = \overline{Q}^n_0 + \overline{A}^n - \overline{S}^n \circ \overline{C}'^n + \frac{\sqrt{n}}{b_n}(\lambda_n - \mu_n)\mathbf{e} - \overline{Q}^n,$$

it follows by (6.1.14) that

$$\mu_n \overline{C}^n = \overline{Q}^n_0 + \overline{A}^n - \overline{S}^n \circ \overline{C}'^n + \frac{\sqrt{n}}{b_n}(\lambda_n - \mu_n)\mathbf{e} - \mathcal{R}\left(\overline{Q}^n_0 + \overline{A}^n - \overline{S}^n \circ \overline{C}'^n + \frac{\sqrt{n}}{b_n}(\lambda_n - \mu_n)\mathbf{e}\right).$$

Therefore, by (6.1.10)

$$\mu_n |\overline{C}^n_t| \le 2 \sup_{s \le t} \left| \overline{Q}^n_0 + \overline{A}^n_s - \overline{S}^n \circ \overline{C}'^n_s + \frac{\sqrt{n}}{b_n}(\lambda_n - \mu_n)s \right|, \ t \in \mathbb{R}_+.$$

The convergences $\overline{Q}^n_0 \overset{P_n^{1/b_n^2}}{\to} q_0$ and $(\overline{A}^n, \overline{S}^n) \overset{ld}{\to} (A, S)$, the fact that A and S are proper idempotent processes, the inequality $\overline{C}'^n_t \le t$, and (6.1.9) imply that

$$\lim_{a \to \infty} \limsup_{n \to \infty} P_n^{1/b_n^2}(\mu_n |\overline{C}^n_t| > a) = 0.$$

Hence, by (6.1.12) and the facts that $\sqrt{n}/b_n \to \infty$ and $\mu_n \to \mu > 0$

$$\int_0^t \mathbf{1}(\overline{Q}^n_s = 0)\, ds \overset{P_n^{1/b_n^2}}{\to} 0 \quad \text{as} \quad n \to \infty,\ t \in \mathbb{R}_+, \qquad (6.1.15)$$

which implies by (6.1.13) that $\overline{C}'^n \overset{P_n^{1/b_n^2}}{\to} \mathbf{e}$. Then "the time-change theorem" (Lemma 3.2.11) implies by the LD convergence of $(\overline{A}^n, \overline{S}^n)$

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to (A,S) that the sequence $\{(\overline{A}^n, \overline{S}^n \circ \overline{C}'^n), n \in \mathbb{N}\}$ LD converges to (A,S) as well. Since $\overline{Q}_0^n \overset{P_n^{1/b_n^2}}{\to} q_0$, we have that $(\overline{Q}_0^n, \overline{A}^n, \overline{S}^n \circ \overline{C}'^n) \overset{ld}{\to} (q_0, A, S)$ by Lemma 3.1.42. By (6.1.14) and continuity of reflection $\overline{Q}^n$ is a continuous function of $(\overline{Q}_0^n, \overline{A}^n, \overline{S}^n \circ \overline{C}'^n, (\sqrt{n}/b_n)(\lambda_n - \mu_n)\mathbf{e})$. Therefore, the LD convergence of $(\overline{Q}_0^n, \overline{A}^n, \overline{S}^n \circ \overline{C}'^n)$, the near-heavy traffic condition (6.1.9), and the contraction principle yield the required LD convergence of $\{\overline{Q}^n,\ n \in \mathbb{N}\}$. The idempotent process Q is Luzin-continuous since A and S are Luzin-continuous and $\mathcal{R}$ is continuous. □

Remark 6.1.2. *If we assume, in addition to the hypotheses of Theorem 6.1.1, that* $\overline{W}_0^n \overset{P_n^{1/b_n^2}}{\to} q_0/\mu$, *then* $(\overline{Q}^n, \overline{D}^n, \overline{W}^n, \overline{C}^n, \overline{H}^n, \overline{L}^n) \overset{ld}{\to} (Q, D, W, C, H, L)$, *where*

$$D_t = \sigma_A W_{A,t} - Q_t + ct,\ W_t = \frac{Q_t}{\mu},$$
$$C_t = \frac{\sigma_A W_{A,t} - \sigma_S W_{S,t} + ct}{\mu} - W_t,$$
$$H_t = W(t/\mu),\ L_t = -\frac{D_{t/\mu}}{\mu}.$$

The following lemma gives an explicit expression for the idempotent distribution of Q. Let Π^Q denote the idempotent distribution of Q and $I^Q(\mathbf{q}) = -\ln \Pi^Q(\mathbf{q})$ be the associated rate function.

Lemma 6.1.3. *Let* $\sigma_A^2 + \sigma_S^2 > 0$. *The rate function* I^Q *is given by*

$$I^Q(\mathbf{q}) = \frac{1}{2(\sigma_A^2 + \sigma_S^2)} \int_0^\infty \mathbf{1}(\mathbf{q}_t > 0)(\dot{\mathbf{q}}_t - c)^2\, dt + \frac{\mathbf{1}(c > 0)c^2}{2(\sigma_A^2 + \sigma_S^2)} \int_0^\infty \mathbf{1}(\mathbf{q}_t = 0)\, dt,$$

if $\mathbf{q}$ *is a non-negative and absolutely continuous function such that* $\mathbf{q}_0 = q_0$, *and* $I_Q(\mathbf{q}) = \infty$ *otherwise.*

For a proof, we need the following result.

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Lemma 6.1.4. *Let* $\mathbf{z} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ *be non-negative and* $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ *be absolutely continuous. Then* $\mathbf{z} = \mathcal{R}(\mathbf{x})$ *if and only if* $\mathbf{z}$ *is absolutely continuous and there exists an absolutely continuous function* $\mathbf{y} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R})$ *such that*

$$\dot{\mathbf{z}}_t = \dot{\mathbf{x}}_t + \dot{\mathbf{y}}_t \quad a.e.$$

and

$$\mathbf{y}_0 = 0, \ \dot{\mathbf{y}}_t \in \mathbb{R}_+ \quad a.e., \ \mathbf{z}_t \dot{\mathbf{y}}_t = 0 \quad a.e.$$

Also $\dot{\mathbf{z}}_t = 0$ *a.e. on the set* $\{t : \mathbf{z}_t = 0\}$.

Proof. Sufficiency of the condition follows by the definition of the reflection mapping. Conversely, if $\mathbf{y} = \mathcal{R}(\mathbf{x}) - \mathbf{x}$, then $\mathbf{y}_t - \mathbf{y}_s \le \int_s^t |\dot{\mathbf{x}}_u| \, du$ for $0 \le s \le t$, so $\mathbf{y}$ is absolutely continuous. The other conditions on $\mathbf{y}$ follow from the definition of reflection. For the final part, note that a.e. $\dot{\mathbf{z}}_t = \lim_{h \to 0} (\mathbf{z}_{t+h} - \mathbf{z}_t)/h$. The numerator in the latter fraction being non-negative since $\mathbf{z}_t = 0$ implies that the fraction is non-negative for h positive and non-positive for h negative. Hence, the limit is zero. □

Proof of Lemma 6.1.3. Let $\sigma^2 = \sigma_A^2 + \sigma_S^2$. By Corollary 2.4.11 we may assume that $\sigma_A W_A + \sigma_S W_S = \sigma W_s$, where W is an idempotent Wiener process. By Theorem 6.1.1 and the definition of the image idempotent measure

$$\Pi^Q(\mathbf{q}) = \sup\{\Pi^W(w), \ \mathbf{q} = \mathcal{R}(q_0 + \sigma w + c\mathbf{e})\}.$$

Therefore, $\mathbf{q}_0 = q_0$ and $\mathbf{q}$ is absolutely continuous Π^Q-a.e. For these $\mathbf{q}$ by the definition of an idempotent Wiener process, Lemma 6.1.4 and Lemma A.2 in Appendix A

$$I^Q(\mathbf{q}) = \inf_{\substack{\mathbf{y}:\, \mathbf{y}_0 = 0, \dot{\mathbf{y}}_t \in \mathbb{R}_+, \\ \mathbf{1}(\mathbf{q}_t > 0)\dot{\mathbf{y}}_t = 0, \\ \dot{\mathbf{q}}_t = \sigma \dot{w}_t + c + \dot{\mathbf{y}}_t}} \frac{1}{2} \int_0^\infty \dot{w}_t^2 \, dt$$

$$= \frac{1}{2\sigma^2} \int_0^\infty \inf_{\substack{\dot{\mathbf{y}}_t :\, \dot{\mathbf{y}}_t \in \mathbb{R}_+, \\ \mathbf{1}(\mathbf{q}_t > 0)\dot{\mathbf{y}}_t = 0}} \left(\dot{\mathbf{q}}_t - c - \dot{\mathbf{y}}_t\right)^2 dt$$

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$$= \frac{1}{2\sigma^2}\int_0^\infty \mathbf{1}(\mathbf{q}_t > 0)\big(\dot{\mathbf{q}}_t - c\big)^2\,dt$$

$$+ \frac{1}{2\sigma^2}\int_0^\infty \mathbf{1}(\mathbf{q}_t = 0)\inf_{y\in\mathbb{R}_+}\big(\dot{\mathbf{q}}_t - c - y\big)^2\,dt$$

$$= \frac{1}{2\sigma^2}\int_0^\infty \mathbf{1}(\mathbf{q}_t > 0)\big(\dot{\mathbf{q}}_t - c\big)^2\,dt + \frac{\mathbf{1}(c>0)c^2}{2\sigma^2}\int_0^\infty \mathbf{1}(\mathbf{q}_t = 0)\,dt.$$

□

The next lemma formulates the LD convergence conditions in the hypotheses of Theorem 6.1.1 in terms of interarrival and service times. Let $U_k^n = \inf\{t \in \mathbb{R}_+ : A_t^n \geq k\}\,, k \in \mathbb{Z}_+$, and $\overline{U}_t^n = \big(U_{\lfloor nt\rfloor}^n - \mu_n^{-1}nt\big)/\big(b_n\sqrt{n}\big)$. Recalling also (6.1.1) we have by Lemma 3.2.13 the following.

Lemma 6.1.5. *The LD convergence* $(\overline{A}^n, \overline{S}^n) \overset{ld}{\to} (\sigma_A W_A, \sigma_S W_S)$ *holds if and only if the sequence* $\{(\overline{U}^n, \overline{V}^n),\, n \in \mathbb{N}\}$ *LD converges to* $(\sigma_A \lambda^{-3/2} W_A, \sigma_S \mu^{-3/2} W_S)$.

We now specify the results to the case of $GI/GI/1$ queues, i.e., we assume that the A^n and S^n are renewal processes. Let us denote by u_i^n, $i \in \mathbb{N}$, the time between the ith and $(i+1)$th arrivals and by v_i^n, $i \in \mathbb{N}$, the service time of the ith customer in the nth system. By hypothesis the sequences $\{u_i^n,\, i \in \mathbb{N}\}$ and $\{v_i^n,\, i \in \mathbb{N}\}$ are independent i.i.d. Theorem 4.4.8 provides us with the following way of checking the convergence requirements of Lemma 6.1.5.

Lemma 6.1.6. *Let either one of the following conditions hold:*

(i) $\sup_n E_n(u_1^n)^{2+\epsilon} < \infty$, $\sup_n E_n(v_1^n)^{2+\epsilon} < \infty$ *for some* $\epsilon > 0$, *and* $b_n^2/\ln n \to 0$;

(ii) $\sup_n E_n \exp(\alpha(u_1^n)^\beta) < \infty$, $\sup_n E_n \exp(\alpha(v_1^n)^\beta) < \infty$ *for some* $\alpha > 0, 0 < \beta \leq 1$, *and* $b_n^{2-\beta}/n^{\beta/2} \to 0$.

If $E_n u_1^n \to \lambda^{-1}$, $E_n v_1^n \to \mu^{-1}$, $Var_n u_1^n \to \sigma_A^2/\lambda^3$, *and* $Var_n v_1^n \to \sigma_S^2/\mu^3$, *then* $(\overline{U}^n, \overline{V}^n) \overset{ld}{\to} (\sigma_A \lambda^{-3/2} W_A, \sigma_S \mu^{-3/2} W_S)$.

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We now establish LD convergence for stationary waiting times. Let partial-sum processes $U'^n = (U'^n_k,\, k \in \mathbb{Z}_+)$ and $V^n = (V^n_k,\, k \in \mathbb{Z}_+)$ be given by

$$U'^n_k = \sum_{i=1}^{k} u^n_i,\ U'^n_0 = 0,\ \ V^n_k = \sum_{i=1}^{k} v^n_i,\ V^n_0 = 0, \qquad (6.1.16)$$

so that, as above, V^n_k, for $k \in \mathbb{N}$, is the cumulative service time of the first k customers. The equation for waiting times is

$$H^n_{k+1} = H^n_1 + V^n_k - U'^n_k - \min_{1 \le i \le k} (H^n_1 + V^n_i - U'^n_i) \wedge 0. \qquad (6.1.17)$$

We recall that if $E_n v^n_1 < E u^n_1$, then the waiting times H^n_k converge in distribution as $k \to \infty$ to the proper random variable $\sup_{k \in \mathbb{Z}_+}(V^n_k - U'^n_k)$ (see, e.g., Borovkov [15]). We denote the latter by H^n_0 and let $\overline{H}^n_0 = H^n_0/(b_n\sqrt{n})$.

Theorem 6.1.7. *Let either one of conditions (i) or (ii) of Lemma 6.1.6 hold. Let* $(\sqrt{n}/b_n)(Eu^n_1 - Ev^n_1) \to c' > 0$, $Var_n u^n_1 \to \sigma^2_U$, $Var_n v^n_1 \to \sigma^2_V$, *where* $\sigma^2_U + \sigma^2_V > 0$, *as* $n \to \infty$. *Then the sequence* $\{\overline{H}^n_0,\, n \in \mathbb{N}\}$ *LD converges in distribution to an exponentially distributed* $\mathbb{R}_+$*-valued idempotent variable with density* $\Pi(x) = \exp\big(-2c'x/(\sigma^2_U + \sigma^2_V)\big)$, $x \in \mathbb{R}_+$.

Proof. Since H^n_0 is distributed as $\sup_{k \in \mathbb{Z}_+}(V^n_k - U'^n_k)$, we have, for a Borel subset A of $\mathbb{R}_+$,

$$\left| P_n(\overline{H}^n_0 \in A) - P_n\left(\frac{1}{b_n\sqrt{n}} \sup_{0 \le k \le \lfloor nt \rfloor} (V^n_k - U'^n_k) \in A \right) \right| \\ \le P_n\big(\sup_{k > \lfloor nt \rfloor} (V^n_k - U'^n_k) \ge 0 \big).$$

Let

$$\tilde{U}'^n = (\tilde{U}'^n_t,\, t \in \mathbb{R}_+),\ \tilde{U}'^n_t = \frac{1}{b_n\sqrt{n}}(U'^n_{\lfloor nt \rfloor} - Eu^n_1 nt),$$
$$\tilde{V}^n = (\tilde{V}^n_t,\, t \in \mathbb{R}_+),\ \tilde{V}^n_t = \frac{1}{b_n\sqrt{n}}(V^n_{\lfloor nt \rfloor} - Ev^n_1 nt).$$

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Since by Theorem 4.4.8 $(\tilde{U}'^n, \tilde{V}^n) \overset{ld}{\to} (\sigma_U W_A, \sigma_V W_S)$, $(E_n u_1^n - E_n v_1^n)\sqrt{n}/b_n \to c'$ by hypotheses, and

$$\frac{1}{b_n\sqrt{n}} \sup_{0\le k\le \lfloor nt\rfloor} (V_k^n - U'^n_k) = \sup_{0\le s\le t}\left(\overline{V}_s^n - \tilde{U}_s'^n - (E_n u_1^n - E_n v_1^n)\frac{\sqrt{n}}{b_n}s\right),$$

by the contraction principle and the fact that $\sigma_U W_A + \sigma_V W_S = \sigma W$, where $\sigma^2 = \sigma_A^2 + \sigma_S^2$ and W is an idempotent Wiener process, we have that

$$\frac{1}{b_n\sqrt{n}} \sup_{0\le k\le \lfloor nt\rfloor} (V_k^n - U'^n_k) \overset{ld}{\to} \sup_{0\le s\le t} (\sigma W_s - c' s).$$

Let ξ_t denote the idempotent variable on the right-hand side and $\xi = \sup_{s\in\mathbb{R}_+}(\sigma W_s - c's)$. It is an easy exercise to check that ξ has idempotent distribution Π in the statement of the theorem; in particular, it is a Luzin idempotent variable. We show that ξ_t converges to ξ as $t\to\infty$ in idempotent distribution. The map $w \to \sup_{0\le s\le t}(\sigma w_s - c's)$ from $\mathbb{C}(\mathbb{R}_+,\mathbb{R})$ to $\mathbb{R}$ is continuous, so ξ_t is a Luzin idempotent variable as well. Assuming that both ξ and ξ_t are defined on $(\mathbb{C}(\mathbb{R}_+,\mathbb{R}), \Pi^W)$, we have that $\xi - \xi_t$ monotonically converges to zero Π^W-almost everywhere, so by Theorem 1.3.10 the convergence is actually in deviability Π^W and by Lemma 1.10.7 $\xi_t \overset{id}{\to} \xi$. Thus, by Lemma 3.1.37 the required would follow by

$$\lim_{t\to\infty}\limsup_{n\to\infty} P_n^{1/b_n^2}\big(\sup_{k>\lfloor nt\rfloor}(V_k^n - U'^n_k) \ge 0\big) = 0. \qquad (6.1.18)$$

Denoting $\delta_n = E_n(u_1^n - v_1^n)$ and $\xi_i^n = v_i^n - u_i^n + \delta_n$, we have, since $\delta_n > 0$,

$$P_n\big(\sup_{k>\lfloor nt\rfloor}(V_k^n - U'^n_k)\ge 0\big) \le \sum_{l=\lfloor \log_2(nt)\rfloor}^{\infty} P_n\left(\max_{k=2^l+1,\dots,2^{l+1}}\left(\sum_{i=1}^k \xi_i^n - k\delta_n\right)\ge 0\right)$$

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$$\leq \sum_{l=\lfloor \log_2(nt) \rfloor}^{\infty} P_n\left(\sum_{i=1}^{2^l} \xi_i^n \geq 2^{l-1}\delta_n\right)$$
$$+ \sum_{l=\lfloor \log_2(nt) \rfloor}^{\infty} P_n\left(\max_{k=1,\dots,2^l} \sum_{i=1}^{k} \xi_i^n \geq 2^{l-1}\delta_n\right)$$
$$\leq 2 \sum_{l=\lfloor \log_2(nt) \rfloor}^{\infty} P_n\left(\max_{k=1,\dots,2^l} \sum_{i=1}^{k} \xi_i^n \geq 2^{l-1}\delta_n\right).$$

Limit (6.1.18) now follows by Lemma A.3 in Appendix A and the near-heavy traffic condition $(\sqrt{n}/b_n)\delta_n \to c' > 0$ as $n \to \infty$. □

6.1.2 Idempotent diffusion approximation for queueing networks

We now extend some of the results of the preceding subsection to the queueing-network set-up. We consider a sequence of networks indexed by n. The nth network has a homogeneous customer population and consists of K FIFO single server stations. The network is open so customers arrive from outside and eventually leave. For the nth network, let $A_t^{n,k}, k = 1, \dots, K$, denote the cumulative number of customers who arrive at station k from outside the network during the interval $[0, t]$, and let $S_t^{n,k}, k = 1, \dots, K$, denote the cumulative number of customers who complete service at station k during the first t units of busy time of that station. We call $A^n = (A^{n,k}, k = 1, \dots, K)$, where $A^{n,k} = (A_t^{n,k}, t \in \mathbb{R}_+)$, and $S^n = (S^{n,k}, k = 1, \dots, K)$, where $S^{n,k} = (S_t^{n,k}, t \in \mathbb{R}_+)$, the arrival process and service process, respectively (note that some of the entries in A^n may equal zero). We associate with the stations of the network the processes $R^{n,k} = (R^{n,kl}, l = 1, \dots, K)$, $k = 1, \dots, K$, where $R^{n,kl} = (R_m^{n,kl}, m \in \mathbb{N})$, and $R_m^{n,kl}$ denotes the cumulative number of customers among the first m customers who depart station k that go directly to station l. The process $R^n = (R^{n,kl}, k, l = 1, \dots, K)$ is referred to as the routing process. We consider the processes $A^{n,k}$, $S^{n,k}$ and $R^{n,k}$ as random elements of the respective Skorohod spaces $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^K)$; accordingly, A^n, S^n and R^n are considered as random elements of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^K)$, $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^K)$ and $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{K\times K})$, respectively. We assume that the data associated with the nth network is defined on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$.

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We next introduce normalized and time-scaled versions of the arrival process, service process and routing process. Let $\lambda_{n,k} \in \mathbb{R}_+$, $\mu_{n,k} \in \mathbb{R}_+$, and $p_{kl} \in [0,1]$, $k = 1, \ldots, K, l = 1, \ldots, K$. We define

$$\overline{A}_t^{n,k} = \frac{A_{nt}^{n,k} - \lambda_{n,k} nt}{b_n \sqrt{n}}, \ \overline{S}_t^{n,k} = \frac{S_{nt}^{n,k} - \mu_{n,k} nt}{b_n \sqrt{n}}, \tag{6.1.19}$$

$$\overline{R}_t^{n,kl} = \frac{R_{\lfloor nt \rfloor}^{n,kl} - p_{kl} \lfloor nt \rfloor}{b_n \sqrt{n}}, \tag{6.1.20}$$

where, as above, $b_n \to \infty$ and $b_n/\sqrt{n} \to 0$, and let $\overline{A}^n = (\overline{A}^{n,k}, k = 1, \ldots, K)$, $\overline{S}^n = (\overline{S}^{n,k}, k = 1, \ldots, K)$, $\overline{R}^{n,k} = (\overline{R}^{n,kl}, l = 1, \ldots, K), k = 1, \ldots, K$, and $\overline{R}^n = (\overline{R}^{n,kl}, k, l = 1, \ldots, K)$. Again the latter processes are considered as random elements of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^K)$, $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^K)$, $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^K)$, and $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{K \times K})$, respectively. Also we denote $\lambda_n = (\lambda_{n,k}, k = 1, \ldots, K)$, $\mu_n = (\mu_{n,k}, k = 1, \ldots, K)$ and $P = (p_{kl}, k = 1, \ldots, K, l = 1, \ldots, K)$. Elements of $\mathbb{R}^K$ are regarded as column-vectors.

Our main concern here is the queue-length process $Q^n = (Q^{n,k}, k = 1, \ldots, K)$, where $Q^{n,k} = (Q_t^{n,k}, t \in \mathbb{R}_+)$, $Q_t^{n,k}$ denoting the number of customers at station k at time t. The associated normalized and time-scaled process $\overline{Q}^n = (\overline{Q}^{n,k}, k = 1, \ldots, K)$ is defined by

$$\overline{Q}_t^{n,k} = \frac{Q_{nt}^{n,k}}{b_n \sqrt{n}}. \tag{6.1.21}$$

In analogy with the hypotheses of Subsection 6.1.1 we assume that $\lambda_n \to \lambda = (\hat{\lambda}_1, \ldots, \hat{\lambda}_K)$ and $\mu_n \to \mu = (\hat{\mu}_1, \ldots, \hat{\mu}_K)$ as $n \to \infty$, where μ is a component-wise positive vector, and that "the near-heavy traffic condition" holds: for some $c \in \mathbb{R}^K$

$$\frac{\sqrt{n}}{b_n}(\lambda_n - (\mathbf{E}_K - P^T)\mu_n) \to c \quad \text{as } n \to \infty, \tag{6.1.22}$$

in particular,

$$\lambda = (\mathbf{E}_K - P^T)\mu. \tag{6.1.23}$$

(Recall that $\mathbf{E}_K$ denotes the identity $K \times K$ matrix.) We also assume that the spectral radius of the matrix P is less than unity.

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We recall that the skew reflection mapping $\mathcal{R}_P$, Harrison and Reiman [59], Reiman [115], is defined as the map from $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^K)$ into $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^K)$ associating to each $\mathbf{x} = (\mathbf{x}_t,\, t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+,\mathbb{R}^K)$ such that $\mathbf{x}_0^k \in \mathbb{R}_+$, $k = 1,\ldots,K$, a function $\mathbf{z} = (\mathbf{z}_t,\, t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+,\mathbb{R}^K)$ such that

1. $\mathbf{z} = \mathbf{x} + (\mathbf{E}_K - P^T)\mathbf{y}$,
2. $\mathbf{y}$ is componentwise increasing and $\mathbf{y}_0^k = 0, k = 1,\ldots,K$,
3. $\mathbf{z}_t^k \in \mathbb{R}_+$ and $\displaystyle\int_0^\infty \mathbf{z}_t^k \, d\mathbf{y}_t^k = 0,\ k = 1,\ldots,K.$

The map $\mathcal{R}_P$ is well defined and Lipshitz continuous for the locally uniform and Skorohod topologies on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^K)$, Harrison and Reiman [59], Reiman [115], Chen and Whitt [23].

As in Subsection 6.1.1 all LD convergences below refer to the rate $r_n = b_n^2$ and the Skorohod topology. We recall the notation introduced at the end of Section 3.2. If $\mathbf{x} \in \mathbb{D}(\mathbb{R}_+,\mathbb{R}^K)$ has componentwise increasing $\mathbb{R}_+$-valued paths, then, for $\mathbf{y} \in \mathbb{D}(\mathbb{R}_+,\mathbb{R}^K)$, we denote $\mathbf{y} \circ \mathbf{x} = ((\mathbf{y}^k_{\mathbf{x}_t^k}, k = 1,\ldots,K),\, t \in \mathbb{R}_+)$; analogously, if $\mathbf{r}_t = (\mathbf{r}_t^{kl}, k,l = 1,\ldots,K) \in \mathbb{R}^{K\times K}$, then $\mathbf{r} \circ \mathbf{x}_t = (\mathbf{r}^{kl}_{\mathbf{x}_t^k}, k,l = 1,\ldots,K)$. For vectors $\alpha = (\alpha^1,\ldots,\alpha^K) \in \mathbb{R}^K$ and $\beta = (\beta^1,\ldots,\beta^K) \in \mathbb{R}^K$, we denote $\alpha \otimes \beta = (\alpha_1\beta^1,\ldots,\alpha^K\beta^K) \in \mathbb{R}^K$. Let $\mathbf{1}$ denote the K-vector with all the entries equal to 1.

Theorem 6.1.8. *Let the near-heavy-traffic condition (6.1.22) hold. Let* $\overline{Q}_0^n \overset{P_n^{1/b_n^2}}{\to} q_0$. *Let the sequence* $\{(\overline{A}^n, \overline{S}^n, \overline{R}^n), n \in \mathbb{N}\}$ *LD converge in* $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^{K\times K})$ *to an idempotent process* (A,S,R), *where* $A = (A^1,\ldots,A^K)$, $S = (S^1,\ldots,S^K)$, *and* $R = ({R^1}^T,\ldots,{R^K}^T)$ *are defined by*

$$A = \Sigma_A W_A,\ S = \Sigma_S W_S,\ \ R^k = \Sigma_R^k W_R^k,\ k = 1,\ldots,K,$$

W_A, W_S, W_R^k, $k = 1,\ldots,K$, *being mutually independent* K*–dimensional idempotent Wiener processes and* $\Sigma_A, \Sigma_S, \Sigma_R^k$, $k = 1,\ldots,K$, *being* $K \times K$ *matrices. Then* $\overline{Q}^n \overset{ld}{\to} Q$, *where* Q *is a Luzin-continuous idempotent process given by*

$$Q = \mathcal{R}_P(q_0 + A + (R \circ \mu e)^T\mathbf{1} - (\mathbf{E}_K - P^T)S + c\mathbf{e}).$$

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Proof. The proof is a straightforward extension of the proof of Theorem 6.1.1. In analogy with (6.1.3), (6.1.4) and (6.1.2), we have that for $k = 1, \dots, K$

$$Q_t^{n,k} = Q_0^{n,k} + A_t^{n,k} + \sum_{l=1}^{K} R^{n,lk} \circ D_t^{n,l} - D_t^{n,k},$$

where $D_t^{n,k} = S^{n,k}_{\int_0^t \mathbf{1}(Q_s^{n,k}>0)ds}$. Introducing

$$\overline{C}'^{n,k}_t = \int_0^t \mathbf{1}(\overline{Q}_s^{n,k} > 0)ds, \; \overline{D}'^{n,k}_t = \frac{D_{nt}^{n,k}}{n},$$

we then have by (6.1.19), (6.1.20) and (6.1.21) that

$$\overline{Q}_t^{n,k} = \overline{Q}_0^{n,k} + \overline{A}_t^{n,k} + \sum_{l=1}^{K} \overline{R}^{n,lk} \circ \overline{D}'^{n,l}_t$$
$$+ \sum_{l=1}^{K} p_{lk} \overline{S}^{n,l} \circ \overline{C}'^{n,l}_t - \overline{S}^{n,k} \circ \overline{C}'^{n,k}_t + \frac{\sqrt{n}}{b_n}(\lambda_{n,k} + \sum_{l=1}^{K} p_{lk}\mu_{n,l} - \mu_{n,k})t$$
$$+ \frac{\sqrt{n}}{b_n}\left(\mu_{n,k}\int_0^t \mathbf{1}(\overline{Q}_s^{n,k} = 0)ds - \sum_{l=1}^{K} p_{lk}\mu_{n,l}\int_0^t \mathbf{1}(\overline{Q}_s^{n,l} = 0)ds\right),$$

or in vector form

$$\overline{Q}^n = \overline{Q}_0^n + \overline{A}^n + (\overline{R}^n \circ \overline{D}'^n)^T \mathbf{1} - (\mathbf{E}_K - P^T)\overline{S}^n \circ \overline{C}'^n$$
$$+ \frac{\sqrt{n}}{b_n}(\lambda_n - (\mathbf{E}_K - P^T)\mu_n)\mathbf{e} - (\mathbf{E}_K - P^T)\mu_n \otimes \overline{C}^n, \quad (6.1.24)$$

where $\overline{C}_t^n = (\overline{C}_t^{n,k}, k = 1, \dots, K)$, $\overline{C}_t^{n,k} = -\left(\sqrt{n}/b_n\right)\int_0^t \mathbf{1}(\overline{Q}_s^{n,k} = 0)ds$, $\overline{C}'^n_t = (\overline{C}'^{n,k}_t, k = 1, \dots, K)$, and $\overline{D}'^n_t = (\overline{D}'^{n,k}_t, k = 1, \dots, K)$. Hence, by the definition of the reflection map $\mathcal{R}_P$

$$\overline{Q}^n = \mathcal{R}_P\big(\overline{Q}_0^n + \overline{A}^n + (\overline{R}^n \circ \overline{D}'^n)^T \mathbf{1} - (\mathbf{E}_K - P^T)\,\overline{S}^n \circ \overline{C}'^n$$
$$+ \frac{\sqrt{n}}{b_n}(\lambda_n - (\mathbf{E}_K - P^T)\mu_n)\mathbf{e}\big), \quad (6.1.25)$$

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so that from (6.1.24)

$$\begin{aligned}
&(\mathbf{E}_K - P^T)\,\mu_n \otimes \overline{C}^n \\
&= \overline{Q}_0^n + \overline{A}^n + (\overline{R}^n \circ \overline{D}'^n)^T\,\mathbf{1} - (\mathbf{E}_K - P^T)\,\overline{S}^n \circ \overline{C}'^n \\
&+ \frac{\sqrt{n}}{b_n}(\lambda_n - (\mathbf{E}_K - P^T)\mu_n)\mathbf{e} \\
&- \mathcal{R}_P\big(\overline{Q}_0^n + \overline{A}^n + (\overline{R}^n \circ \overline{D}'^n)^T\,\mathbf{1} - (\mathbf{E}_K - P^T)\,\overline{S}^n \circ \overline{C}'^n \\
&\qquad\qquad + \frac{\sqrt{n}}{b_n}(\lambda_n - (\mathbf{E}_K - P^T)\mu_n)\mathbf{e}\big).
\end{aligned}$$

Since $\mathcal{R}_P$ is a bounded map (in the sense that if $\mathbf{z} = \mathcal{R}_P(\mathbf{x})$, then $\sup_{0\le s\le t}\mathbf{z}_s \le K(t)\sup_{0\le s\le t}|\mathbf{x}_s|$, where $K(t)$ depends only on t), the convergences $\overline{Q}_0^n \overset{P_n^{1/b_n^2}}{\to} q_0$, $(\overline{A}^n, \overline{S}^n, \overline{R}^n) \overset{ld}{\to} (\overline{A}, \overline{S}, \overline{R})$, and $\sqrt{n}/b_n \to \infty$, the near-heavy-traffic condition (6.1.22), and the facts that $\mathbf{E}_K - P^T$ is nonsingular and μ is component-wise positive yield by the argument of the proof of (6.1.15)

$$\int_0^t \mathbf{1}(\overline{Q}_s^{n,k} = 0)ds \overset{P_n^{1/b_n^2}}{\to} 0 \text{ as } n \to \infty,\ k = 1, \dots, K, t \in \mathbb{R}_+,$$

implying that $\overline{C}'^{n,k} \overset{P_n^{1/b_n^2}}{\to} \mathbf{e}$ as $n \to \infty$. Then, since $\overline{D}_t'^{n,k} = S^{n,k} \circ \overline{C}'^{n,k}/n$ and $S^{n,k}/n \overset{P_n^{1/b_n^2}}{\to} \mu\mathbf{e}$, we have that $\overline{D}'^n \overset{P_n^{1/b_n^2}}{\to} \mu\mathbf{e}$, so by Lemma 3.2.11 $(\overline{Q}_0^n, \overline{A}^n, \overline{S}^n \circ \overline{C}'^n, \overline{R}^n \circ \overline{D}'^n) \overset{ld}{\to} (q_0, A, S, R \circ \mu^{-1}\mathbf{e})$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^{K\times K})$. The claim now follows by (6.1.22), (6.1.25), continuity of the reflection and the contraction principle. The idempotent process Q is Luzin-continuous since the idempotent processes A, S and R are Luzin-continuous and $\mathcal{R}_P$ is continuous. ☐

Remark 6.1.9. *One can also prove LD convergences for waiting and sojourn times. Let* $W_t^{n,k}, k = 1, \dots, K,$ *denote the virtual waiting time at station* k *at time* t. *We define* $\overline{W}_t^{n,k} = W_{nt}^{n,k}/(b_n\sqrt{n})$ *and let* $\overline{W}_n = ((\overline{W}_t^{n,k}, k = 1, \dots, K), t \in \mathbb{R}_+)$. *For a vector* $\mathbf{k} = (k_1, \dots, k_l)$, *where* $k_i \in \{1, 2, \dots, K\}$, *let* $A_t^{n,\mathbf{k}}$ *denote the number of customers with the routing* $(k_1, k_2, \dots, k_l)$ *who exogenously arrive by* t, $Y_m^{n,\mathbf{k}}$ *denote the sojourn time of the* m *th exogenous customer with*

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the routing $(k_1, k_2, \ldots, k_l)$, *and* $\overline{Y}_t^{n,\mathbf{k}} = Y_{\lfloor nt \rfloor+1}^{n,\mathbf{k}}/(b_n\sqrt{n})$, $\overline{Y}^{n,\mathbf{k}} = (\overline{Y}_t^{n,\mathbf{k}}, t \in \mathbb{R}_+)$, $\overline{A}'^{n,\mathbf{k}} = (A_{nt}^{n,\mathbf{k}}/n, t \in \mathbb{R}_+)$. *If, in addition to the hypotheses of the theorem,* $\overline{W}^n \overset{P_n^{1/b_n^2}}{\to} w_0$, *where* $q_0 = \mu \otimes w_0$, *then* $(\overline{Q}^n, \overline{W}^n) \overset{ld}{\to} (Q, W)$, *where* $Q = \mu \otimes W$. *If, in addition,* $\overline{A}'_{n,\mathbf{k}} \overset{P_n^{1/b_n^2}}{\to} \lambda_{\mathbf{k}}\mathbf{e}$ *as* $n \to \infty$, *where* $\lambda_{\mathbf{k}} > 0$, *then* $(\overline{W}_n, \overline{Y}_{n,\mathbf{k}}) \overset{ld}{\to} (W, Y)$, *where* $Y \circ (\lambda_{\mathbf{k}}\mathbf{e}) = \sum_{i=1}^{l} W_{k_i}$.

We now give an explicit expression for the idempotent distribution of Q. We define some more notation. For a subset J of $\{1, 2, \ldots, K\}$, we set $F_J = \{\alpha = (\alpha^1, \ldots, \alpha^K) \in \mathbb{R}_+^K : \alpha^k = 0, k \in J, \alpha^k > 0, k \notin J\}$ and $\overline{F}_J = \{\alpha = (\alpha^1, \ldots, \alpha^K) \in \mathbb{R}_+^K : \alpha^k = 0, k \in J\}$; $\mathbf{1}_J$ denotes the K-vector with entries from J equal to 1 and the rest of the entries equal to 0; J^c denotes the complement of J. Let also $\mathbb{R}_+^{K,0}$ denote the interior of $\mathbb{R}_+^K$, and $\mathcal{K}$ the set of all the subsets of $\{1, 2, \ldots, K\}$ except the empty set. We introduce the positive semi-definite symmetric matrix

$$\Gamma = \Sigma_A \Sigma_A^T + (\mathbf{E}_K - P^T)\Sigma_S \Sigma_S^T (\mathbf{E}_K - P) + \sum_{k=1}^{K} \hat{\mu}_k \Sigma_{R,k} \Sigma_{R,k}^T.$$

Lemma 6.1.10. *Let* Γ *be positive definite. Then the idempotent distribution of* Q *has rate function*

$$I_Q(\mathbf{q}) = \frac{1}{2} \int_0^\infty \mathbf{1}(\mathbf{q}_t \in \mathbb{R}_+^{K,0})(\dot{\mathbf{q}}_t - r) \cdot \Gamma^{-1}(\dot{\mathbf{q}}_t - r)\, dt$$

$$+ \sum_{J \in \mathcal{K}} \frac{1}{2} \int_0^\infty \mathbf{1}(\mathbf{q}_t \in F_J) \inf_{y \in \overline{F}_{J^c}} \Big((\dot{\mathbf{q}}_t \otimes \mathbf{1}_{J^c} - r - (\mathbf{E}_K - P^T)y) \cdot \Gamma^{-1}$$

$$(\dot{\mathbf{q}}_t \otimes \mathbf{1}_{J^c} - r - (\mathbf{E}_K - P^T)y)\Big)\, dt$$

when $\mathbf{q} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^k)$ *is absolutely continuous and* $\mathbf{q}_0 = q_0$, *and* $I_Q(\mathbf{q}) = \infty$ *otherwise.*

For a proof we need the following lemma which extends Lemma 6.1.4 and has a similar proof.

Lemma 6.1.11. *Let* $\mathbf{z} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^K)$ *have* $\mathbb{R}_+$*-valued entry functions and* $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^K)$ *be absolutely continuous. Then* $\mathbf{z} = \mathcal{R}_P(\mathbf{x})$ *if*

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and only if $\mathbf{z}$ *is absolutely continuous and there exists an absolutely continuous function* $\mathbf{y} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^K)$ *such that*

$$\dot{\mathbf{z}}_t = \dot{\mathbf{x}}_t + (\mathbf{E}_K - P^T)\dot{\mathbf{y}}_t \quad a.e.$$

and

$$\mathbf{y}_0^k = 0,\ \dot{\mathbf{y}}_t^k \in \mathbb{R}_+ \quad a.e.,\ \mathbf{z}_t^k \dot{\mathbf{y}}_t^k = 0 \quad a.e.,\ k = 1, 2, \ldots, K.$$

Also $\dot{\mathbf{z}}_t^k = 0$ *a.e. on the set* $\{t \in \mathbb{R}_+ : \mathbf{z}_t^k = 0\}$, $k = 1, 2, \ldots, K$.

Proof of Lemma 6.1.10. By the definition of Q and Lemma 6.1.11 $I^Q(\mathbf{q}) = \infty$ unless $\mathbf{q}_0 = q_0$ and $\mathbf{q}$ is absolutely continuous. Since the idempotent Wiener processes W_A, W_S and W_R^k, $k = 1, \ldots, K$, are mutually independent, we have by Corollary 2.4.11 that

$$A + (R \circ \mu \mathbf{e})^T \mathbf{1} - (\mathbf{E}_K - P^T) S \Sigma_A W_A + \sum_{k=1}^{K} \Sigma_R^k W_R^k \circ (\hat{\mu}^k \mathbf{1e}) - (\mathbf{E}_K - P^T)\Sigma_S W_S = \Gamma^{1/2} W,$$

where W is a K-dimensional idempotent Wiener process. Therefore,

$$I^Q(\mathbf{q}) = \inf_{\substack{w \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^K): \\ \mathbf{q} = \mathcal{R}_P(q_0 + \Gamma^{1/2} w + c\mathbf{e})}} \frac{1}{2} \int_0^\infty |\dot{w}_t|^2 \, dt$$
$$= \frac{1}{2} \int_0^\infty \inf_{y \in \mathbb{R}_+^K : \mathbf{q}_t^k y^k = 0} \big(\dot{\mathbf{q}}_t - c - (\mathbf{E}_K - P^T) y\big) \cdot \Gamma^{-1} \big(\dot{\mathbf{q}}_t - c - (\mathbf{E}_K - P^T) y\big) \, dt,$$

which obviously coincides with the expression for I^Q in the statement of the lemma. ☐

Let us now consider the i.i.d. case. Let, for some K', $\hat{\lambda}_k > 0$ when $k = 1, \ldots, K'$, and $\overline{A}_t^{n,k} = 0$ when $k = K' + 1, \ldots, K$. Let the processes $\overline{A}^{n,k}, k = 1, \ldots, K'$, $\overline{S}^{n,k}, k = 1, \ldots, K$, and $\overline{R}^{n,k}, k = 1, \ldots, K$, be mutually independent for each n. Let the processes $\overline{A}^{n,k}, k = 1, \ldots, K'$, and $\overline{S}^{n,k}, k = 1, \ldots, K$, be renewal processes with times between renewals having finite second moments.

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Let $\hat{u}^{n,k}, k = 1, \dots, K'$, denote the generic exogenous interarrival time and $\hat{v}^{n,k}, k = 1, \dots, K$, the generic service time, for station k. Let, in addition, the routing mechanism not depend on n and be i.i.d. at each station with p_{kl} being the probability of going directly from station k to station l.

Lemma 6.1.12. *Let under the above hypotheses, as $n \to \infty$,*

$$E_n \hat{u}^{n,k} \to 1/\hat{\lambda}_k, \ \mathit{Var}\ \hat{u}^{n,k} \to \sigma^2_{u,k}, \ k = 1, \dots, K',$$
$$E_n \hat{v}^{n,k} \to 1/\hat{\mu}_k, \ \mathit{Var}\ \hat{v}^{n,k} \to \sigma^2_{v,k}, \ k = 1, \dots, K,$$

and either one of the following conditions be met:

(i) $\sup_n E_n(\hat{u}^{n,k})^{2+\epsilon} < \infty$, $k = 1, \dots, K'$, *and* $\sup_n E_n(\hat{v}^{n,k})^{2+\epsilon} < \infty$, $k = 1, \dots, K$, *for some* $\epsilon > 0$, *and* $b_n^2/\ln n \to 0$*;*

(ii) $\sup_n E_n \exp(\alpha(\hat{u}^{n,k})^{\beta}) < \infty$, $k = 1, \dots, K'$, *and* $\sup_n E_n \exp(\alpha(\hat{v}^{n,k})^{\beta}) < \infty$, $k = 1, \dots, K$, *for some* $\alpha > 0$ *and* $0 < \beta \le 1$, *and* $b_n^{2-\beta}/n^{\beta/2} \to 0$.

Then the LD convergence $(\overline{A}^n, \overline{S}^n, \overline{R}^n) \overset{ld}{\to} (A, S, R)$ *in the hypotheses of Theorem 6.1.8 holds for*

$$\Sigma_A \Sigma_A^T = \operatorname{diag}(\sigma^2_{A,1}, \dots, \sigma^2_{A,K}),$$
$$\Sigma_S \Sigma_S^T = \operatorname{diag}(\sigma^2_{S,1}, \dots, \sigma^2_{S,K}),$$
$$\left(\Sigma_R^k {\Sigma_R^k}^T\right)_{l,m} = \begin{cases} p_{kl}(1 - p_{kl}), & \text{if } m = l, \\ -p_{kl}p_{km}, & \text{if } m \ne l, \end{cases}$$
$$k, l, m = 1, \dots, K.$$

where

$$\sigma^2_{A,k} = \sigma^2_{u,k}\hat{\lambda}_k^3, \ k = 1, \dots, K', \ \sigma^2_{A,k} = 0, \ k = K' + 1, \dots, K,$$
$$\sigma^2_{S,k} = \sigma^2_{v,k}\hat{\mu}_k^3, \ k = 1, \dots, K.$$

Proof. Since the $\overline{A}^n$, $\overline{S}^n$ and $\overline{R}^{n,k}$, $k = 1, \dots, K$, are mutually independent, by Lemma 3.1.42 it is sufficient to prove the entry-wise convergence, i.e., $\overline{A}^n \overset{ld}{\to} \Sigma_A W_A$, $\overline{S}^n \overset{ld}{\to} \Sigma_S W_S$ and $\overline{R}^n_k \overset{ld}{\to} \Sigma_{R,k} W_{R,k}$, $k = 1, \dots, K$. All of them follow by Theorem 4.4.8. In more detail, for the LD convergence of the $\overline{R}^{n,k}$ we write

$$\overline{R}^{n,k}_t = \sum_{i=1}^{\lfloor nt \rfloor} \frac{\zeta_i^{n,k} - p_k}{b_n \sqrt{n}},$$

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where $p_k = (p_{kl}, l = 1, \dots, K)$ and $\zeta_i^{n,k}$, $i \in \mathbb{N}$, are i.i.d. K-vectors, which have one entry equal to 1 and the rest equal to 0, the probability of the lth entry being equal to 1 being p_{kl}. Clearly, the conditions of Theorem 4.4.8 are met with $b_n\sqrt{n}$ as b_n and

$$(\Sigma)_{l,m} = \begin{cases} p_{kl}(1-p_{kl}) & \text{if } m = l, \\ -p_{kl}p_{km} & \text{if } m \neq l. \end{cases}$$

□

6.2 Very large and moderate deviations for many server queues

In this section we derive results on LD convergence for many server queues. We consider a sequence of many server queues with exponential service times and Poisson arrival processes, which may be non-time-homogeneous. Arriving customers who find no available servers form a queue and are served in the order of arrival. At time t the nth queueing system has K_t^n homogeneous servers in parallel, arrival rate λ_t^n and service rate μ_t^n. We assume that the following expansions hold

$$\lambda_t^n = n\lambda_{0,t} + \sqrt{n}b_n\lambda_{1,t} + O(\sqrt{n}), \tag{6.2.1a}$$

$$\mu_t^n = \mu_{0,t} + \frac{b_n}{\sqrt{n}}\mu_{1,t} + O(1/\sqrt{n}), \tag{6.2.1b}$$

$$K_t^n = n\alpha_{0,t} + \sqrt{n}b_n\alpha_{1,t} + O(\sqrt{n}), \tag{6.2.1c}$$

where $b_n \to \infty$, $b_n/\sqrt{n} \to 0$, the functions $\lambda_{0,t}$, $\lambda_{1,t}$, $\mu_{0,t}$, $\mu_{1,t}$, $\alpha_{0,t}$, and $\alpha_{1,t}$ are Lebesgue measurable, the functions $\lambda_{0,t}$, $\lambda_{1,t}$, $\mu_{0,t}$, $\mu_{1,t}$, and $\alpha_{1,t}$ are bounded on bounded intervals, and the O's are uniform in t over bounded intervals. We do not rule out the case $\alpha_{0,t} = \infty$, which corresponds to an infinite server queue.

Let $A^n = (A_t^n, t \in \mathbb{R}_+)$ and $B^{n,k} = (B_t^{n,k}, t \in \mathbb{R}_+)$, $k \in \mathbb{N}$, be independent Poisson processes of respective rates λ_t^n and μ_t^n at time t. We assume that the objects associated with the nth system are defined on a complete probability space $(\Omega_n, \mathcal{F}_n, P_n)$. All the processes are considered as random elements of $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. Denoting by Q_t^n the number of customers in the nth system at time t, we have

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that distributionally the process $Q^n = (Q^n_t,\, t \in \mathbb{R}_+)$ satisfies the equation

$$Q^n_t = Q^n_0 + A^n_t - \sum_{k=1}^{K^n_t} \int_0^t \mathbf{1}(Q^n_{s-} \geq k)\, dB^{n,k}_s. \tag{6.2.2}$$

Let

$$M^n_t = A^n_t - \int_0^t \lambda^n_s\, ds + \sum_{k=1}^{K^n_t} \int_0^t \mathbf{1}(Q^n_{s-} \geq k)\, \big(dB^{n,k}_s - \mu^{n,k}_s ds\big). \tag{6.2.3}$$

Then $M^n = (M^n_t,\, t \in \mathbb{R}_+)$ is a local martingale with respect to the filtration $\mathbb{F}^n = (\mathcal{F}^n_t,\, t \in \mathbb{R}_+)$, where $\mathcal{F}^n_t = \cap_{\epsilon>0}\mathcal{G}^n_{t+\epsilon}$ and $\mathcal{G}^n_t$ is the sub-σ-algebra of $\mathcal{F}^n$ generated by Q^n_0, A^n_s, $B^{n,k}_s$, $s \in [0,t], k = 1, \ldots, K$, and sets of P_n-measure zero. The predictable quadratic-variation process of M^n has the form

$$\langle M^n \rangle_t = \int_0^t \lambda^n_s\, ds + \int_0^t Q^n_s \wedge K^n_s\, \mu^n_s\, ds. \tag{6.2.4}$$

We write equation (6.2.2) in the form

$$Q^n_t = Q^n_0 + \int_0^t \lambda^n_s\, ds - \int_0^t Q^n_s \wedge K^n_s\, \mu^n_s\, ds + M^n_t. \tag{6.2.5}$$

The following auxiliary result is standard. We denote by $\xrightarrow{P_n}$ convergence in probability.

Lemma 6.2.1. *Let $Q^n_0/n \xrightarrow{P_n} q_0 \in \mathbb{R}_+$ as $n \to \infty$. Then, for $T \in \mathbb{R}_+$,*

$$\sup_{t\in[0,T]} \Big|\frac{Q^n_t}{n} - q_t\Big| \xrightarrow{P_n} 0,$$

where $q = (q_t,\, t \in \mathbb{R}_+)$ is the solution to the differential equation

$$\dot{q}_t = \lambda_{0,t} - \mu_{0,t}(q_t \wedge \alpha_{0,t}). \tag{6.2.6}$$

Remark 6.2.2. *Equation* (6.2.6) *has a unique solution by Caratheodory's theorem, see, e.g., Coddington and Levinson [26].*

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Proof of Lemma 6.2.1. It is easy to check that $A^n_t/n^2 \xrightarrow{P_n} 0$. By (6.2.4) $\langle M^n\rangle_t \le \int_0^t \lambda^n_s\,ds + \int_0^t Q^n_s\mu^n_s\,ds \le \int_0^t \lambda^n_s\,ds + Q^n_0 t + \int_0^t A^n_s\mu^n_s\,ds$ so by (6.2.1a), (6.2.1b) and (6.2.1c) $\langle M^n\rangle_t/n^2 \xrightarrow{P_n} 0$ as $n \to \infty$, which implies by the Lenglart-Rebolledo inequality that $\sup_{t\in[0,T]}|M^n_t|/n \xrightarrow{P_n} 0$ as $n\to\infty$. Applying a standard tightness argument to (6.2.5) and using the fact the functions $\lambda_{0,t}$, $\lambda_{1,t}$, $\mu_{0,t}$, and $\mu_{1,t}$ are bounded on bounded intervals, we conclude that the sequence of laws of the processes Q^n/n is relatively compact in distribution, all the limit points satisfying equation (6.2.6) with probability 1. The solution of the latter equation being unique completes the proof. □

The next result gives an idempotent diffusion approximation for Q^n. Let

$$X^n_t = \frac{\sqrt{n}}{b_n}\Big(\frac{Q^n_t}{n} - q_t\Big) \tag{6.2.7}$$

and $X^n = (X^n_t,\, t \in \mathbb{R}_+)$.

Theorem 6.2.3. *Let $X^n_0 \xrightarrow{P_n^{1/b_n^2}} x_0 \in \mathbb{R}$ as $n\to\infty$. If, in addition, $\inf_{s\in[0,t]}\big(\lambda_{0,s}+\mu_{0,s}(q_s\wedge\alpha_{0,s})\big) > 0$, $t\in\mathbb{R}_+$, then $X^n \xrightarrow[b_n^2]{ld} X$ as $n\to\infty$ for the Skorohod topology, where $X = (X_t,\, t\in\mathbb{R}_+)$ is the idempotent diffusion specified by the equation*

$$\begin{aligned}\dot X_t &= \lambda_{1,t} - \mu_{1,t}(q_t\wedge\alpha_{0,t})\\ &-\mu_{0,t}\big(\mathbf{1}(q_t<\alpha_{1,t})X_t + \mathbf{1}(q_t=\alpha_{0,t})(X_t\wedge\alpha_{1,t}) + \mathbf{1}(q_t>\alpha_{0,t})\alpha_{1,t}\big)\\ &\qquad + \sqrt{\lambda_{0,t}+\mu_{0,t}(q_t\wedge\alpha_{0,t})}\,\dot W_t, \quad X_0 = x_0,\end{aligned}$$

with $W = (W_t,\, t\in\mathbb{R}_+)$ being an idempotent Wiener process.

Proof. By (6.2.5), (6.2.6) and (6.2.7) we can write

$$\begin{aligned}X^n_t = X^n_0 &+ \int_0^t \frac{\sqrt{n}}{b_n}\Big(\frac{\lambda^n_s}{n} - \lambda_{0,s}\Big)\,ds\\ &- \int_0^t \Big(\Big(q_s + X^n_s\frac{b_n}{\sqrt{n}}\Big)\wedge\frac{K^n_s}{n}\Big)\frac{\sqrt{n}}{b_n}(\mu^n_s - \mu_{0,s})\,ds\end{aligned}$$

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$$- \frac{\sqrt{n}}{b_n} \int_0^t \Big(\big(q_s + X_s^n \frac{b_n}{\sqrt{n}}\big) \wedge \frac{K_s^n}{n} - q_s \wedge \alpha_{0,s} \Big) \mu_{0,s}\, ds + \frac{1}{\sqrt{n} b_n} M_t^n .$$

Therefore, the first characteristic of X^n without truncation is

$$B_t'^n = \int_0^t b_s^n(X_s^n)\, ds,$$

where

$$b_s^n(u) = \frac{\sqrt{n}}{b_n}\big(\frac{\lambda_s^n}{n} - \lambda_{0,s}\big) - \Big(\big(q_s + u\frac{b_n}{\sqrt{n}}\big) \wedge \frac{K_s^n}{n}\Big) \frac{\sqrt{n}}{b_n}(\mu_s^n - \mu_{0,s})$$
$$- \frac{\sqrt{n}}{b_n}\Big(\big(q_s + u\frac{b_n}{\sqrt{n}}\big) \wedge \frac{K_s^n}{n} - q_s \wedge \alpha_{0,s}\Big)\mu_{0,s}.$$

The modified second characteristic without truncation $\tilde{C}'^n = (\tilde{C}_t'^n,\, t \in \mathbb{R}_+)$ coincides with the predictable quadratic-variation process of $(M_t^n/(\sqrt{n} b_n),\, t \in \mathbb{R}_+)$ and by (6.2.4) has the form

$$\tilde{C}_t'^n = \frac{1}{b_n^2} \int_0^t \tilde{c}_s'^n(X_s^n)\, ds,$$

where

$$\tilde{c}_s'^n(u) = \frac{\lambda_s^n}{n} + \mu_s^n \Big(\big(q_s + \frac{b_n}{\sqrt{n}} u\big) \wedge \frac{K_s^n}{n} \Big).$$

Easy calculations show that for $t \in \mathbb{R}_+$ and $v \in \mathbb{R}_+$

$$\lim_{n\to\infty} \int_0^t \operatorname*{ess\,sup}_{|u|\le v} |b_s^n(u) - b_s(u)|\, ds = 0,$$
$$\lim_{n\to\infty} \int_0^t \operatorname*{ess\,sup}_{|u|\le v} |\tilde{c}_s'^n(u) - c_s|\, ds = 0,$$

where

$$b_s(u) = \lambda_{1,s} - \mu_{1,s}(q_s \wedge \alpha_{0,s}) - \mu_{0,s}\big(\mathbf{1}(q_s < \alpha_{0,s})u + \mathbf{1}(q_s = \alpha_{0,s})(u \wedge \alpha_{1,s}) + \mathbf{1}(q_s > \alpha_{0,s})\alpha_{1,s}\big), \tag{6.2.8}$$

$$c_s = \lambda_{0,s} + \mu_{0,s}(q_s \wedge \alpha_{0,s}). \tag{6.2.9}$$

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Thus, the convergence conditions of Theorem 5.4.4 are satisfied. The moment conditions are satisfied since the jumps of X^n are bounded above by $1/(b_n\sqrt{n})$. Also the law of the semimaxingale with local characteristics $(b, c, 0, 0)$ is uniquely specified by Theorem 2.8.21. Therefore, by Theorem 5.4.4 $X^n \overset{ld}{\to} X$ at rate b_n^2 as $n \to \infty$. □

Remark 6.2.4. *We note that the idempotent distribution of* X *has density*

$$\Pi^X(\mathbf{x}) = \exp\Bigl(-\frac{1}{2}\int_0^\infty \frac{\bigl(\dot{\mathbf{x}}_t - b_t(\mathbf{x}_t)\bigr)^2}{c_t}\,dt\Bigr)$$

if $\mathbf{x}$ *is absolutely continuous and* $\mathbf{x}_0 = x_0$, *and* $\Pi^X(\mathbf{x}) = 0$ *otherwise, where* b_t *and* c_t *are defined by the respective equalities* (6.2.8) *and* (6.2.9).

We consider now very large deviations. Let us assume, in addition, that $\inf_{s\in[0,t]} \alpha_{0,s} > 0$, $t \in \mathbb{R}_+$. Let a process $Y^n = (Y_t^n,\, t \in \mathbb{R}_+)$ be defined by $Y_t^n = Q_t^n/n$. Let $\mathcal{N}_1 = (\mathcal{N}_1(t),\, t \in \mathbb{R}_+)$ and $\mathcal{N}_2 = (\mathcal{N}_2(t),\, t \in \mathbb{R}_+)$ be independent Poisson idempotent processes on an idempotent probability space (Ω, Π). Let $Y = (Y_t,\, t \in \mathbb{R}_+)$ be a Luzin solution of the equation

$$Y_t = y_0 + \mathcal{N}_1\Bigl(\int_0^t \lambda_{0,s}\,ds\Bigr) - \mathcal{N}_2\Bigl(\int_0^t (Y_s \wedge \alpha_{0,s})\mu_{0,s}\,ds\Bigr),\ y_0 \in \mathbb{R}_+,$$

such that the idempotent distribution of Y has density

$$\Pi^Y(\mathbf{x}) = \exp\Bigl(-\int_0^\infty \sup_{\lambda\in\mathbb{R}}\bigl(\lambda\dot{\mathbf{x}}_t - (e^\lambda - 1)\lambda_{0,t} - (e^{-\lambda} - 1)(\mathbf{x}_t \wedge \alpha_{0,t})\mu_{0,t}\bigr)\,dt\Bigr)$$

if $\mathbf{x}_0 = y_0$ and $\mathbf{x}$ is absolutely continuous, and $\Pi^Y(\mathbf{x}) = 0$ otherwise. It is well defined by Theorems 2.6.33, 2.8.10 and 2.8.29.

Theorem 6.2.5. *If* $Y_0^n \overset{P_n^{1/n}}{\to} y_0$, *then* $Y^n \xrightarrow[n]{ld} Y$ *as* $n \to \infty$ *for the Skorohod topology.*

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Proof. By (6.2.5)

$$Y_t^n = Y_0^n + \int_0^t \frac{\lambda_s^n}{n}\, ds - \int_0^t \Bigl(Y_s^n \wedge \frac{K_s^n}{n}\Bigr)\mu_s^n\, ds + \frac{1}{n} M_t^n .$$

Therefore, Y^n is a squarely integrable semimartingale. Its first characteristic without truncation is given by

$$B_t'^n = \int_0^t \Bigl(\frac{\lambda_s^n}{n} - \Bigl(Y_s^n \wedge \frac{K_s^n}{n}\Bigr)\mu_s^n\Bigr)\, ds,$$

the predictable measure of jumps in view of (6.2.2) is given by

$$\nu^n([0,t],\Gamma) = n \int_0^t \Bigl(\frac{\lambda_s^n}{n}\, \mathbf{1}\Bigl(\frac{1}{n} \in \Gamma\Bigr) + \mu_s^n \Bigl(Y_s^n \wedge \frac{K_s^n}{n}\Bigr)\, \mathbf{1}\Bigl(-\frac{1}{n} \in \Gamma\Bigr)\Bigr)\, ds,$$

and the modified second characteristic without truncation in view of (6.2.4) is given by

$$\tilde{C}_t'^n = \frac{1}{n} \int_0^t \Bigl(\frac{\lambda_s^n}{n} + \Bigl(Y_s^n \wedge \frac{K_s^n}{n}\Bigr)\mu_s^n\Bigr)\, ds.$$

Then the convergence conditions of Theorem 5.4.3 hold with $h(x) = x$ for

$$b_s(u) = \lambda_{0,s} - (u \wedge \alpha_{0,s})\mu_{0,s},\ \tilde{c}_s(u) = \lambda_{0,s} + (u \wedge \alpha_{0,s})\mu_{0,s},$$
$$\nu_s(\Gamma) = \lambda_{0,s}\, \mathbf{1}(1 \in \Gamma) + (u \wedge \alpha_{0,s})\mu_{0,s}\, \mathbf{1}(-1 \in \Gamma).$$

The Cramér condition holds since the jumps of Y^n are not greater than $1/n$. Thus, by Theorem 5.4.3 the sequence of laws of the Y^n is $\mathbb{C}(\mathbb{R}_+,\mathbb{R})$-exponentially tight of order n and every LD accumulation point for LD convergence of rate n is the law of a semimaxingale with local characteristics $(b,0,\nu,0)$. By Theorem 2.8.29 the latter law is unique, hence, it is the LD limit of the laws of the Y^n. □

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Appendix A

Auxiliary lemmas

This Appendix contains lemmas we referred to in the main body of the book.

We first prove the fact from convex analysis used in the proof of Lemma 1.11.5. For it we adopt the usual definitions and notation from convex analysis, Rockafellar [117]. For a subset A of a Euclidean space, $\mathrm{cl}\, A$ denotes its closure, $\mathrm{ri}\, A$ the relative interior, $\mathrm{rb}\, A = \mathrm{cl}\, A \backslash \mathrm{ri}\, A$ the relative boundary, and $\mathrm{conv}\, A$ the convex hull of A. Let f be a function from $\mathbb{R}^d$, $d \in \mathbb{N}$, into $]-\infty, \infty]$. Its conjugate (or the Legendre–Fenchel transform) f^* is defined by

$$f^*(\lambda) = \sup_{x \in \mathbb{R}^d} \big(\lambda \cdot x - f(x)\big), \ \lambda \in \mathbb{R}^d,$$

and the bipolar f^{**} of f is defined as the conjugate of f^*:

$$f^{**}(x) = \sup_{\lambda \in \mathbb{R}^d} \big(\lambda \cdot x - f^*(\lambda)\big), \ x \in \mathbb{R}^d.$$

Obviously, f^{**} is convex and lower semi-continuous.

We denote by $\mathrm{epi}\, f$ the epigraph of f: $\mathrm{epi}\, f = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \geq f(x)\}$, and let $\mathrm{dom}\, f = \{x \in \mathbb{R}^d : f(x) < \infty\}$ denote the effective domain of f. The convex hull $\mathrm{conv}\, f$ of f is defined by $\mathrm{epi}(\mathrm{conv}\, f) = \mathrm{conv}(\mathrm{epi}\, f)$ and the lower semi-continuous hull $\mathrm{cl}\, f$ by $\mathrm{epi}(\mathrm{cl}\, f) = \mathrm{cl}(\mathrm{epi}\, f)$. If f is convex, then $\partial f(x)$ denotes the subdifferential of f at x. We say that f is essentially strictly convex if it is strictly convex on every convex subset of the set of those x for which the

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set $\partial f(x)$ is nonempty. If f is essentially strictly convex, then it is strictly convex on $\mathrm{ri}(\mathrm{dom}\, f^{**})$, Rockafellar [117].

Lemma A.1. *If $f : \mathbb{R}^d \to]-\infty, \infty]$ is a lower semi-continuous function and its bipolar f^{**} is strictly convex on $\mathrm{ri}(\mathrm{dom}\, f^{**})$, then $f = f^{**}$.*

Proof. It is obvious that $f^{**} \leq f$. So we prove the opposite inequality. By Rockafellar [117, Corollary 12.1.1 and the argument below] we have

$$f^{**} = \mathrm{cl}(\mathrm{conv}\; f). \tag{A.1}$$

We first prove that

$$f^{**}(x) \geq f(x), \; x \in \mathrm{ri}(\mathrm{dom}\, f^{**}). \tag{A.2}$$

Assume the contrary, i.e., that for some $x_0 \in \mathrm{ri}(\mathrm{dom}\, f^{**})$ and $\gamma > 0$ we have

$$f(x_0) > f^{**}(x_0) + \gamma. \tag{A.3}$$

Since $x_0 \in \mathrm{ri}(\mathrm{dom}\, f^{**})$, by Rockafellar [117, Theorem 23.4] the set $\partial f^{**}(x_0)$ is nonempty. Let $\lambda_0 \in \partial f^{**}(x_0)$. Then by the definition of f^{**}

$$f^{**}(x) \geq \lambda_0 \cdot x - f^*(\lambda_0) \tag{A.4}$$

and by Rockafellar [117, Theorem 23.5]

$$f^{**}(x_0) = \lambda_0 \cdot x_0 - f^*(\lambda_0). \tag{A.5}$$

It is easy to see that strict convexity of f^{**} implies that

$$f^{**}(x) > \lambda_0 \cdot x - f^*(\lambda_0), \, x \neq x_0. \tag{A.6}$$

Indeed, if for some $x \neq x_0$ we had equality in (A.4), then by convexity of f^{**} and (A.5) $f^{**}(z) \leq \lambda_0 \cdot z - f^*(\lambda_0)$, for every $z \in [x_0, x[$, which together with (A.4) would yield $f^{**}(z) = \lambda_0 \cdot z - f^*(\lambda_0)$, $z \in [x_0, x[$. On the other hand, $[x_0, x[\subset \mathrm{ri}(\mathrm{dom}\, f^{**})$ (by Rockafellar [117, Theorem 6.1] and since $x \in \mathrm{dom}\, f^{**}$ if there is equality in (A.4)). Thus, f^{**} would fail to be strictly convex on $\mathrm{ri}(\mathrm{dom}\, f^{**})$, and (A.6) is proved.

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By lower semi-continuity of f we can choose $\varepsilon > 0$ such that $\varepsilon|\lambda_0| < \gamma/3$ and

$$f(x) > f(x_0) - \frac{\gamma}{3},\ |x - x_0| < \varepsilon. \tag{A.7}$$

For this ε, we choose $\delta > 0, \delta < \gamma/3$, satisfying the inclusion

$$\{x : \lambda_0 \cdot x - f^*(\lambda_0) + \delta \geq f^{**}(x)\} \subset \{x : |x - x_0| < \varepsilon\}. \tag{A.8}$$

In order to show that such a δ exists, let us denote by A_δ the set on the left of (A.8). Then by (A.5) and (A.6)

$$\bigcap_{\delta > 0} A_\delta = \{x_0\}. \tag{A.9}$$

The hyperplane in $\mathbb{R}^d \times \mathbb{R}$ defined by the equation $y = \lambda_0 \cdot x - f^*(\lambda_0) + \delta,\ x \in \mathbb{R}^d, y \in \mathbb{R}$, is parallel to the hyperplane $y = \lambda_0 \cdot x - f^*(\lambda_0)$. The latter in view of (A.5) and (A.6) has with epi f^{**} the only point x_0 in common. Then by Rockafellar [117, Corollary 8.4.1] the sets A_δ are bounded. They are closed since f^{**} is lower semi-continuous. Thus, the A_δ are compacts and (A.9) easily implies that for all $\delta > 0$ small enough $A_\delta \subset \{x : |x - x_0| < \varepsilon\}$ proving (A.8).

For the chosen δ and ε, we define

$$f_{\delta,\varepsilon}(x) = \max\bigl(f^{**}(x), \lambda_0 \cdot x - f^*(\lambda_0) + \delta\bigr). \tag{A.10}$$

Obviously, $f_{\delta,\varepsilon}$ is convex, lower semi-continuous and $f_{\delta,\varepsilon}(x_0) > f^{**}(x_0)$ by (A.5). If we show that

$$f_{\delta,\varepsilon}(x) \leq f(x),\ x \in \mathbb{R}^d, \tag{A.11}$$

this would contradict (A.1), and (A.2) would be proved. It is clear that (A.11) holds on the set $\{x : |x - x_0| \geq \varepsilon\}$ since $f_{\delta,\varepsilon}(x) = f^{**}(x)$ for these x by (A.8) and (A.10). If $|x - x_0| < \varepsilon$, then using (A.5), (A.3) and (A.7) we have

$$\begin{aligned} \lambda_0 \cdot x - f^*(\lambda_0) + \delta &= \lambda_0 \cdot (x - x_0) + f^{**}(x_0) + \delta \\ &< \varepsilon|\lambda_0| + f(x_0) - \gamma + \delta \leq \varepsilon|\lambda_0| + f(x) - \frac{2\gamma}{3} + \delta < f(x) \end{aligned}$$

(the latter inequality holds by the choice of ε and δ). Since, as we noted, $f^{**} \leq f$, this proves (A.11) on $\{x : |x - x_0| < \varepsilon\}$. Thus (A.2) is proved.

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Now if $x \in \mathrm{rb}(\mathrm{dom}\, f^{**})$ we have by Rockafellar [117, Theorem 7.5] in view of lower semi-continuity of f^{**} that for arbitrary $z \in \mathrm{ri}(\mathrm{dom}\, f^{**})$

$$f^{**}(x) = \lim_{\theta \uparrow 1} f^{**}\big((1-\theta)z + \theta x\big). \tag{A.12}$$

By Rockafellar [117, Theorem 6.1] $[z, x[\subset \mathrm{ri}(\mathrm{dom}\, f^{**})$, and then by the part just proved

$$f^{**}\big((1-\theta)z+\theta x\big) = f\big((1-\theta)z+\theta x\big),\ 0 \le \theta < 1,$$

so that by lower semi-continuity of f and (A.12) we have that $f^{**}(x) \ge f(x)$ proving the assertion of the lemma for $x \in \mathrm{cl}(\mathrm{dom}\, f^{**})$.

Finally, for $x \notin \mathrm{cl}(\mathrm{dom}\, f^{**})$ we obviously have $f(x) = f^{**}(x) = \infty$. □

For the next lemma we recall that Λ_0 denotes the set of all $\mathbb{R}^d$–valued piecewise constant functions $(\lambda(t), t \in \mathbb{R}_+)$ of the form

$$\lambda(t) = \sum_{i=1}^{k} \lambda_i \, \mathbf{1}(t \in (t_{i-1}, t_i]),$$

where $0 \le t_0 < t_1 < \ldots < t_k,\ \lambda_i \in \mathbb{R}^d,\ i = 1, \ldots, k,\ k \in \mathbb{N}$.

Lemma A.2. *Let $f(t,\lambda), t \in \mathbb{R}_+, \lambda \in \mathbb{R}^d$, be an $\mathbb{R}$-valued function, which is Lebesgue measurable in t, continuous in λ, and is such that $f(t,0) = 0$ and $\int_0^T f(t,\lambda)dt$ is well defined for $T \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}^d$.*

Then for $T \in \mathbb{R}_+$

$$\int_0^T \sup_{\lambda \in \mathbb{R}^d} f(t,\lambda) \; dt = \sup_{(\lambda(t)) \in \Lambda_0} \int_0^T f(t, \lambda(t)) \; dt. \tag{A.13}$$

Proof. We denote $F(t) = \sup_{\lambda \in \mathbb{R}^d} f(t,\lambda)$. Since the supremum may be taken over the rational λ in view of continuity of $f(t,\lambda)$ in λ, the function $F(t)$ is Lebesgue measurable and non-negative, so that the integral on the left-hand side of (A.13) is well defined.

Given arbitrary $\varepsilon > 0$, we introduce the set

$$A_\varepsilon = \{(t,\lambda) \in [0,T] \times \mathbb{R}^d : \frac{1}{\epsilon} \ge f(t,\lambda) \ge (F(t) - \varepsilon)^+ \wedge \frac{1}{\varepsilon}\}.$$

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By a measurable selection theorem, see, e.g., Clarke [25], Ethier and Kurtz [48], there exists an $\mathbb{R}^d$–valued Lebesgue measurable function $\tilde{\lambda}_\varepsilon(t)$ such that

$$\frac{1}{\epsilon} \geq f(t, \tilde{\lambda}_\varepsilon(t)) \geq (F(t) - \varepsilon)^+ \wedge \frac{1}{\varepsilon},\ t \in [0, T].$$

By Luzin's theorem there exists a continuous function $\lambda_\varepsilon(t)$ such that $\int_0^T \mathbf{1}\big(\tilde{\lambda}_\varepsilon(t) \neq \lambda_\varepsilon(t)\big)\, dt < \epsilon^2$. Then

$$\int_0^T f(t, \lambda_\varepsilon(t)) \vee 0\, dt \geq \int_0^T f(t, \tilde{\lambda}_\varepsilon(t))\, dt - \epsilon \geq \int_0^T (F(t) - \varepsilon)^+ \wedge \frac{1}{\varepsilon}\, dt - \epsilon.$$

Since $(\lambda_\epsilon(t))$ is continuous, it can be approximated by functions from Λ_0. Since $f(t, \lambda)$ is continuous in λ and $f(t, 0) = 0$, by Fatou's lemma there exists a function $\lambda_0 \in \Lambda_0$ such that

$$\int_0^T f(t, \lambda_0(t))\, dt \geq \int_0^T f(t, \lambda_\varepsilon(t)) \vee 0\, dt - \epsilon.$$

Thus, since $\epsilon > 0$ is arbitrary,

$$\int_0^T \sup_{\lambda \in \mathbb{R}^d} f(t, \lambda)\ \ dt \leq \sup_{(\lambda(t)) \in \Lambda_0} \int_0^T f(t, \lambda(t))\ dt.$$

The reverse inequality is obvious. ☐

We state and prove the lemma used in the proof of Theorem 6.1.7.

Lemma A.3. *Let $\{\xi_i^n, i \in \mathbb{N}\}, n \in \mathbb{N}$, be a triangular array of row-wise i.i.d. real-valued r.v. with zero mean on respective probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$. Let $b_n \to \infty$ as $n \to \infty$, and $\alpha > 0$.*

(i) If $b_n/\sqrt{n} \to 0$ as $n \to \infty$ and, for some $\varepsilon > 0$, we have $\sup_n E_n |\xi_1^n|^{2+\varepsilon} < \infty$, then there exist n_0, $t_0 > 0$, $C_1 > 0$, and $C_2 > 0$ such that, for all $t \geq t_0$ and $n \geq n_0$,

$$P_n\left(\max_{k=1,\ldots,\lfloor nt \rfloor} \frac{1}{b_n \sqrt{n}} \sum_{i=1}^k \xi_i^n > \alpha t \right) \leq \exp(-C_1 b_n^2 \sqrt{t}) + C_2 \frac{b_n^{2+\varepsilon}}{n^{\varepsilon/2}} \frac{1}{t^{\varepsilon/2}}. \quad \text{(A.14)}$$

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(ii) If, for some $\gamma > 0$ *and* $\beta \in (0,1]$, *we have* $\sup_n E_n \exp(\gamma |\xi_1^n|^\beta) < \infty$ *and* $b_n^{2-\beta}/n^{\beta/2} \to 0$ *as* $n \to \infty$, *then there exist* n_0', $t_0' > 0$, $C_1' > 0$ *and* $C_2' > 0$ *such that, for all* $t \geq t_0'$ *and* $n \geq n_0'$,

$$P_n\left(\max_{k=1,\dots,\lfloor nt\rfloor} \frac{1}{b_n\sqrt{n}} \sum_{i=1}^k \xi_i^n > \alpha t\right) \leq \exp(-C_1' b_n^2 \sqrt{t}) + \exp(-C_2'(b_n\sqrt{nt})^\beta). \quad \text{(A.15)}$$

Proof. The argument uses the ideas of the proof of Theorem 4.4.8. Let the conditions of (i) hold. We first prove that there exist $C_1 > 0$ and t_0 such that for $t \geq t_0$

$$P_n\left(\max_{k=1,\dots,\lfloor nt\rfloor} \frac{1}{b_n\sqrt{n}} \sum_{i=1}^k \xi_i^n \mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_i^n| \leq \sqrt{t}\right) > \alpha t\right) \leq \exp(-C_1 b_n^2 \sqrt{t}). \quad \text{(A.16)}$$

We denote $B = \sup_n E_n|\xi_1^n|^{2+\varepsilon} + 1$ and

$$\hat{\xi}_i^n = \frac{b_n}{\sqrt{n}}\left(\xi_i^n \mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_i^n| \leq \sqrt{t}\right) - E_n \xi_i^n \mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_i^n| \leq \sqrt{t}\right)\right). \quad \text{(A.17)}$$

By Doob's inequality (see, e.g., Liptser and Shiryaev [79, Theorem 1.9.1]), for $\lambda > 0$,

$$P_n\left(\max_{k=1,\dots,\lfloor nt\rfloor} \frac{1}{b_n^2} \sum_{i=1}^k \hat{\xi}_i^n > \frac{\alpha t}{2}\right) \leq \frac{\left(E_n e^{2\lambda \hat{\xi}_1^n}\right)^{\lfloor nt\rfloor}}{e^{\lambda b_n^2 \alpha t}}. \quad \text{(A.18)}$$

Since $E_n\hat{\xi}_1^n = 0$, $|\hat{\xi}_1^n| \leq 2\sqrt{t}$ and $E_n(\hat{\xi}_1^n)^2 \leq E_n(\xi_1^n)^2 b_n^2/n$, it follows that

$$E_n e^{2\lambda\hat{\xi}_1^n} \leq 1 + 2\lambda^2 e^{4\lambda\sqrt{t}} E_n(\hat{\xi}_1^n)^2 \leq 1 + 2\lambda^2 e^{4\lambda\sqrt{t}} \frac{b_n^2}{n} B,$$

so

$$\left(E_n e^{2\lambda\hat{\xi}_1^n}\right)^{\lfloor nt\rfloor} \leq \exp(2\lambda^2 e^{4\lambda\sqrt{t}} B t b_n^2).$$

Choosing in (A.18) $\lambda = 1/\sqrt{t}$, we obtain for $t \geq t_0 = (4e^4 B/\alpha)^2$ and $C_1 = \alpha/2$

$$P_n\left(\max_{k=1,\dots,\lfloor nt\rfloor} \frac{1}{b_n^2} \sum_{i=1}^k \hat{\xi}_i^n > \frac{\alpha t}{2}\right) \leq \exp(-C_1 b_n^2 \sqrt{t}). \quad \text{(A.19)}$$

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Now note that, since $E_n \xi_1^n = 0$,

$$\left| E_n \xi_1^n \, \mathbf{1}\left(\frac{b_n}{\sqrt{n}} |\xi_1^n| \le \sqrt{t} \right) \right| = \left| E_n \xi_1^n \, \mathbf{1}\left(\frac{b_n}{\sqrt{n}} |\xi_1^n| > \sqrt{t} \right) \right| \le \frac{b_n^{1+\varepsilon}}{n^{(1+\varepsilon)/2}} \frac{B}{t^{(1+\varepsilon)/2}},$$

hence,

$$\frac{\lfloor nt \rfloor}{b_n \sqrt{n}} \left| E_n \xi_1^n \, \mathbf{1}\left(\frac{b_n}{\sqrt{n}} |\xi_1^n| \le a \right) \right| \le \frac{b_n^{\varepsilon}}{n^{\varepsilon/2}} B t^{(1-\varepsilon)/2},$$

so, by the fact that $b_n/\sqrt{n} \to 0$ as $n \to \infty$ and (A.17), for all n large enough and $t \ge t_0$

$$P_n \left(\max_{k=1,\dots,\lfloor nt \rfloor} \frac{1}{b_n \sqrt{n}} \sum_{i=1}^{k} \xi_i^n \, \mathbf{1}\left(\frac{b_n}{\sqrt{n}} |\xi_i^n| \le \sqrt{t} \right) > \alpha t \right) \le P_n \big(\max_{k=1,\dots,\lfloor nt \rfloor} \frac{1}{b_n^2} \sum_{i=1}^{k} \hat{\xi}_i^n > \frac{\alpha t}{2} \big),$$

which together with (A.19) proves (A.16).

The estimate (A.14) now follows by (A.16) and the inequalities

$$P_n \left(\max_{k=1,\dots,\lfloor nt \rfloor} \frac{1}{b_n \sqrt{n}} \sum_{i=1}^{k} \xi_i^n > \alpha t \right) \le P_n \left(\max_{k=1,\dots,\lfloor nt \rfloor} \frac{1}{b_n \sqrt{n}} \sum_{i=1}^{k} \xi_i^n \, \mathbf{1}\left(\frac{b_n}{\sqrt{n}} |\xi_i^n| \le \sqrt{t} \right) > \alpha t \right) + P_n \left(\max_{k=1,\dots,\lfloor nt \rfloor} \frac{b_n}{\sqrt{n}} |\xi_i^n| > \sqrt{t} \right) \quad \text{(A.20)}$$

and

$$P_n \left(\max_{k=1,\dots,\lfloor nt \rfloor} \frac{b_n}{\sqrt{n}} |\xi_i^n| > \sqrt{t} \right) \le \lfloor nt \rfloor P_n \left(\frac{b_n}{\sqrt{n}} |\xi_1^n| > \sqrt{t} \right) \le \lfloor nt \rfloor \frac{b_n^{2+\varepsilon}}{n^{1+\varepsilon/2}} \frac{B}{t^{1+\varepsilon/2}}.$$

Part (i) is proved.

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For part (ii), we write

$$P_n\left(\max_{k=1,\ldots,\lfloor nt\rfloor}\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{k}\xi_i^n > \alpha t\right)$$
$$\le P_n\left(\max_{k=1,\ldots,\lfloor nt\rfloor}\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{k}\xi_i^n\,\mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_i^n|\le\sqrt{t}\right) > \frac{\alpha t}{2}\right)$$
$$+ P_n\left(\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}|\xi_i^n|\,\mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_i^n|>\sqrt{t}\right) > \frac{\alpha t}{2}\right). \quad \text{(A.21)}$$

Noting that the conditions of part (ii) imply the conditions of part (i), we estimate the first term on the right of (A.21) with the help of (A.16). For the second, we use the inequality

$$P_n\left(\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}|\xi_i^n|\,\mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_i^n|>\sqrt{t}\right) > \frac{\alpha t}{2}\right)$$
$$\le P_n\left(\frac{1}{b_n\sqrt{n}}\max_{k=1,\ldots,\lfloor nt\rfloor}|\xi_i^n| > \sqrt{t}\right)$$
$$+ P_n\Big(\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}|\xi_i^n|\,\mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_i^n|>\sqrt{t}\right)$$
$$\mathbf{1}\left(\frac{1}{b_n\sqrt{n}}|\xi_i^n|\le\sqrt{t}\right) > \frac{\alpha t}{2}\Big). \quad \text{(A.22)}$$

We first work with the second probability on the right. We have, for $\lambda > 0$ by Chebyshev's inequality

$$P_n\Big(\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}|\xi_i^n|\,\mathbf{1}\Big(\frac{b_n}{\sqrt{n}}|\xi_i^n|>\sqrt{t}\Big)$$
$$\mathbf{1}\Big(\frac{1}{b_n\sqrt{n}}|\xi_i^n|\le\sqrt{t}\Big) > \frac{\alpha t}{2}\Big)$$

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$$
\begin{aligned}
&\le \Big(E_n \exp\Big(2\lambda \frac{b_n}{\sqrt{n}}|\xi_1^n|\,\mathbf{1}\Big(\frac{b_n}{\sqrt{n}}|\xi_1^n| > \sqrt{t}\Big) \\
&\mathbf{1}\Big(\frac{1}{b_n\sqrt{n}}|\xi_1^n| \le \sqrt{t}\Big)\Big)\Big)^{\lfloor nt \rfloor} \exp(-\lambda b_n^2 \alpha t) \\
&\le \exp\Big(nt \log E_n\Big(\exp\Big(2\lambda \frac{b_n}{\sqrt{n}}|\xi_1^n|\Big)\,\mathbf{1}\Big(\frac{b_n}{\sqrt{n}}|\xi_1^n| > \sqrt{t}\Big) \\
&\qquad \mathbf{1}\Big(\frac{1}{b_n\sqrt{n}}|\xi_1^n| \le \sqrt{t}\Big)\Big) - \lambda b_n^2 \alpha t\Big). \qquad \text{(A.23)}
\end{aligned}
$$

Next, for $0 < \beta < 1, c > 0$ and $\lambda c \le \gamma/4$,

$$
\begin{aligned}
&E_n \exp\left(2\lambda \frac{b_n}{\sqrt{n}}|\xi_1^n|\right) \mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_1^n| > \sqrt{t}\right) \mathbf{1}\left(\frac{1}{b_n\sqrt{n}}|\xi_1^n| \le \sqrt{t}\right) \\
&\le E_n \exp\left(2\lambda \frac{b_n}{\sqrt{n}}|\xi_1^n|\right) \mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_1^n|^{1-\beta} > c\right) \\
&\mathbf{1}\left(\frac{1}{b_n\sqrt{n}}|\xi_1^n| \le \sqrt{t}\right) \\
&+ E_n \exp\left(2\lambda \frac{b_n}{\sqrt{n}}|\xi_1^n|\right) \mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_1^n|^{1-\beta} \le c\right) \mathbf{1}\left(\frac{b_n}{\sqrt{n}}|\xi_1^n| > \sqrt{t}\right) \\
&\le \exp\left(2\lambda b_n^2 \sqrt{t} - \gamma \left(\frac{c\sqrt{n}}{b_n}\right)^{\frac{\beta}{1-\beta}}\right) E_n \exp\left(\gamma |\xi_1^n|^\beta\right) \\
&\quad + E_n \exp\left(\left(2\lambda c + \frac{\gamma}{2}\right)|\xi_1^n|^\beta\right) \exp\left(-\frac{\gamma}{2}\left(\frac{\sqrt{nt}}{b_n}\right)^\beta\right). \qquad \text{(A.24)}
\end{aligned}
$$

Taking $\lambda = 1/(2\sqrt{t})$ and $c = \gamma\sqrt{t}/2$, and using the condition $n^{\beta/2}/b_n^{2-\beta} \to \infty$ as $n \to \infty$, we conclude that the rightmost side of (A.24) is not greater than $\exp\left(-\tilde{C}\left(\sqrt{nt}/b_n\right)^\beta\right)$ for some $\tilde{C} > 0$. Substituting the estimate into (A.23) and again using the convergence $n^{\beta/2}/b_n^{2-\beta} \to \infty$ implies that, for all n and t large enough,

$$
\begin{aligned}
&P_n\Big(\frac{1}{b_n\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} |\xi_i^n|\,\mathbf{1}\Big(\frac{b_n}{\sqrt{n}}|\xi_i^n| > \sqrt{t}\Big) \\
&\mathbf{1}\Big(\frac{1}{b_n\sqrt{n}}|\xi_i^n| \le \sqrt{t}\Big) > \frac{\alpha t}{2}\Big) \\
&\qquad\qquad \le \exp\left(-C_1'' b_n^2 \sqrt{t}\right).
\end{aligned}
$$

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By a similar argument, this bound is seen to hold for $\beta = 1$ as well.

Finally, the first term on the right of (A.22) is estimated as

$$P_n\left(\frac{1}{b_n\sqrt{n}} \max_{i=1,\dots,\lfloor nt\rfloor} |\xi_i^n| > \sqrt{t}\right) \le nt\frac{E_n e^{\gamma|\xi_1^n|^\beta}}{e^{\gamma(b_n\sqrt{nt})^\beta}}$$
$$\le \exp(-C_2'(b_n\sqrt{nt})^\beta).$$

Substituting the estimates into (A.21) finishes the proof of (ii). □

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Appendix B

Notes and remarks

Part I

This part considers idempotent analogues of the constructions of probability theory. They also belong to the realm of possibility theory so one can replace the adjective "idempotent" with "possibilistic" (or, perhaps, "fuzzy"). The observation of the analogy between certain probabilistic and "max-plus" constructions seems to have first been made in Baccelli et al. [8].

Section 1.1

It appears that maxitive measures were first introduced by Shilkret [119], who also studied properties such as convergence, Egorov's theorem, and others. Idempotent measures are known as possibility measures in fuzzy measure theory, see, e.g., Dubois and Prade [40], Wang and Klir [133], de Cooman, Kerre and Vanmassenhove [33], Pap [101] (who also uses the term "maxitive measure"), Mesiar [86], and references therein; and as A-measures in idempotent measure theory, see Kolokoltsov and Maslov [73] (del Moral in Kolokoltsov and Maslov [73] uses the name "performance measure"). Another name is "cost measure", see Akian, Quadrat and Viot [2, 3]. Both Wang and Klir [133] and Kolokoltsov and Maslov [73] use the requirement of τ-maxitivity as a definition and call the property fuzzy additivity and complete additivity, respectively. Pap [101] uses the name "complete maxitivity". Some authors replace the τ-maxitivity

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property by σ-maxitivity, see, e.g., Akian [1], Pap [101]. In the topological setting similar objects have been studied by Norberg [95], and O'Brien and Vervaat [97]. The latter authors use the name "sup-measure", which is explained by the "sup-representation" (1.1.2), and require certain inner and outer regularity properties rather than τ-smoothness. Possibility measures with inner and outer regularity properties on topological spaces have been considered by Janssen, de Cooman and Kerre [68].

Our usage of the concept of τ-smoothness is consistent with the one adopted in measure theory, see Topsøe [125], Vakhania, Tarieladze and Chobanyan [126]. A τ-smooth idempotent measure is a specific case of a Choquet capacity, see, e.g., Meyer [88] or Neveu [94]. Our study uses some of the ideas as well as the terminology of the theory of Choquet capacities. The definition of a maxitive set function is due to Norberg [95].

Since the collection $\mathcal{E}_{iu}$ contains the collection of $\mathcal{E}$-analytic (or Suslin) subsets of Ω, see, e.g., Kuratowski and Mostowski [76], Meyer [88] or Neveu [94] for the definition, Theorem 1.1.7 is a (very simple) analogue of Choquet's theorem, Meyer [88, T19], Neveu [94]. The definition of a paving is borrowed from Meyer [88].

Theorem 1.1.9 is in the theme of Meyer [88, Theorem IIIT23] and Wang and Klir [133, Theorem 4.9], and runs parallel to the result on the extension of a measure from a ring to a σ-ring, see, e.g., Halmos [58]. The proof uses the construction of Wang and Klir. One can also obtain the same extension of μ by applying the construction used by Meyer [88, IIIT23]: define, for $A \in \mathcal{E}_u$,

$$\mu^*(A) = \sup_{\substack{F \in \mathcal{E} \\ F \subset A}} \mu(F),$$

and, for arbitrary $B \subset \Omega$, let

$$\mu^*(B) = \inf_{\substack{A \in \mathcal{E}_u \\ A \supset B}} \mu^*(A).$$

However, checking the necessary properties is more complicated. The latter approach is better suited to σ-maxitive measures, cf. Akian [1].

Objects that we call τ-algebras have been known in possibility theory as ample fields or complete fields, see Wang [132], De Cooman and Kerre [31, 32], and Wang and Klir [133]. Both our definition

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of and notation for atoms are consistent with Wang [132], and De Cooman and Kerre [31, 32]. Most of the properties of τ-algebras stated in this section can be found in these papers. Corollary 1.1.22 has been prompted by Neveu [94, Proposition I.6.1].

Section 1.2

Functions measurable with respect to ample fields are called fuzzy variables in fuzzy measure theory, see Wang [132], De Cooman and Kerre [31, 32], and Wang and Klir [133]; Janssen, de Cooman and Kerre [68] use the name "possibilistic variables". Measurability properties for more general set functions are considered in Pap [101]. Images of possibility measures are considered in Wang [132], de Cooman, Kerre and Vanmassenhove [33].

Lemma 1.2.7 is an analogue of Doob's theorem on representation of measurable functions, see, e.g., Meyer [88, IT18]. The proof is also along the lines of the proof given in Meyer [88, IT18].

O'Brien and Vervaat [97] distinguish between tightness and classical tightness. We need only the latter concept for which we reserve the name "tightness".

The notion of Luzin measurability with respect to idempotent measures is an analogue of Luzin measurability in measure theory, see Schwartz [118], Vakhania, Tarieladze and Chobanyan [126]. Theorem 1.2.14 is an abstract version of a result in large deviation theory (cf., e.g., Deuschel and Stroock [36]).

Section 1.3

Modes of convergence have also been studied by Shilkret [119] and del Moral in Kolokoltsov and Maslov [73]. Extensions of many of the results of the section to more general set functions are given in Pap [101]. For analogues in probability theory see, e.g., Shiryaev [120].

Section 1.4

The notion of idempotent integral has been introduced by Shilkret [119], who also studied its basic properties, but the name seems to

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be due to Maslov [84, 85]. Similar constructions appear in Norberg [95], see also Vervaat [130]. More general integrals are studied in fuzzy measure theory, see Dubois and Prade [40], de Cooman, Kerre and Vanmassenhove [33], Wang and Klir [133], Wu, Wang, and Ma [137], Pap [101], de Cooman and Kerre [31], Mesiar [86], Guo, Zhang, and Wu [56], Mesiar and Pap [87], and references therein; integrals of lattice-valued functions have been considered in Akian [1], de Cooman and Kerre [31], Mesiar and Pap [87], Pap [101].

Lemma 1.4.5 also holds for so called pan integrals, Wang and Klir [133]. Part 1 of Lemma 1.4.7 appears in Kolokoltsov and Maslov [73], Theorem 1.4.11 is stated by del Moral in Kolokoltsov and Maslov [73], it also appears in Puhalskii [111]. Sufficiency of the existence of the function F for uniform maximability in Corollary 1.4.14 is stated by del Moral in Kolokoltsov and Maslov [73], who also studies convergence of idempotent integrals. Our analysis of the convergence properties is based on Puhalskii [111]. For convergence properties in a more general setting see Pap [101].

The proof of Theorem 1.4.22 uses the ideas of the proof of Daniell's theorem in Meyer [88, III.2.24]. For another form of the Daniell property see Pap [101].

Section 1.5

Products of idempotent measures have been studied in more generality in fuzzy set theory, see, e.g., de Cooman, Kerre and Vanmassenhove [33], Janssen, de Cooman and Kerre [68], and references therein; they have also been analysed by Kolokoltsov and Maslov [73]. Products of ample fields have been considered by Wang [132].

Section 1.6

Exposition is based on Puhalskii [111], who however conditions on collections of analytic sets rather than τ-algebras. The notions of independence and conditioning for idempotent variables in the fuzzy set theory context have been studied in Wang [132], de Cooman, Kerre and Vanmassenhove [33]. Similar properties as well as a definition of conditional idempotent expectation with respect to σ-algebras are considered by del Moral in Kolokoltsov and Maslov [73] (see Re-

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mark 1.6.26). For other approaches see Akian, Quadrat and Viot [2, 3].

Absolute continuity has been studied in the fuzzy set theory context, see, e.g., Mesiar [86] and references therein; a general treatment appears in Pap [101]; however, the definitions are stated for σ-algebras and the weaker notion of absolute continuity (cf. Remark 1.6.30).

Section 1.7

The concept of Luzin measurability for idempotent variables on topological spaces has been introduced in Puhalskii [111], for the measure-theoretic analogue see Schwartz [118], and Vakhania, Tarieladze and Chobanyan [126].

The first result in the theme of Theorems 1.7.21, 1.7.23, and 1.7.25 seems to be due to Choquet [24]. Theorems 1.7.21 and 1.7.25 appear in Breyer and Gulinski [17]; our proof of Theorem 1.7.25 uses their idea of invoking the Stone-Čzech compactification. Puhalskii [107, 108] proves a similar result for metric spaces under the additional condition of sub-additivity of the functional V. Kolokoltsov and Maslov [73, Theorem 1.5, ch.1] prove the result of Theorem 1.7.21 for a locally compact normal space and functions with values in an idempotent metric semiring. They also prove the stated representation for the case where V is a continuous homeomorphism from $C_b^+(E)$ equipped with the topology of pointwise convergence to an idempotent metric semiring and E is Tihonov. Akian [1] considers the same characterisation in terms of continuity for integrals of lattice-valued functions. Algebraic versions appear in Litvinov, Maslov and Shpiz [80, 81].

Section 1.8

In this section we use some of the ideas of Schwartz [118]. The result in Lemma 1.8.3 is a special case of a result in large deviation theory due to Dawson and Gärtner [28], see also Dembo and Zeitouni [35]. The setting of Theorem 1.8.6 for regular possibility measures has been considered by Janssen, de Cooman and Kerre [68].

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Sections 1.9 and 1.10

The results are modelled after weak convergence theory of probability measures, see Billingsley [11], Parthasarathy [102], Topsøe [125, 124], Vakhania, Tarieladze and Chobanyan [126]. For a prototype see Vervaat [129]. The setting of metric spaces is studied in Puhalskii [108]. For facts about uniform spaces used in the proof of Theorem 1.9.2 see, e.g., Engelking [47]. Theorem 1.9.28 is an analogue of Ranga Rao's result, see Vakhania, Tarieladze and Chobanyan [126]. More general compactness results and other properties of the vague topology are in O'Brien and Vervaat [97] and O'Brien and Watson [99].

For the definitions of the Prohorov and Kantorovich-Wasserstein metrics for probability measures see, e.g., Dudley [41]. Jiang and O'Brien [69] define the Prohorov metric on a space of set functions that includes idempotent probability measures and probability measures and show, in particular, that it metrises convergence of sequences in the narrow topology; they also show that the Kantorovich-Wasserstein metric has this property for sequences of the exponentials of rate functions and address the issue of characterising tight collections as totally bounded sets.

For the definition and properties of Mosco convergence see Mosco [92], Zabell [138] and references therein.

Section 1.11

Kolokoltsov and Maslov [73] refer to the Laplace-Fenchel transform as the Fourier-Legendre transform. The inversion formula appears in Puhalskii [108]. Lemma 1.11.19 is also taken from the latter paper. For required facts from convex analysis see Rockafellar [117] and Appendix A.

Sections 2.1 – 2.6

The results and approaches are analogous to those in stochastic calculus, see Dellacherie [34], Elliott [45], Ikeda and Watanabe [66], Jacod and Shiryaev [67], Liptser and Shiryaev [79], Meyer [88], Neveu [94], Øksendal [100], and Stroock and Varadhan [123]. Idempotent martingales have been considered by Del Moral in Kolokoltsov and Maslov [73] (for conditional expectations with respect to σ-algebras).

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For other approaches see Akian, Quadrat and Viot [2, 3]. "Possibilistic" processes have been studied in Janssen, de Cooman and Kerre [68].

Theorems 2.2.26 and 2.2.27 are taken from Puhalskii [108]. Section 2.3 is based on Puhalskii [111]. The definitions of the idempotent Wiener and Poisson processes in Section 2.4 are motivated by the fact that the associated rate functions appear in the large deviation principles for Wiener and Poisson processes, respectively, see, e.g., Borovkov [13], Freidlin and Wentzell [51].

For the properties of the pseudo-inverses of matrices see, e.g., Campbell and Meyer [20].

Theorem 2.6.22 is in essence the Picard-Lindelöf-Caratheodory theorem, see, e.g., Coddington and Levinson [26], Hartman [60].

Sections 2.7

The results are based on Puhalskii [109, 111]. We follow the ideas of stochastic calculus, see, e.g., Liptser and Shiryaev [79], Jacod and Shiryaev [67]; in particular, the definition of a semimaxingale is analogous to the exponential characterisation of semimartingales. Del Moral in Kolokoltsov and Maslov [73] has considered idempotent semimartingales for conditional expectations with respect to σ-algebras.

Lemma 2.7.5 is in essence due to Liptser and Shiryaev [79, Theorem 6.2.3], whose argument also applies to the proof.

Theorem 2.7.16 admits a revealing interpretation in terms of Orlicz spaces, Krasnosel'skii and Rutickii [75]. Specifically, for $\mathbf{x} \in \mathbb{C}$ and $t \in \mathbb{R}_+$, let $L^*_{\hat{g}(\mathbf{x})}(0,t)$ denote the set of all functions $f(s), s \le t$, such that

$$\int_0^t \hat{g}_s\left(\frac{1}{\alpha}f(s);\mathbf{x}\right)\,ds < \infty,$$

for some $\alpha > 0$. $L^*_{\hat{g}(\mathbf{x})}(0,t)$ is easily seen to be a vector space. Let for $f \in L^*_{\hat{g}(\mathbf{x})}(0,t)$

$$\|f\|_{L^*_{\hat{g}(\mathbf{x})}(0,t)} = \inf\Big(\alpha > 0 : \int_0^t \hat{g}_s\left(\frac{1}{\alpha}f(s);\mathbf{x}\right)\,ds \le 1\Big).$$

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This can be shown to define a seminorm on $L^*_{\hat{g}(\mathbf{x})}(0,t)$, which is a norm if $\hat{g}_s(\lambda;\mathbf{x}) \neq 0$ for $\lambda \neq 0$ (cf. Krasnosel'skii and Rutickii [75]). Let us assume, for the moment, that $\hat{g}_s(\lambda;\mathbf{x})$ does not depend on s: $\hat{g}_s(\lambda;\mathbf{x}) = \hat{g}(\lambda;\mathbf{x})$. Then the above norm is called the Luxembourg norm and $L^*_{\hat{g}(\mathbf{x})}(0,t)$ is called an Orlicz space, Krasnosel'skii and Rutickii [75]. Also in this case the set of functions, for which (2.7.18) holds, is the closure of the space of bounded functions in the Luxembourg norm, Krasnosel'skii and Rutickii [75]. In analogy with Krasnosel'skii and Rutickii [75], we denote this set by $E_{\hat{g}(\mathbf{x})}(0,t)$. We then have the following insight into the statement of Theorem 2.7.16. Let $L_{\hat{g}(\mathbf{x})}(0,t)$ be the set of functions f such that

$$\int_0^t \hat{g}(f(s);\mathbf{x})\,ds < \infty.$$

Then, by Krasnosel'skii and Rutickii [75], we have the strict inclusions

$$E_{\hat{g}(\mathbf{x})}(0,t) \subset L_{\hat{g}(\mathbf{x})}(0,t) \subset L^*_{\hat{g}(\mathbf{x})}(0,t),$$

unless $\hat{g}(\lambda;\mathbf{x})$ satisfies the weak growth condition (or the Δ_2–condition):

$$\limsup_{\lambda\to\infty} \frac{\hat{g}(2\lambda;\mathbf{x})}{\hat{g}(\lambda;\mathbf{x})} < \infty.$$

If $g(\lambda;\mathbf{x})$ has the semimartingale representation (2.7.55), then the weak growth condition means that $K_s(\mathbb{R}^d;\mathbf{x}) = L_s(\mathbb{R}^d;\mathbf{x}) = 0$, i.e., it holds only in "the diffusion case". Hence, generally, the class of functions $\overline{\lambda}$, for which we have proved that $Z(\overline{\lambda})$ is a Π-local exponential maxingale, is smaller than the class defined by the condition

$$\int_0^t |g_s(\lambda(s,\mathbf{x});\mathbf{x})|\,ds < \infty.$$

We do not know if Theorem 2.7.16 can be extended to a larger set of functions $\overline{\lambda}$. We also note that the proof of Theorem 2.7.16 has been prompted by the methods of the theory of Orlicz spaces.

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Section 2.8

The results are based on Puhalskii [112]. Our conditions for uniqueness of solutions to maxingale problems are similar to conditions required for corresponding martingale problems. It is thus instructive to compare our results with those for martingale problems in Ikeda and Watanabe [66], Jacod and Shiryaev [67], Stroock and Varadhan [123].

The setting of Theorem 2.8.5 corresponds to the situation where a martingale problem is specified by a deterministic triplet of predictable characteristics so that the associated process is a process with independent increments; the problem then has a unique solution, see, e.g., Jacod and Shiryaev [67, Theorem III.2.16].

The function $\Lambda(s, \mathbf{x}; y)$ in the hypotheses of Theorem 2.8.27 exists and equals the gradient $\nabla h_s(y; \mathbf{x})$ if the latter exists for (almost all) $s \in \mathbb{R}_+, y \in G$ and $\mathbf{\Pi}_x$-almost all $\mathbf{x}$, and is bounded on the sets $[0, t] \times K \times G_m$, where $t \in \mathbb{R}_+, m \in \mathbb{N}$ and K is compact in $\mathbb{C}$.

The role of conditions I and II and the conditions in Theorem 2.8.33 is analogous to the role of conditions A–E in Wentzell [134]. A distinctive feature of our conditions is that they are stated only in terms of the cumulant and do not invoke its Fenchel–Legendre transform (as in Wentzell [134]). We believe that this makes the conditions easier to check. Also we relax the requirements on boundedness and continuity of the cumulant.

Condition (2.8.14) is analogous to condition III in Liptser and Puhalskii [78].

The regularisation approach of Lemma 2.8.26 applied later in the section has earlier been used in Wentzell [134], and Liptser and Puhalskii [78] in the large deviation setting.

For background on the notions used in Lemma 2.8.31 see Aubin and Cellina [5], von Leichtweiss [131], and Rockafellar [117]. For measurable-selection theorems see Clarke [25, Theorem 4.1.1], Ethier and Kurtz [48], or Dellacherie [34, IT37].

Part II

Exposition is based on Puhalskii [106] – [114]. Standard manuals on large deviation theory are Freidlin and Wentzell [51], Varadhan [128], Deuschel and Stroock [36], Dembo and Zeitouni [35].

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Section 3.1

Some of the results of this section are large deviation convergence versions of the results formulated in the setting of the large deviation principle, see Varadhan [128], Stroock [122], Deuschel and Stroock [36], Dembo and Zeitouni [35], Bryc [18], Dinwoodie [37]. For an approach from the point of view of convergence of capacities see O'Brien and Vervaat [97], O'Brien [96], O'Brien and Watson [99]. Since this section considers similar issues as Section 1.9, most of the comments to that section apply here. In particular, there are many analogies with results in weak convergence theory, see, Billingsley [11], Parthasarathy [102], Topsøe [125, 124], Vakhania, Tarieladze and Chobanyan [126]. The definition of the large deviation convergence and the term itself have been introduced in Puhalskii [108, 109]. Properties of a more general type of convergence have been considered by Pap [101], Mesiar and Pap [87].

Theorem 3.1.3 for the setting of metric spaces has appeared in Puhalskii [107]. It combines a number of earlier results. The fact that part 3 implies part 2 is "Varadhan's lemma", Varadhan [127], who also proves Lemma 3.1.12, the converse under the additional condition of exponential tightness is due to Bryc, see Varadhan [128], Bryc [18], Dembo and Zeitouni [35]. Instead of the definition we have adopted for large deviation convergence one could use part 2 of Theorem 3.1.3 in order to define "weak large deviation convergence". It would then be equivalent to the large deviation principle, or "narrow large deviation convergence", cf. Remark 1.9.6.

The name "contraction principle" as given by Varadhan [128] refers to the case of continuous f in Corollary 3.1.15. Corollary 3.1.15 for convergence of sequences has appeared in Puhalskii [106], it is referred to in Puhalskii and Whitt [113] as the extended contraction principle. Theorem 3.1.14 for metric spaces and sequences of probability measures has been proved in Puhalskii [110]; Chaganty [22] proves the statement in the setting of Polish spaces under the assumption that the convergence $f_n(z_n) \to f(z)$ holds for every $z \in E$, a similar type of condition has been used by Dinwoodie and Zabell [38]. Other versions and generalisations have been considered by Deuschel and Stroock [36] and O'Brien [96].

Theorems 3.1.19 and 3.1.28 are analogues of Prohorov's criterion for weak convergence, Prohorov [104], see also Billingsley [11],

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Vakhania, Tarieladze and Chobanyan [126]. Theorem 3.1.28 has appeared in Puhalskii [106]. The proofs here are along the lines of the one in Puhalskii [107], who also mentions the extension to Tihonov spaces. A more general setting has been considered by O'Brien and Vervaat [97] so that Theorems 3.1.19 and 3.1.28 for separable metric spaces follow from Theorem 3.11 and Lemma 5.2 there; an announcement had been made in Vervaat [129]. Lynch and Sethuraman [82] have proved that LD convergence implies exponential tightness for sequences of probability measures on complete separable metric spaces; an extension appears in Jiang and O'Brien [69]. De Acosta [30] has generalised and simplified the proof of Puhalskii [106] to give weaker conditions for part 1 of Theorem 3.1.19 to hold, in particular, extending the result to Hausdorff topological spaces (similarly to Prohorov's tightness criterion in weak convergence theory, see Topsøe [125]) and probabilities on non-Borel σ-algebras. For other versions and extensions see Vervaat [129], O'Brien and Vervaat [98]. The vague large deviation convergence has extensively been studied by O'Brien and Vervaat [97, 98] who prove Theorem 3.1.34.

Jiang and O'Brien [69], considering a more general setting, prove that the Prohorov and the Kantorovich-Wasserstein metrics metrise LD convergence of sequences. They also address the issue of characterising tight collections as totally bounded sets. See also Dembo and Zeitouni [35] for some of the results.

Theorem 3.1.31, which is usually referred to as the Gärtner-Ellis theorem, has been proved by Gärtner [53] for the case where $G(\lambda)$ is smooth and finite everywhere. The extension to the essentially smooth case has been obtained by Freidlin and Wentzell [51] and, later and apparently independently, by Ellis [46]. The name "Gärtner-Ellis" appears to have been first used in Bucklew [19]. Our method of proof follows Puhalskii [109] and O'Brien and Vervaat [98].

Theorem 3.1.32 is an analogue of Ranga Rao's result in weak convergence, see Vakhania, Tarieladze and Chobanyan [126]. In a somewhat different form it appears in Jiang and O'Brien [69]. Part 1 of Lemma 3.1.42 in the form of the LDP extends to regular spaces, see Cegła and Klimek [21].

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Section 3.2

For more background on the concepts and properties concerning stochastic processes on Skorohod spaces the reader is referred to Ethier and Kurtz [48], Jacod and Shiryaev [67], Lindvall [77], Liptser and Shiryaev [79], Skorohod [121], and Whitt [135].

The notion of $\mathbb{C}$-exponential tightness has been introduced in Liptser and Puhalskii [78]. Theorem 3.2.3 is an analogue of Aldous' tightness condition, Aldous [4], and has appeared in Puhalskii [106]. Lemma 3.2.5 can also be used in order to give a somewhat different proof of Aldous' original result. Feng and Kurtz [50] obtain different exponential tightness conditions.

For the Lenglart-Rebolledo inequality, see, e.g., Liptser and Shiryaev [79, Theorem 1.9.3].

Theorems 3.2.8 and 3.2.9, taken from Puhalskii [106, 107, 109, 111], are analogues of the methods of finite-dimensional distributions and the martingale problem in weak convergence theory, cf. Jacod and Shiryaev [67], Liptser and Shiryaev [79], Ethier and Kurtz [48].

Lemmas 3.2.11 and 3.2.13 are LD convergence versions of results in Puhalskii and Whitt [113], which are also more general. See also Lemma 4.2 there for more detail about the proof of Lemma 3.2.13. Prototypes for weak convergence are in Whitt [135].

Chapter 4

The method of finite-dimensional distributions is essentially an adaptation of projective limit arguments, see Dawson and Gärtner [28], to the setting of stochastic processes. It was first used by Varadhan [127], see also Dembo and Zeitouni [35]. Our exposition follows Puhalskii [108, 109]. The main results, both in content and form, are analogous to results on convergence in distribution of a sequence of semimartingales to a process with independent increments in Liptser and Shiryaev [79] and Jacod and Shiryaev [67]. For results on the LDP for processes with independent increments see Varadhan [127], Borovkov [13], Mogulskii [90, 91], Lynch and Sethuraman [82], de Acosta [29], Puhalskii [106].

Lemma 4.1.1 is an analogue of results in Liptser and Shiryaev [79, Chapter 2, §3], Jacod and Shiryaev [67, Chapter 2, §2d] for complex-valued stochastic exponentials. Theorem 4.1.2 is an ana-

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logue of Jacod and Shiryaev [67, Theorem VIII.2.30]. Lemma 4.2.6 is a variation on the theme of Polya's theorem, see, e.g., Liptser and Shiryaev [79, Problem 5.3.2].

The proof of Theorem 4.2.11 uses the method of the proof of Theorem 5.4.1 in Liptser and Shiryaev [79]. In particular the fundamental decomposition (LS^{ϕ}) originates in Lemma 5.4.1 there. In the proof of Theorem 4.2.11 condition ($\hat{\nu}$) has been used only while proving convergence γ). Since γ) clearly holds under the condition

$$
(\log \hat{\nu}) \qquad \begin{aligned} &\frac{1}{r_\phi} \sum_{0<s\le t} \Big(f(r_\phi x) \bullet \nu_s^{\phi} - \ln\big(1 + f(r_\phi x) \bullet \nu_s^{\phi}\big)\Big) \\ &- \int_0^t \Big(f(x) \bullet L_s - \ln\big(1 + f(x) \bullet L_s\big)\Big) ds \overset{P_\phi^{1/r_\phi}}{\to} 0 \\ &\qquad\qquad \text{as } \phi \in \Phi,\ t \in U,\ f \in \mathcal{C}_b, \end{aligned}
$$

and, as the proof shows, condition ($\hat{\nu}$) implies condition ($\log \hat{\nu}$), it follows that the theorem holds when condition ($\hat{\nu}$) is replaced by condition ($\log \hat{\nu}$). One can show that these two conditions are actually equivalent.

Corollary 4.3.5 and its proof are analogous to Proposition VIII.3.40 in Jacod and Shiryaev [67].

Condition (L_2) and Lemma 4.3.9 originate from Djellout [39]. The argument of the proof of Theorem 4.4.6 follows Djellout [39].

LDPs for partial-sum processes have been studied in Varadhan [127], Borovkov [13], and Mogulskii [90, 91].

Conditions (4.4.10) have been found by Ibragimov and Linnik [61, Theorem 13.1.1] who studied exact asymptotics in the nonfunctional case and showed that the moment condition in (4.4.10) is in a certain sense necessary, see [61, Theorem 13.1.2]. Mogulskii [90, Theorem 1] has established a functional LDP under (4.4.10) for the setting of Theorem 4.4.6. Logarithmic asymptotics for sums of i.i.d.r.v. under (4.4.9) can also be derived from the estimates of the convergence rate in the CLT in Ibragimov and Linnik [61, Chapter 3] and Petrov [103, Chapter 5].

Example 4.4.13 is motivated by Theorem VIII.3.43 in Jacod and Shiryaev [67].

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Chapter 5

The results are based on Puhalskii [111, 112] (note, however, that our condition (NE) is somewhat stronger than in Puhalskii [111], so a correction needs to be made in that paper). The majoration and continuity conditions are similar to those used in weak convergence theory, see, e.g., Jacod and Shiryaev [67, VI.3.34]. Theorems 5.3.3, 5.3.4 and 5.3.5 are analogues of respective Theorems 8.2.1, 8.4.2 and 8.3.1 in Liptser and Shiryaev [79]. For the case where A_t^ϕ is continuous Theorem 5.3.7 is an analogue of Theorem 8.4.1 in Liptser and Shiryaev [79]. Theorem 5.4.3 is an analogue of Theorems IX.4.8 and IX.4.15 in Jacod and Shiryaev [67].

The setting of Liptser and Puhalskii [78], who have considered large deviations for quasi-continuous processes with the Cramér condition on the jumps and linearly growing coefficients, is a special case of the setting of Theorem 5.3.3 for the case where conditions $(A)_{loc} + (a)_{loc}$ are checked by checking $(I_e)_{loc}$. In particular, Theorem 2.2 there follows by Theorem 5.3.7. As for Theorem 2.1 of Liptser and Puhalskii [78], it is not quite clear if it follows from our results since both results require some implicit conditions on the rate functions, which are difficult to compare. However, for the case of explicit sufficient conditions given by Theorem 9.1 in Liptser and Puhalskii [78], one can obtain Theorem 2.1 there as a corollary of Theorems 5.3.3 and 2.8.33.

The Markov setting has been analysed in Freidlin and Wentzell [51], Wentzell [134], Dupuis and Ellis [43], Feng [49], Feng and Kurtz [50], see also Azencott [7], Baldi [9], Baldi and Chaleyat-Maurel [10], Friedman [52], Makhno [83], Micami [89], Narita [93], Remillard and Dawson [116]. Diffusions with dependence on the past have been considered by Cutland [27]. Markov processes with discontinuous statistics have been studied by Blinovskii and Dobrushin [12], Dupuis, Ellis and Weiss [44], Dupuis and Ellis [42], Korostelev and Leonov [74], Ignatyuk, Malyshev and Scherbakov [65].

Theorem 5.4.1, taken from Puhalskii [111], is a large-deviation analogue of a result that derives convergence of Markov semigroups from convergence of associated generators, see, e.g., Ethier and Kurtz [48]. Feng [49] has obtained a similar result for Markov processes with values in general metric spaces by the methods of nonlinear semigroup convergence, see Feng and Kurtz [50] for further develop-

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ments in this direction.

Theorem 5.4.4 relaxes the assumptions of Theorems 4.4.1, 4.4.2, 4.4.2′, and 4.4.3 in Wentzell [134], and Theorem 5.4.2 in combination with Theorem 2.8.33 relaxes the assumptions of Wentzell's Theorems 4.3.1, 3.2.3 and 3.2.3'. The main improvements are that we do not require that either $b_s(u)$, or $c_s(u)$, or $\nu_s(dx;u)$, or $g_s(\lambda;u)$ be either continuous in the time variable or bounded, and the associated convergences may take place only locally uniformly, not necessarily uniformly on the entire space.

Theorem 5.4.8 has been motivated by Theorem 4.5.2 in Wentzell [134]. A similar result can be proved for discrete-time processes.

Example 2 is prompted by Problem 8.4.1 in Liptser and Shiryaev [79]. The proof of (5.4.11) is based on Liptser's idea (private communication).

Section 6.1

The results are based on Puhalskii [114], where more detail is given. They complement results on diffusion approximation for queues in Kingman [72], Prohorov [105], Iglehart and Whitt [64], Borovkov [15], and Reiman [115]. Theorem 6.1.7 is an analogue of diffusion approximation results in Prohorov [105] and Borovkov [15]; the proof borrows ideas used in these proofs. The results of Subsection 6.1.2 are inspired by Reiman [115].

Section 6.2

Theorem 6.2.3 complements diffusion approximation results by Iglehart [62, 63], Borovkov [14, 16], Halfin and Whitt [57], and Whitt [136]. For other results on large deviation asymptotics for many server queues see Glynn [55] and Zajic [139].

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